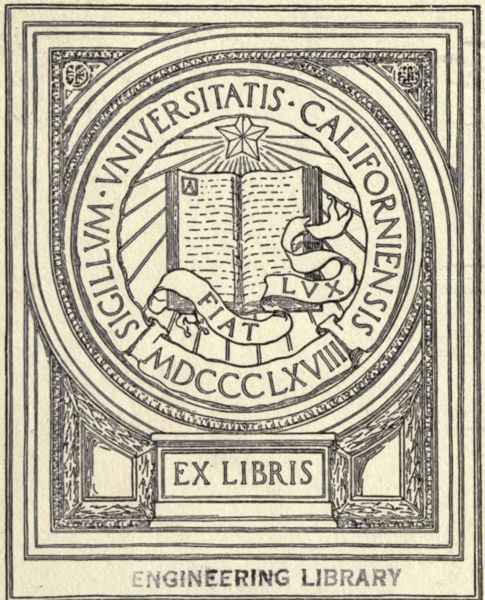

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PREFACE.

The object of this book is to set forth in a compact form those parts of the Science of Mechanics which are practically applicable to Structures and Machines. Its plan is sufficiently explained by the Table of Contents, by the Introduction, and by the initial articles of the six parts into which the body of the treatise is divided.

This work, like others of the same class, contains facts and principles that have been long and widely known, mingled with others, of which some are the results of the labours of recent discoverers, some have been published only in scientific Transactions and periodicals, not generally circulated, or in oral lectures, and some are now published for the first time. I have endeavoured, to the best of my knowledge, to mention in their proper places the authors of recent discoveries and improvements, and to refer to scientific papers which have furnished sources of information.

A branch of Mechanics not usually found in elementary treatises is explained in this work, viz., that which relates to the equilibrium of stress, or internal pressure, at a point in a solid mass, and to the general theory of the elasticity of solids. It is the basis of a sound knowledge of the principles of the stability of earth, and of the strength and stiffness of materials; but, so far as I know, the only elementary treatise on it that has hitherto been published is that of M. Lamé, entitled *Leçons sur la Théorie mathématique de l'Élasticité des Corps solides.*

In treating of the stability of arches, the lateral pressure of the load is taken into account. So far as I know, the only author who has hitherto done so in an exact manner, is M. Yvon-Villarceaux, in the *Mémoires des Savans étrangers.*

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The principle of the transformation of structures and its applications have hitherto appeared in the *Proceedings of the Royal Society* alone.

The correct laws of the flow of elastic fluids (first investigated by Dr. Joule and Dr. Thomson), and the true equations of the action of steam and other vapours against pistons, as deduced from the principles of thermodynamics, by Professor Clausius and myself, contemporaneously, are now for the first time stated and applied in an elementary manual.

Other portions of the work, which are wholly or partly new, are indicated in their places.

In the arrangement of this treatise an effort has been made to adhere as rigidly as possible to a methodical classification of its subjects; and, in particular, care has been taken to keep in view the distinction between the comparison of motions with each other, and the relations between motions and forces, which was first pointed out by Monge and Ampère, and which Mr. Willis has so successfully applied to the subject of mechanism. The observing of that distinction is highly conducive to the correct understanding and ready application of the principles of Mechanics.

W. J. M. R.

GLASGOW UNIVERSITY, *May*, 1858.

ADVERTISEMENT TO THE TENTH EDITION.

The *Tenth Edition* has been carefully revised, and new matter, bearing on subjects treated of in the text, has been added to the Appendix. The Index has also been enlarged, and rendered more suitable for reference.

For various notes and suggestions, the Editor begs to thank, amongst others, Prof. Eddy, of Cincinnati University, and Mr. Arthur W. Thompson, B.Sc., late of the Imperial College of Engineering, Japan.

W. J. M.

GLASGOW, *Jan., 1882.*

CONTENTS.

Page
PRELIMINARY DISSERTATION ON THE HARMONY OF THEORY AND PRACTICE IN MECHANICS, ... 1
INTRODUCTION,—Definition of General Terms, and Division of the Subject, . 13

Article Page
1. Mechanics, ... 13
2. Applied Mechanics, . . . 13
3. Matter, ... 13
4. Bodies,—Solid, Liquid, Gaseous, 13
5. Material or Physical Volume, . 13
6. Material or Physical Surface, . 13
7. Line, Point, Physical Point, Measures of Length, . . . 13
8. Rest, ... 14
9. Motion, ... 14
10. Fixed Point, ... 14
11. Cinematics, ... 15
12. Force, ... 15
13. Equilibrium or Balance, . . 15
14. Statics and Dynamics, . . 15
15. Structures and Machines, . . 15
16. General Arrangement of the Subject, 16

PART I.—PRINCIPLES OF STATICS.

CHAPTER I.—BALANCE AND MEASUREMENT OF FORCES ACTING IN ONE STRAIGHT LINE.

17. Forces, how Determined, . . 17
18. Place of Application — Point of Application, ... 17
19. Supposition of Perfect Rigidity, . 18
20. Direction—Line of Action, . . 18
21. Magnitude—Units of Force, British and French, ... 18
22. Resultant of Forces acting in one Line, 18
23. Representation of Forces by Lines, 19
24. Pressure, ... 20

CHAPTER II.—THEORY OF COUPLES AND OF THE BALANCE OF PARALLEL FORCES.

SECTION 1.—*On Couples with the Same Axis.*

25. Couples, ... 21
26. Force of a Couple—Arm or Leverage—Moment, . . . 21
27. Tendency of a Couple—Plane and Axis of a Couple—Right-handed and Left-handed Couples, . 21
28. Equivalent Couples of equal Force and Leverage, . . . 21
29. Moment of a Couple, . . . 22
30. Addition of Couples of equal Force, 22
31. Equivalent Couples of equal Moment, ... 22
32. Resultant of Couples with the same Axis, ... 23
33. Equilibrium of Couples having the same Axis, ... 23
34. Representation of Couples by Lines, 23

SECTION 2.—*On Couples with Different Axes.*

35. Resultant of two Couples with different Axes, ... 24
36. Equilibrium of three Couples with different Axes in the same Plane, 25
37. Equilibrium of any number of Couples, ... 25

SECTION 3.—*On Parallel Forces.*

38. Balanced Parallel Forces in general, 25
39. Equilibrium of three Parallel Forces in one Plane—Principle of Lever, 26
40. Resultant of two Parallel Forces, . 26
41. Resultant of a Couple, and a single Force in Parallel Planes, . . 27
42. Moment of a Force with respect to an Axis, ... 27
43. Equilibrium of any system of Parallel Forces in one Plane, . 28
44. Resultant of any number of Parallel Forces in one Plane, . . 28
45. Moments of a Force with respect to a pair of Rectangular Axes, 29
46. Equilibrium of any system of Parallel Forces, ... 30
47. Resultant of any number of Parallel Forces, ... 30

SECTION 4.—*On Centres of Parallel Forces.*

48. Centre of a pair of Parallel Forces, 31
49. Centre of any system of Parallel Forces, ... 32
50. Co-ordinates of Centre of Parallel Forces, ... 32

CHAPTER III.—BALANCE OF INCLINED FORCES.

SECTION 1.—*Inclined Forces applied at One Point.*

51. Parallelogram of Forces, . . 35
52. Equilibrium of three Forces acting through one Point in one Plane, 36
53. Equilibrium of any system of Forces acting through one Point, 36

Page
54. Parallelopiped of Forces, . . . 37
55. Resolution of a Force into two Components, ... 37
56. Resolution of a Force into three Components, ... 37
57. Rectangular Components, . . . 38

SECTION 2.—*Inclined Forces applied to a System of Points.*

58. Forces acting in one Plane,—Graphic Solution, ... Page 39
59. Forces acting in one Plane,—Solution by Rectangular Co-ordinates, 40
60. Any system of Forces, . . . 41

CHAPTER IV.—ON PARALLEL PROJECTIONS IN STATICS.

61. Parallel Projection of a Figure defined, ... 45
62. Geometrical Properties of Parallel Projections, ... 45
63. Application to Parallel Forces, . 46
64. Application to Centres of Parallel Forces, ... 46
65. Application to Inclined Forces acting through one Point, . 46
66. Application to any system of Forces, 47

CHAPTER V.—ON DISTRIBUTED FORCES.

67. Restriction of the subject to Parallel Distributed Forces, . . 48
68. Intensity of a Distributed Force, 48

SECTION 1.—*Of Weight, and Centres of Gravity.*

69. Specific Gravity, ... 49
70. Centre of Gravity, ... 49
71. Centre of Gravity of a Homogeneous Body having a Centre of Figure, ... 49
72. Bodies having Planes or Axes of Symmetry, ... 49
73. System of Symmetrical Bodies, . 50
74. Homogeneous Body of any Figure, 51
75. Centre of Gravity found by Addition, ... 53
76. Centre of Gravity found by Subtraction, ... 53
77. Centre of Gravity altered by Transposition, ... 54
78. Centres of Gravity of Prisms and Flat Plates, ... 54
79. Body with similar Cross-Sections, 56
80. Curved Rod, ... 57
81. Approximate Computation of Integrals, ... 58
82. Centre of Gravity found by Projection, ... 61
83. Examples of Centres of Gravity, 63
84. Heterogeneous Body, . . . 68
85. Centre of Gravity found experimentally, ... 68

SECTION 2.—*Of Stress, and its Resultants and Centres.*

86. Stress—its Nature and Intensity, 68
87. Classes of Stress, ... 69
88. Resultant of Stress,—its Magnitude, ... 70
89. Centre of Stress, or of Pressure, . 71
90. Centre of Uniform Stress, . . 73
91. Moment of Uniformly-Varying Stress,—Neutral Axis, . . 73
92. Moment of Bending Stress, . 75
93. Moment of Twisting Stress, . 76
94. Centre of Uniformly-Varying Stress,—Conjugate Axis, . 76
95. Moments of Inertia of a Surface, 77

SECTION 3.—*Of Internal Stress: its Composition and Resolution.*

96. Internal Stress in general, . . 82
97. Simple Stress and its Normal Intensity, ... 83
98. Reduction of Simple Stress to an Oblique Plane, . . . 83
99. Resolution of Oblique Stress into Normal and Tangential Components, 84
100. Compound Stress, . . . 84
101. Pair of Conjugate Stresses, . 85
102. Three Conjugate Stresses, . 85
103. Planes of Equal Shear, or Tangential Stress, . . . 87
104. Stress on three Rectangular Planes, 88
105 Tetraedron of Stress, . . 90
106. Transformation of Stress, . . 92
107. Principal Axes of Stress, . . 93
108. Stress Parallel to one Plane, . 95
109. Principal Axes of Stress Parallel to one Plane, ... 98
110. Equal Principal Stresses—Fluid Pressure, ... 99
111. Opposite Principal Stresses composing Shear, ... 101
112. Ellipse of Stress,—Problems, . 101
113. Combined Stresses in one Plane, 110

SECTION 4.—*Of the Internal Equilibrium of Stress and Weight, and the principles of Hydrostatics.*

114. Varying Internal Stress, . . 112
115. Causes of Varying Stress, . . 112
116. General Problem of Internal Equilibrium, ... 113
117. Equilibrium of Fluids, . . 116
118. Equilibrium of a Liquid, . . 118
119. Equilibrium of different Fluids in contact with each other, . . 118
120. Equilibrium of a Floating Body, 120
121. Pressure on an Immersed Body, 122
122. Apparent Weights, . . . 123
123. Relative Specific Gravities, . 124
124. Pressure on an Immersed Plane, 125
125. Pressure in an Indefinite Uniformly Sloping Solid, . . 126
126. On the Parallel Projection of Stress and Weight, . . . 127

CHAPTER VI.—ON STABLE AND UNSTABLE EQUILIBRIUM.

Page
127. Stable and Unstable Equilibrium of a Free Body, ... 128
128. Stability of a Fixed Body, . 128

PART II.—THEORY OF STRUCTURES.

CHAPTER I.—DEFINITIONS AND GENERAL PRINCIPLES.

129. Structures—Pieces—Joints, . 129
130. Supports—Foundations, . . 129
131. Conditions of Equilibrium of a Structure, ... 129
132. Stability, Strength, and Stiffness, 130

CHAPTER II.—STABILITY.

133. Resultant Gross Load, . . 131
134. Centre of Resistance of a Joint, 131
135. Line of Resistance, . . . 131
136. Joints Classed, ... 131

SECTION 1.—*Equilibrium and Stability of Frames.*

137. Frame, ... 132
138. Tie, ... 132
139. Strut, ... 133
140. Treatment of the Weight of a Bar, 133
141. Beam under Parallel Forces, . 133
142. Beam under Inclined Forces, . 134
143. Load supported by three Parallel Forces, ... 135
144. Load supported by three Inclined Forces, ... 135
145. Frame of Two Bars—Equilibrium, 136
146. Frame of Two Bars—Stability, 136
147. Treatment of Distributed Loads, 137
148. Triangular Frame, . . . 137
149. Triangular Frame under Parallel Forces, ... 138
150. Polygonal Frame—Equilibrium, 139
151. Open Polygonal Frame, . . 140
152. Polygonal Frame—Stability. . 140
153. Polygonal Frame under Parallel Forces, ... 141
154. Open Polygonal Frame under Parallel Forces, . . . 142
155. Bracing of Frames, . . . 142
156. Rigidity of a Truss, . . . 144
157. Variations of Load on Truss, . 144
158. Bar common to several Frames, 145
159. Secondary Trussing—Examples from Roofs, ... 145
160. Compound Trusses, . . . 148
161. Resistance of Frame at a Section, 150
162. Half-lattice Girder, any Load, 153
163. Half-lattice Girder, Uniform Load 156
164. Lattice Girder, any Load, . . 160
165. Lattice Girder, Uniform Load, . 161
166. Transformation of Frames, . 162

SECTION 2.—*Equilibrium of Chains, Cords, Ribs, and Linear Arches.*

167. Equilibrium of a Cord, . . 162
168. Cord under Parallel Loads, . 164
169. Cord under uniform Vertical Load, 164
170. Suspension Bridge with Vertical Rods, ... 168
171 Flexible Tie, ... 169
172. Suspension Bridge with Sloping Rods, ... 171
173. Extrados and Intrados, . . 173
174. Cord with Horizontal Extrados, 175
175. Catenary, ... 177
176. Centre of Gravity of a Flexible Structure, ... 180
177. Transformation of Cords and Chains, ... 180
178. Linear Arches or Ribs, . . 182
179. Circular Arch for Uniform Fluid Pressure, ... 183
180. Elliptical Arches for Uniform Pressures, ... 184
181. Distorted Elliptic Arch, . 186
182. Arches for Normal Pressure in general, ... 189
183. Hydrostatic Arch (see also p. 353) 190
184. Geostatic Arches, . . . 196
185. Stereostatic Arch, . . . 198
186. Pointed Arches, ... 203
187. Total Conjugate Thrust of Linear Arches—Point and Angle of Rupture, ... 203
188. Approximate Hydrostatic and Geostatic Arches, . . . 207

SECTION 3.—*On Frictional Stability.*

189. Friction — distinguished from Adhesion, ... 209
190. Law of Solid Friction, . . 209
191. Angle of Repose, ... 210
192. Table of Co-efficients of Friction and Angles of Repose, . 211
193. Frictional Stability of Plane Joints, 211
194. Frictional Stability of Earth, . 212
195. Mass of Earth with Plane Surface, 214
196. Principle of Least Resistance, . 215
197. Earth Loaded with its own Weight, 216
198. Pressure of Earth against a Vertical Plane, ... 218
199. Supporting Power of Earth-Foundations, ... 219
200. Abutting Power of Earth, . 220
201. Table of Examples, . . . 221
202. Frictional Tenacity or Bond of Masonry and Brickwork, . 222
203. Friction of Screws, Keys, and Wedges, ... 226
204. Friction of Rest and Friction of Motion, ... 226

SECTION 4.—*On the Stability of Abutments and Vaults.*

Page
205. Stability at a Plane Joint, . 226
206. Stability of a series of Blocks —Line of Resistance—Line of Pressures, ... 230
207. Analogy of Blockwork and Framework, ... 231
208. Transformation of Blockwork Structures, ... 232
209. Frictional Stability of a Transformed Structure, . . . 233
210. Structure not laterally pressed, 233
211. Moment of Stability, . . 233
212. Abutments classed, . . . 235
213. Buttresses in general, . . 235
214. Rectangular Buttress, . . 238
215. Towers and Chimneys, . . 240
216. Dams or Reservoir-Walls, . 243
217. Retaining Walls in general, . 249
218. Rectangular Retaining Walls, . 252
219. Trapezoidal Walls, . . . 254
Page
220. Battering Walls of Uniform Thickness, ... 254
221. Foundation Courses of Retaining Walls, ... 255
222. Counterforts, ... 255
223. Arches of Masonry, . . . 256
224. Line of Pressure in an Arch—Condition of Stability, . . 257
225. Angle, Joint, and Point of Rupture, 259
226. Thrust of an Arch of Masonry, . 260
227. Abutments of Arches, . . 261
228. Skew Arches, ... 261
229. Groined Vaults, . . . 262
230. Clustered Arches, . . . 263
231. Piers of Arches, . . . 263
232. Open and Hollow Piers and Abutments, ... 263
233. Tunnels, ... 264
234. Domes, ... 265
235. Strength of Abutments and Vaults, 268
235A. Transformation of Structures in Masonry, ... 268

CHAPTER III.—STRENGTH AND STIFFNESS.

SECTION 1.—*Summary of General Principles.*

236. Theory of Elasticity, . . 270
237. Elasticity defined, . . . 270
238. Elastic Force or Stress, . . 270
239. Fluid Elasticity, . . . 270
240. Liquid Elasticity, . . . 271
241. Rigidity or Stiffness, . . . 271
242. Strain and Fracture, . . 272
243. Perfect and Imperfect Elasticity —Plasticity, ... 272
244. Strength, Ultimate and Proof—Toughness—Stiffness—Spring, 273
245. Determination of Proof Strength, 274
246. Working Stress, . . . 274
247. Factors of Safety, . . . 274
248. Divisions of Mathematical Theory of Elasticity, ... 275
249. Resolution and Composition of Strains, ... 275
250. Displacements, ... 276
251. Analogy of Stresses and Strains, 276
252. Potential Energy of Elasticity, . 277
253. Co-efficients of Elasticity, . . 277
254. Co-efficients of Pliability, . . 277
255. Axes of Elasticity, . . . 278
256. Isotropic Solid, . . . 278
257. Modulus of Elasticity, . . 279
258. Examples of Co-efficients, . . 279
259. General Problem of Internal Equilibrium of an Elastic Solid, 280

SECTION 2.—*On Relations between Strain and Stress.*

260. Ellipse of Strain, . . . 280
261. Ellipsoid of Strain, . . . 283
262. Transverse Elasticity of an Isotropic Substance, . . . 284
263. Cubic Elasticity, . . . 285
264. Fluid Elasticity, . . . 285

SECTION 3.—*On Resistance to Stretching and Tearing.*

265. Stiffness and Strength of a Tie-bar, 286
266. Resilience, or Spring of a Tie-Bar —Modulus of Resilience, . 287
267. Sudden Pull, ... 287
268. Explanation of Table of the Resistance of Materials to Stretching and Tearing.—(see page 646), ... 288
269. Additional data—Welded Joint —Iron Wire Ropes—Hempen Cables—Leather Belts—Chain Cables, ... 288
270. Strength of Rivetted Joints, . 288
271. Thin Hollow Cylinders—Boilers —Pipes, ... 289
272. Thin Hollow Spheres, . . 290
273. Thick Hollow Cylinder, . . 290
274. Cylinder of Strained Rings, . 294
275. Thick Hollow Sphere, . . 295
276. Boiler Stays, ... 296
277. Suspension-Rod of Uniform Strength, ... 297

SECTION 4.—*On Resistance to Shearing.*

278. Condition of Uniform Intensity, 298
279. Explanation of Table of Resistance of Materials to Shearing and Distortion—(see p. 648), 299
280. Economy of Material in Bolts and Rivets, ... 299
281. Fastenings of Timber Ties, . 301

SECTION 5.—*On Resistance to Direct Compression and Crushing.*

282. Resistance to Compression, . 302
283. Modes of Crushing—Splitting—Shearing—Bulging—Buckling —Cross-breaking, . . . 302

Page
284. Explanation of Table of the Resistance of Materials to Crushing by a Direct Thrust—(see p. 648), ... 304
285. Unequal Distribution of the Pressure, ... 304
286. Limitations, ... 306
287. Crushing and Collapsing of Tubes, 306

SECTION 6.—*On Resistance to Bending and Cross-Breaking.*

288. Shearing Force and Bending Moment. 307
289. Beams Fixed at one end only, . 310
290. Beams Supported at both ends, 310
291. Moments of Flexure in terms of Load and Length, . . . 311
292. Uniform Moment of Flexure—Railway Carriage Axles, . . 312
293. Resistance of Flexure, . . 312
294. Transverse Strength of Beams in General, ... 315
295. Transverse Strength in Terms of Breadth and Depth, . . 316
296. Explanation of Table of Moduli of Rupture.—(see p. 649), 317
297. Modulus of Rupture of Cast Iron Beams, ... 318
298. Section of Equal Strength for Cast Iron Beams, . . . 319
299. Beams of Uniform Strength, . 320
300. Proof Deflection of Beams, . 322
301. Deflection found by Graphic Construction, ... 326
302. Proportion of the greatest Depth of a Beam to the Span, . . 327
303. Slope and Deflection of a Beam under any load, . . . 328
304. Deflection with Uniform Moment, 330
305. Resilience or Spring of a Beam, 330
306. Suddenly applied Transverse Load, 332
307. Beam Fixed at both ends, . . 332
308. Beam Fixed at one end, and Supported at both, . . . 338
309. Shearing Stress in Beams, . 338
310. Lines of Principal Stress in Beams, 341
311. Direct Vertical Stress, . . 342
312. Small effect of Shearing Stress upon Deflection, . . . 342
313. Partially-loaded Beam, . . 344
314. Allowance for Weight of Beam, 346
315. Limiting Length of Beam, . 347
316. Sloping Beam, ... 348
317. Originally Curved Beam, . . 348
318. Expansion and Contraction of Long Beams, ... 348
319. Elastic Curve, ... 349
319A. Hydrostatic Arch, . . . 353

SECTION 7.—*On Resistance to Twisting and Wrenching.*

320. Twisting Moment, . . . 353
321. Strength of a Cylindrical Axle. 353
322. Angle of Torsion of a Cylindrical Axle, ... 356
323. Resilience of a Cylindrical Axle, 357
324. Axles not Circular in Section, . 358
325. Bending and Twisting combined; Crank and Axle, . . . 358
326. Teeth of Wheels, . . . 359

SECTION 8.—*On Crushing by Bending.*

327. Introductory Remarks, . . 360
328. Strength of Iron Pillars and Struts, 361
329. Connecting Rods, Piston Rods, . 363
330. Comparison of Cast and Wrought Iron Pillars, ... 363
331. Mr. Hodgkinson's Formulæ for the Ultimate Strength of Cast Iron Pillars, ... 363
332. Wrought Iron Framework, . 364
333. Wrought Iron Cells, . . . 364
334. Sides of Plate Iron Girders, . 365
335. Timber Posts and Struts, . . 365

SECTION 9.—*On Compound Girders, Frames, and Bridges.*

336. Compound Girders in General, . 366
337. Plate Iron Girders, . . . 366
338. Half-Lattice and Lattice Beams, 369
339. Bowstring Girder, . . . 369
340. Stiffened Suspension Bridges, . 370
341. Ribbed Arches, . . . 376

SECTION 10.—*Miscellaneous Remarks on Strength and Stiffness.*

342. Effects of Temperature, . . 376
343. Effects of Repeated Meltings on Cast Iron, ... 376
344. Effects of Ductility, . . . 376
345. Internal Friction, . . . 377
346. Concluding remarks on Strength and Stiffness, ... 377

PART III.—PRINCIPLES OF CINEMATICS, OR THE COMPARISON OF MOTIONS.

347. Division of the Subject, ... 379

CHAPTER I.—MOTIONS OF POINTS.

SECTION 1.—*Uniform Motion of a Pair of Points.*

348. Fixed and Nearly Fixed Directions, 379
349. Motion of a Pair of Points, . 380
350. Fixed Point and Moving Point, 381
351. Component and Resultant Motions, 381
352. Measurement of Time, . . 381
353. Uniform Velocity, . . . 382
354. Uniform Motion, . . . 382

SECTION 2.—*Uniform Motion of several Points.*

355. Motion of Three Points, . . 383
356. Motions of a Series of Points, . 383

Page
357. Parallelopiped of Motions, . 384
358. Comparative Motion, . . 384

SECTION 3.—*Varied Motion of Points.*

359. Velocity and Direction of Varied Motion, ... 385
360. Components of Varied Motion, . 386
361. Uniformly-Varied Velocity, . 386
Page
362. Varied Rate of Variation of Velocity, ... 387
363. Uniform Deviation, . . . 387
364. Varying Deviation, . . . 388
365. Resultant Rate of Variation, . 388
366. Rates of Variation of Component Velocities, ... 388
367. Comparison of Varied Motions, 389

CHAPTER II.—MOTIONS OF RIGID BODIES.

SECTION 1.—*Rigid Bodies, and their Translation.*

368. Rigid Bodies—how understood, 390
369. Translation or Shifting, . . 390

SECTION 2.—*Simple Rotation.*

370. Rotation defined — Centre of Rotation, ... 390
371. Axis of Rotation, . . . 390
372. Plane of Rotation—Angle of Rotation, ... 391
373. Angular Velocity, . . . 391
374. Uniform Rotation, . . . 391
375. Rotation common to all parts of body, ... 391
376. Right and Left-handed Rotation, 391
377. Relative Motion of a pair of Points in a Rotating Body, . . 392
378. Cylindrical Surface of equal Velocities, ... 392
379. Comparative Motions of two Points relatively to an Axis, . 393
380. Components of Velocity of a Point in a Rotating Body, . . 393

SECTION 3.—*Combined Rotations and Translations.*

381. Property of all Motions of Rigid Bodies, ... 394
382. Helical Motion, . . . 394
383. To find the Motion of a Rigid Body from the Motions of three of its Points, ... 395
384. Special cases, ... 396
385. Rotation combined with Translation in the same Plane, . . 397
386. Rolling Cylinder—Trochoids, . 398
387. Plane rolling on Cylinder—Spiral Paths, ... 398
388. Combined Parallel Rotations, . 399
389. Cylinder rolling on Cylinder—Epitrochoids, ... 400
390. Curvature of Epitrochoids, . 401
391. Equal and Opposite Parallel Rotations combined, . . . 404
392. Rotations about Intersecting Axes combined, ... 404
393. Rolling Cones, ... 405
394. Analogy of Rotations and Single Forces, ... 405
395. Comparative Motions in Compound Rotations, . . . 406

SECTION 4.—*Varied Rotation.*

396. Variation of Angular Velocity, . 406
397. Change of the Axis of Rotation, 407
398. Components of Varied Rotation, 407

CHAPTER III.—MOTIONS OF PLIABLE BODIES, AND OF FLUIDS.

399. Division of the Subject, ... 408

SECTION 1.—*Motions of Flexible Cords.*

400. General Principles, . . . 408
401. Motions classed, . . . 409
402. Cord guided by Surfaces of Revolution, ... 409

SECTION 2.—*Motions of Fluids of Constant Density.*

403. Velocity and Flow, . . . 410
404. Principle of Continuity, . . 411
405. Flow in a Stream, . . . 411
406. Pipes, Channels, Currents, and Jets, ... 411
407. Radiating Current, . . . 412
408. Vortex, Eddy, or Whirl, . . 412
409. Steady Motion, ... 412
410. Unsteady Motion, . . . 413
411. Motion of Piston, . . . 413
412. General Differential Equations of Continuity, ... 413
413. General Differential Equations of Steady Motion, . . . 414
414. General Differential Equations of Unsteady Motion, . . . 415
415. Equations of Displacement, . 415
416. Wave, ... 416
417. Oscillation, ... 416

SECTION 3.—*Motions of Fluids of Varying Density.*

418. Flow of Volume and Flow of Mass, ... 417
419. Principle of Continuity, . . 417
420. Stream, ... 418
421. Steady Motion, ... 419
422. Pistons and Cylinders, . . 419
423. General Differential Equations, ... 419

---

424. Motions of Connected Bodies, . 420

PART IV.—THEORY OF MECHANISM.

CHAPTER I.—DEFINITIONS AND GENERAL PRINCIPLES.

Page
425. Theory of Pure Mechanism defined, 421
426. General Problem of Mechanism stated, ... 422
427. Frame — Moving Pieces — Connectors, ... 422
428. Bearings, ... 422
429. Motions of Primary Moving Pieces, 423
430. Motions of secondary Moving Pieces, 423
Page
431. Elementary Combinations in Mechanism, ... 423
432. Line of Connection, . . . 424
433. Principle of Connection, . . 424
434. Adjustments of Speed, . . 424
435. Train of Mechanism, . . 425
436. Aggregate Combinations in Mechanism, ... 425

CHAPTER II.—ON ELEMENTARY COMBINATIONS AND TRAINS OF MECHANISM.

SECTION 1.—*Rolling Contact.*

437. Pitch-Surfaces, . . . 426
438. Smooth Wheels, Rollers, Racks, 426
439. General Conditions of Rolling Contact, ... 426
440. Circular Cylindrical Wheels, . 427
441. Straight Rack and Circular Wheel, 427
442. Bevel Wheels, ... 428
443. Non-Circular Wheels, . . 428

SECTION 2.—*Sliding Contact.*

444. Skew-Bevel Wheels, . . 430
445. Grooved Wheels, or Frictional Gearing, ... 431
446. Teeth of Wheels—Definitions and General Principles, . . 432
447. Pitch and Number of Teeth, . 433
448. Hunting Cog, ... 434
449. Trains of Wheelwork, . . 434
450. Principle of Sliding Contact, . 436
451. Teeth of Spur-Wheels and Racks —General Principle, . . 438
452. Teeth described by rolling curves, 438
453. Sliding of a pair of Teeth (see also 455, 458, 462A), . . 439
454. Arc of Contact on Pitch Lines, . 440
455. Length of a Tooth; Sliding of Teeth, 440
456. Inside Gearing, ... 441
457. Involute Teeth for Circular Wheels, 441
458. Sliding of Involute Teeth, . . 443
459. Addendum for Involute Teeth, . 443
460. Smallest Pinion with Involute Teeth, ... 443
461. Epicycloidal Teeth—least Pinion, 444
462. Addendum for Epicycloidal Teeth, 445
462A. Sliding of Epicycloidal Teeth, 445
463. Approximate Epicycloidal Teeth, 445
464. Teeth of Wheel and Trundle, . 447
465. Dimensions of Teeth, . . 447
466. Mr. Sang's process for Describing Teeth, ... 448
467. Teeth of Bevel Wheels, . . 448
468. Teeth of Skew-Bevel Wheels, . 449
469. Teeth of Non-Circular Wheels, . 449
470. Cam, or Wiper, . . . 449
471. Screws—Pitch, ... 449
472. Normal and Circular Pitch, . 450
473. Screw Gearing, . . . 451
474. Hooke's Gearing, . . . 451
475. Wheel and Screw, . . . 452
476. Relative Sliding of Pair of Screws, 453
477. Oldham's Coupling, . . . 453

SECTION 3.—*Connection by Bands.*

478. Bands classed: Belts, Cords, Chains, 454
479. Principle of Connection by Bands, 454
480. Pitch Surface of a Pulley or Drum, 455
481. Circular Pulleys and Drums, . 455
482. Length of an Endless Belt, . 456
483. Speed Cones, ... 457

SECTION 4.—*Linkwork.*

484. Definitions, ... 458
485. Principles of Connection, . . 458
486. Dead Points, ... 458
487. Coupling of Parallel Axes, . 459
488. Comparative Motion of Connected Points, ... 459
489. Eccentric, ... 460
490. Length of Stroke, . . . 460
491. Hooke's Universal Joint, . . 461
492. Double Hooke's Joint, . . 462
493. Click, ... 462

SECTION 5.—*Reduplication of Cords.*

494. Definitions, ... 462
495. Velocity-Ratio, . . . 463
496. Velocity of any Ply, . . . 463
497. White's Tackle, . . . 463

SECTION 6.—*Hydraulic Connection.*

498. General Principle, . . . 464
499. Valves, Pumps, Working Cylinder, 464
500. Hydraulic Press, . . . 464
501. Hydraulic Hoist, . . . 465

SECTION 7.—*Trains of Mechanism.*

502. Trains of Elementary Combinations, ... 465

CHAPTER III.—ON AGGREGATE COMBINATIONS.

503. General Principles, . . . 466
504. Differential Windlass, . . 466
505. Compound Screws, . . . 467
506. Link Motion, ... 468
507. Parallel Motions, . . . 469
508. Epicyclic Trains, . . . 473

PART V.—PRINCIPLES OF DYNAMICS.

Page
509. Division of the Subject, ... 475

CHAPTER I.—UNIFORM MOTION UNDER BALANCED FORCES.

Page
510. First Law of Motion, . . . 476
511. Effort, Resistance, Lateral Force, 476
512. Conditions of Uniform Motion, . 477
513. Work, ... 477
514. Energy, ... 477
515. Energy and Work of Varying Forces, ... 477
516. Dynamometer, or Indicator, . 478
517. Energy and Work of Fluid Pressure, ... 478
518. Conservation of Energy, . . 478
519. Principle of Virtual Velocities, . 479
520. Energy of Component Forces and Motions, ... 480

CHAPTER II.—ON THE VARIED TRANSLATION OF POINTS AND RIGID BODIES.

SECTION 1.—*Definitions.*

521. Mass, or Inertia, . . . 482
522. Centre of Mass, . . . 482
523. Momentum, ... 482
524. Resultant Momentum, . . 482
525. Variations and Deviations of Momentum, ... 483
526. Impulse, ... 483
527. Impulse, Accelerating, Retarding, Deflecting, ... 483
528. Relations between Impulse, Energy, and Work, . . 484

SECTION 2.—*Law of Varied Translation.*

529. Second Law of Motion, . . 484
530. General Equations of Dynamics, 484
531. Mass in terms of Weight, . . 485
532. Absolute Unit of Force, . . 486
533. Motion of a Falling Body, . . 486
534. Projectile, Unresisted, . . 487
535. Motion along an Inclined Path, . 489
536. Uniform Effort, or Resistance, . 490
537. Deviating Force, . . . 491
538. Centrifugal Force, . . . 491
539. Revolving Simple Pendulum, . 492
540. Deviating Force in terms of Angular Velocity, . . . 492
541. Rectangular Components of Deviating Force, . . . 493
542. Straight Oscillation, . . . 494
543. Elliptical Oscillations, or Revolutions, ... 495
544. Simple Oscillating Pendulum, . 496
545. Cycloidal Pendulum, . . 497
546. Residual Forces, . . . 498

SECTION 3.—*Transformation of Energy.*

547. Actual Energy defined, *Vis-Viva*, 499
548. Components of Actual Energy, . 499
549. Energy of Varied Motion, . . 499
550. Energy Stored and Restored, . 501
551. Transformation of Energy, . 501
552. Conservation of Energy in Varied Motion, ... 501
553. Periodical Motion, . . . 501
554. Measures of Unbalanced Force, . 501
555. Energy due to Oblique Force, . 502
556. Reciprocating Force, . . 503
557. Total Energy—Initial Energy, . 503

SECTION 4.—*Varied Translation of a System of Bodies.*

558. Conservation of Momentum, . 505
559. Motion of Centre of Gravity, . 505
560. Angular Momentum Defined, . 505
561. Angular Impulse Defined, . 506
562. Relations of Angular Impulse and Angular Momentum, . . 506
563. Conservation of Angular Momentum, ... 506
564. Actual Energy of a System of Bodies, ... 507
565. Conservation of Internal Energy, 508
566. Collision, ... 508
567. Action of Unbalanced External Forces on a System. General Equations, ... 510
568. Determination of Internal Forces —D'Alembert's Principle, . 511
569. Residual External Forces in a System of Bodies, . . . 511

CHAPTER III.—ROTATIONS OF RIGID BODIES.

570. Motion of a Rigid Body in General, ... 513

SECTION 1.—*On Moments of Inertia, Radii of Gyration, Moments of Deviation, and Centres of Percussion.*

571. Moment of Inertia Defined, . 514
572. Moment of Inertia of a System of Physical Points, . . . 514
573. Moment of Inertia of a Rigid Body, 514
574. Radius of Gyration, . . . 515
575. Components of Moments of Inertia, ... 515
576. Moments of Inertia round Parallel Axes compared, . . 516
577. Combined Moments of Inertia, . 517
578. Examples of Moments of Inertia and Radii of Gyration, . 517
579. Moments of Inertia found by Division and Subtraction, . 519
580. Moments of Inertia found by Transformation, . . . 519
581. Centre of Percussion—Centre of Gyration, ... 520

Page
582. No Centre of Percussion, . . 522
583. Moments of Inertia about Inclined Axes, ... 522
584. Principal Axes of Inertia, . . 524
585. Ellipsoid of Inertia, . . . 526
586. Resultant Moment of Deviation, 528

SECTION 2.—*On Uniform Rotation.*

587. Momentum, ... 529
588. Angular Momentum, . . 529
589. Actual Energy of Rotation, . 532
590. Free Rotation, ... 533
591. Uniform Rotation about a Fixed Axis, ... 535
592. Deviating Couple — Centrifugal Couple, ... 535
593. Energy and Work of Couples, . 537

SECTION 3.—*On Varied Rotation.*

594. Law of Varied Rotation, . . 538
Page
595. Varied Rotation about a fixed Axis, 540
596. Analogy of Varied Rotation and Varied Translation, . . 541
597. Uniform Variation. . . . 541
598. Gyration, or Angular Oscillation, 542
599. Single Force applied to a Body with a Fixed Axis, . . 543

SECTION 4.—*Varied Rotation and Translation Combined.*

600. General Principles, . . . 543
601. Properties of the Centre of Percussion, ... 544
602. Fixed Axis, ... 545
603. Deviating Force, . . . 545
604. Compound Oscillating Pendulum—Centre of Oscillation, . . 546
605. Compound Revolving Pendulum, 547
606. Rotating Pendulum, . . 547
607. Ballistic Pendulum, . . . 548

CHAPTER IV.—MOTIONS OF PLIABLE BODIES.

608. Nature of the Subject—Vibration, 552
609. Isochronous Vibration.—Condition of Isochronism, . . 553
610. Vibrations of a Mass held by a light Spring, ... 554
611. Superposition of Small Motions, 555
612. Vibrations not Isochronous, . 557
613. Vibrations of an Elastic Body in General, ... 557
614. Waves of Vibration, . . . 562
615. Velocity of Sound, . . . 563
616. Impact and Pressure; Pile-driving, 564

CHAPTER V.—MOTIONS OF FLUIDS—HYDRODYNAMICS.

617. Division of the Subject, ... 566

SECTION 1.—*Motion of Liquids without Friction.*

618. General Equations, . . . 567
619. Dynamic Head, ... 568
620. General Dynamic Equations in terms of Dynamic Head, . 568
621. Law of Dynamic Head for Steady Motion, ... 568
622. Total Energy, ... 569
623. Free Surface, ... 570
624. Surface of Equal Pressure, . 570
625. Motion in Plane Layers, . . 570
626. Contracted Vein, ... 572
627. Vertical Orifices, ... 572
628. Surfaces of Equal Head, . . 573
629. Radiating Current, . . . 574
630. Free Circular Vortex, . . . 574
631. Free Spiral Vortex, . . . 576
632. Forced Vortex, ... 576
633. Combined Vortex, . . . 576
634. Vertical Revolution, . . . 578

SECTION 2.—*Motions of Gases without Friction.*

635. Dynamic Head in Gases, . . 579
636. Equation of Continuity for a Steady Stream of Gas, . . 581
637. Flow of Gas from an Orifice, . 581
637A. Maximum Flow of Gas, . . 582

SECTION 3.—*Motions of Liquids with Friction.*

638. General Laws of Fluid Friction, 584
639. Internal Fluid Friction, . . 585
640. Friction in an Uniform Stream —Hydraulic Mean Depth, . 586
641. Varying Stream, ... 587
642. Friction in a Pipe running full, . 588
643. Resistance of Mouthpieces, . 589
644. Resistance of Curves and Knees, 589
645. Sudden Enlargement of Channel, ... 589
646. General Problem, . . . 590

SECTION 4.—*Flow of Gases with Friction.*

647. General Law, ... 590

SECTION 5.—*Mutual Impulse of Fluids and Solids.*

648. Pressure of a Jet against a fixed Surface, ... 591
649. Pressure of a Jet against a moving Surface, ... 593
650. Pressure of a Forced Vortex against a Wheel, . . . 595
651. Centrifugal Pumps and Fans, . 597
652. Pressure of a Current, . . 598
653. Resistance of Fluids to Floating and Immersed Bodies, . 598
654. Stability of Floating Bodies—Metacentre of a Ship, . . 600
655. Oscillations of Floating Bodies, . 603
656. Action between a Fluid and a Piston—Work of Air—Work of Steam, ... 604

PART VI.—THEORY OF MACHINES.

Page
657. Nature and Division of the Subject, ... 609

CHAPTER I.—WORK OF MACHINES WITH UNIFORM OR PERIODIC MOTION.

Page

SECTION 1.—*General Principles.*

658. Useful and Lost Work, . . 610
659. Useful and Prejudicial Resistance, 610
660. Efficiency, ... 610
661. Power and Effect:—Horse Power, 610
662. Driving Point:—Train:—Working Point, ... 610
663. Points of Resistance, . . 610
664. Efficiencies of Pieces of a Train, 610
665. Mean Efforts and Resistances, . 611
666. General Equations, . . . 611
667. Equations in terms of Comparative Motion, ... 612
668. Reduction of Forces and Couples, 612

SECTION 2.—*On the Friction of Machines.*

669. Co-efficients of Friction, . . 612
670. Unguents, ... 613
671. Limit of Pressure between Rubbing Surfaces, . . . 613
672. Friction of a Sliding Piece, . 614
673. Moment of Friction, . . . 614
674. Friction of an Axle, . . . 614
675. Friction of a Pivot, . . . 616
676. Friction of a Collar, . . . 616
677. Friction of Teeth, . . . 617
678. Friction of a Band, . . . 617
679. Frictional Gearing, . . . 618
680. Friction Couplings, . . . 618
681. Stiffness of Ropes, . . . 619
682. Rolling Resistance of Smooth Surfaces, ... 619
683. Resistance of Carriages on Roads, ... 619
684. Resistance of Railway Trains, . 620
685. Heat of Friction, . . . 620

CHAPTER II.—VARIED MOTIONS OF MACHINES.

686. Centrifugal Forces and Couples, 621
687. Actual Energy of a Machine, . 621
688. Reduced Inertia, . . . 621
689. Fluctuations of Speed, . . 622
690. Fly-Wheel, ... 623
691. Starting and Stopping—Brakes, 624

CHAPTER III.—ON PRIME MOVERS.

692. Prime Mover defined, . . 625
693. Regulators—Governors, . . 625
694. Prime Movers Classed, . . 625
695. Muscular Strength, . . . 625
696. Water-Pressure Engines, . . 626
697. Water Wheels in General, . . 627
698. Classes of Water Wheels, . . 628
699. Overshot and Breast Wheels, . 628
700. Undershot Wheels, . . . 628
701. Turbines, ... 629
702. Windmills, ... 629
703. Efficiency of Heat Engines in General, ... 629
704. Steam Engines, . . . 630
705. Electrodynamic Engines: Science of Energetics, ... 630

APPENDIX.

Motion of Water in Waves, ... 631
Continuous Girders, ... 634
Reservoir Walls, ... 638
American Bridges, ... 639
Wind Pressure, ... 639
Continuous Brakes, ... 640
Steel Boilers, ... 640
Pulsometer Pump, ... 641
Strength of Steel, ... 641
Ship Resistance, ... 641
Hydraulic Rivetting, &c., ... 642
Safety Valves, ... 642
Coefficient of Contraction, ... 642
Gas Engines, ... 643
Railway Resistance, ... 643
Compound Engines, ... 643
Dimensions of Great Chimneys, ... 644
I. Tables of the Resistance of Materials to Stretching and Tearing, . . . 646
II. Table of the Resistance of Materials to Shearing and Distortion, . . . 648
III. Table of the Resistance of Materials to Crushing, ... 648
IV. Table of the Resistance of Materials to Breaking Across, ... 649
V. Comparative Tables of British and French Measures, ... 651
VI. Table of Specific Gravities of Materials, ... 652
Index, ... 655

PRELIMINARY DISSERTATION

ON THE

HARMONY OF THEORY AND PRACTICE IN MECHANICS.*

THE words, *theory* and *practice*, are of Greek origin: they carry our thoughts back to the time of those ancient philosophers by whom they were contrived; and by whom also they were contrasted and placed in opposition, as denoting two conflicting and mutually inconsistent ideas.

In geometry, in philosophy, in poetry, in rhetoric, and in the fine arts, the Greeks are our masters; and great are our obligations to the ideas and the models which they have transmitted to our times. But in physics and in mechanics their notions were very generally pervaded by a great fallacy, which attained its complete and most mischievous development amongst the mediæval schoolmen, and the remains of whose influence can be traced even at the present day—the fallacy of a *double system of natural laws;* one theoretical, geometrical, rational, discoverable by contemplation, applicable to celestial, ætherial, indestructible bodies, and being an object of the noble and liberal arts; the other practical, mechanical, empirical, discoverable by experience, applicable to terrestrial, gross, destructible bodies, and being an object of what were once called the vulgar and sordid arts.

The so-called physical theories of most of those whose understandings were under the influence of that fallacy, being empty dreams, with but a trace of truth here and there, and at variance with the results of every-day observation on the surface of the planet we inhabit, were calculated to perpetuate the fallacy. The stars were celestial, incorruptible bodies; their orbits were circular and their motions perpetual; such orbits and motions being characteristic of perfection. Objects on the earth's surface were terrestrial and corruptible; their motions being characteristic of imperfection, were in mixed straight and curved lines, and of limited duration. Rational and practical mechanics (as Newton observes in his preface to the *Principia*) were considered as in a measure opposed to each other, the latter being an inferior branch of study, to be cultivated only for the sake of gain or some other material advantage. Archytas of Tarentum might illustrate the truths of geometry by mechanical contrivances; his methods were regarded by his pupil Plato as a lowering of the dignity of science. Archimedes, to the character of the first geometer and arithmetician of his day, might add that of the first mechanician and physicist,—he might, by his unaided strength acting through suitable machinery, move a loaded ship on dry land,—he might contrive and execute deadly engines of war, of which even the Roman soldiers stood in dread,—he might, with an art afterwards regarded as fabulous till it was revived by Buffon, burn fleets with the concentrated sunbeams; but that mechanical knowledge, and that practical skill, which, in our eyes, render that great man so illustrious, were, by men of learning, his contemporaries and successors, regarded as accomplishments of an inferior order, to which the philosopher, from the height of geometrical abstraction, condescended, with a view to the service of the State. In those days the notion arose that scientific men were unfit for the business of life, and various facetious anecdotes were contrived illustrative of this notion, which have been handed down from age to age, and in each age applied, with little variation, to the eminent philosophers of the time.

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* This Dissertation contains the substance of a discourse, "De Concordiâ inter Scientiarum Machinalium Contemplationem et Usum," read before the Senate of the University of Glasgow on the 10th of December, 1855, and of an inaugural lecture, delivered to the Class of Civil Engineering and Mechanics in that University on the 3d of January, 1856.

That the Romans were eminently skilful in many departments of practical mechanics, especially in masonry, road-making, and hydraulics, is clearly established by the existing remains of their magnificent works of engineering and architecture, from many of which we should do well to take a lesson. But the fallacy of a supposed discordance between rational and practical, celestial and terrestrial mechanics, still continued in force, and seems to have gathered strength, and to have attained its full vigour during the middle ages. In those ages, indeed, were erected those incomparable ecclesiastical buildings, whose beauty, depending, as it does, mainly on the nice adjustment of the form, strength, and position of each part, to the forces which it has to sustain, evinces a profound study of the principles of equilibrium on the part of the architects. But the very names of those architects, with few and doubtful exceptions, were suffered to be forgotten; and the principles which guided their work remain unrecorded, and were left to be re-discovered in our own day; for the scholars of those times, despising practice and observation, were occupied in developing and magnifying the numerous errors, and in perverting and obscuring the much more numerous truths, which are to be found in the writings of Aristotle; and those few men who, like Roger Bacon, combined scientific with practical knowledge, were objects of fear and persecution, as supposed allies of the powers of darkness.

At length, during the great revival of learning and reformation of science in the fifteenth, sixteenth, and seventeenth centuries, the system falsely styled Aristotelian was overthrown: so also was the fallacy of a double system of natural laws; and the truth began to be duly appreciated, that sound theory in physical science consists simply of facts, and the deductions of common sense from them, reduced to a systematic form. The science of motion was founded by Galileo, and perfected by Newton. Then it was established that celestial and terrestrial mechanics are branches of one science; that they depend on one and the same system of clear and simple first principles; that those very laws which regulate the motion and the stability of bodies on earth, govern also the revolutions of the stars, and extend their dominion throughout the immensity of space. Then it came to be acknowledged, that no material object, however small,—no force, however feeble,—no phenomenon, however familiar, is insignificant, or beneath the attention of the philosopher; that the processes of the workshop, the labours of the artizan, are full of instruction to the man of science; that the scientific study of practical mechanics is well worthy of the attention of the most accomplished mathematician. Then the notion, that scientific men are unfit for business, began to disappear. It was not court favour, not high connection, not Parliamentary influence, which caused Newton to be appointed Warden, and afterwards Master, of the Mint; it was none of these; but it was the knowledge possessed by a wise minister of the fact, that Newton's skill, both theoretical and practical, in those branches of knowledge which that office required, rendered him the fittest man in all Britain to direct the execution of a great reform of the coinage. Of the manner in which Newton performed the business entrusted to him, we have the following account in the words of Lord Macaulay, an author who cannot be accused of undue partiality to speculative science or its cultivators:—

> "The ability, the industry, and the strict uprightness of the great philosopher, speedily produced a complete revolution throughout the department which was under his direction. He devoted himself to the task with an activity which left him no time to spare for those pursuits in which he had surpassed Archimedes and Galileo. Till the great work was completely done, he resisted firmly, and almost angrily, every attempt that was made by men of science, here or on the Continent, to draw him away from his official duties."*

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* Vol. iv., p. 703.

Then the historian proceeds to detail the results of Newton's exertions, and shows, that within a short time after his appointment, the weekly amount of the coinage of silver was increased to *eightfold* of that which had been looked upon as the utmost practicable amount by his predecessors.

The extension of experimental methods of investigation, has caused even manual skill in practical mechanics, when scientifically exercised, to be duly honoured, and not (as in ancient times) to be regarded as beneath the dignity of science.

As a systematically avowed doctrine, there can be no doubt that the fallacy of a discrepancy between rational and practical mechanics came long ago to an end; and that every well-informed and sane man, expressing a deliberate opinion upon the mutual relations of those two branches of science, would at once admit that they agree in their principles, and assist each other's progress, and that such distinction as exists between them arises from the difference of the *purposes* to which the same body of principles is applied.

If this doctrine had as strong an influence over the actions of men as it now has over their reasonings, it would have been unnecessary for me to describe, so fully as I have done, the great scientific fallacy of the ancients. I might, in fact, have passed it over in silence, as dead and forgotten; but, unfortunately, that discrepancy between theory and practice, which in sound physical and mechanical science is a delusion, has a real existence in the minds of men; and that fallacy, though rejected by their judgments, continues to exert an influence over their acts. Therefore it is that I have endeavoured to trace the prejudice as to the discrepancy of theory and practice, especially in Mechanics, to its origin; and to show that it is the ghost of a defunct fallacy of the ancient Greeks and of the mediæval schoolmen.

This prejudice, as I have stated, is not to be found, at the present day, in the form of a definite and avowed principle: it is to be traced only in its pernicious effects on the progress both of speculative science and of practice, and sometimes in a sort of tacit influence which it exerts on the forms of expression of writers, who have assuredly no intention of perpetuating a delusion. To exemplify the kind of influence last referred to, I shall cite a passage from the same historical work which I recently quoted for a different purpose. Lord Macaulay, in treating of the Act of Toleration of William III., compares, metaphorically, the science of politics to that of mechanics, and then proceeds as follows:—

> "The mathematician can easily demonstrate that a certain power, applied by means of a certain lever, or of a certain system of pulleys, will suffice to raise a certain weight. But his demonstration proceeds on the supposition that the machinery is such as no load will bend or break. If

the engineer who has to lift a great mass of real granite by the instrumentality of real timber and real hemp, should absolutely rely on the propositions which he finds in treatises on Dynamics, and should make no allowance for the imperfection of his materials, his whole apparatus of beams, wheels, and ropes, would soon come down in ruin, and with all his geometrical skill, he would be found a far inferior builder to those painted barbarians who, though they never heard of the parallelogram of forces, managed to pile up Stonehenge."*

It is impossible to read this passage without feeling admiration for the force and clearness (and I may add, for the brilliancy and wit) of the language in which it is expressed; and those very qualities of force and clearness, as well as the author's eminence, render it one of the best examples that can be found to illustrate the lurking influence of the fallacy of a double set of mechanical laws, rational and practical.

In fact, the mathematical theory of a machine,—that is, the body of principles which enables the engineer to compute the arrangement and dimensions of the parts of a machine intended to perform given operations,—is divided by mathematicians, for the sake of convenience of investigation, into two parts. The part first treated of, as being the more simple, relates to the motions and mutual actions of the solid pieces of a machine, and the forces exerted by and upon them, each continuous solid piece being treated as a whole, and of sensibly invariable figure. The second and more intricate part relates to the actions of the forces tending to break or to alter the figure of each such solid piece, and the dimensions and form to be given to it in order to enable it to resist those forces: this part of the theory depends, as much as the first part, on the general laws of mechanics; and it is, as truly as the first part, a subject for the reasonings of the mathematician, and equally requisite for the completeness of the mathematical treatise which the engineer is supposed to consult. It is true, that should the engineer implicitly trust to a pretended mathematician, or an incomplete treatise, his apparatus would come down in ruin, as the historian has stated: it is true also that the same result would follow, if the engineer was one who had not qualified himself, by experience and observation, to distinguish between good and bad materials and workmanship; but the passage I have quoted conveys an idea different from these; for it proceeds on the erroneous supposition, that the first part of the theory of a machine is the whole theory, and is at variance with something else which is independent of mathematics, and which constitutes, or is the foundation of, practical mechanics.

The evil influence of the supposed inconsistency of theory and practice upon speculative science, although much less conspicuous than it was in the ancient and middle ages, is still occasionally to be traced. This it is which opposes the mutual communication of ideas between men of science and men of practice, and which leads scientific men sometimes to employ, on problems that can only be regarded as ingenious mathematical exercises, much time and mental exertion that would be better bestowed on questions having some connection with the arts, and sometimes to state the results of really important investigations on practical subjects in a form too abstruse for ordinary use; so that the benefit which might be derived from their application is for years lost to the public; and valuable practical principles, which might have been anticipated by reasoning, are left to be discovered by slow and costly experience.

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* Vol. iii., p. 84.

But it is on the practice of mechanics and engineering that the influence of the great fallacy is most conspicuous and most fatal. There is assuredly, in Britain, no deficiency of men distinguished by skill in judging of the quality of materials and work, and in directing the operations of workmen,—by that sort of skill, in fact, which is purely practical, and acquired by observation and experience in business. But of that scientifically practical skill which produces the greatest effect with the least possible expenditure of material and work, the instances are comparatively rare. In too many cases we see the strength and the stability, which ought to be given by the skilful arrangement of the parts of a structure, supplied by means of clumsy massiveness, and of lavish expenditure of material, labour, and money; and the evil is increased by a perversion of the public taste, which causes works to be admired, not in proportion to their fitness for their purposes, or to the skill evinced in attaining that fitness, but in proportion to their size and cost.

With respect to those works which, from unscientific design, give way during or immediately after their erection, I shall say little; for, with all their evils, they add to our experimental knowledge, and convey a lesson, though a costly one. But a class of structures fraught with much greater evils exists in great abundance throughout the country:—namely, those in which the faults of an unscientific design have been so far counteracted by massive strength, good materials, and careful workmanship, that a temporary stability has been produced, but which contain within themselves sources of weakness, obvious to a scientific examination only, that must inevitably cause their destruction within a limited number of years.

Another evil, and one of the worst which arises from the separation of theoretical and practical knowledge, is the fact that a large number of persons, possessed of an inventive turn of mind and of considerable skill in the manual operations of practical mechanics, are destitute of that knowledge of scientific principles which is requisite to prevent their being misled by their own ingenuity. Such men too often spend their money, waste their lives, and it may be lose their reason, in the vain pursuit of visionary inventions, of which a moderate amount of theoretical knowledge would be sufficient to demonstrate the fallacy; and for want of such knowledge, many a man who might have been a useful and happy member of society, becomes a being than whom it would be hard to find anything more miserable.

The number of those unhappy persons—to judge from the patent-lists, and from some of the mechanical journals—must be much greater than is generally believed. The most absurd of all their delusions,—that commonly called the perpetual motion, or to speak more accurately, the inexhaustible source of power,—is, in various forms, the subject of several patents in each year.

The ill success of the projects of misdirected ingenuity has very naturally the effect of driving those men of practical skill who, though without scientific knowledge, possess prudence and common sense, to the opposite extreme of caution, and of inducing them to avoid all experiments, and to confine themselves to the careful copying of successful existing structures and machines: a course which, although it avoids risk, would, if generally followed, stop the progress of all improvement. A similar course has sometimes, indeed, been adopted by men possessed of scientific as well as practical skill: such men having, in certain cases, from deference to popular prejudice, or from a dread of being reputed as theorists, considered it advisable to adopt the worse and customary design for a work in preference to a better but unusual design.

Some of the evils which are caused by the fallacy of an incompatibility between theory and practice having been described, it must now be admitted, that at the present time those evils show a decided tendency to decline. The extent of intercourse, and of mutual assistance, between men of science and men of practice, the practical knowledge of scientific men, and the scientific knowledge of practical men, have been for some time steadily increasing; and that combination and harmony of theoretical and practical knowledge—that skill in the application of scientific principles to practical purposes, which in former times was confined to a few remarkable individuals, now tends to become more generally diffused. With a view to promote the diffusion of that kind of skill, Chairs were instituted at periods of from fifteen to ten years ago, in the two Colleges of the University of London, in the University of Dublin, in the three Queen's Colleges of Belfast, Cork, and Galway, and in this University of Glasgow.

For the sake of a parallel, it may here be worth while to refer to another branch of practical science—that of Medicine. From the time of the first establishment of Medical Schools in Universities, there have existed, not only Chairs for the teaching of the purely scientific departments of Medical Science, such as Anatomy and Physiology, but also Chairs for instruction in the art of applying scientific principles to practice, such as those of Surgery, the Practice of Physic, and others. The institution of a Chair of Mechanics and Engineering in a University where there have long existed Chairs of Mathematics and Natural Philosophy, is an endeavour to place Mechanical Science on the same footing with that of Medicine.

Another parallel may be found in an Institution, which, though not a University, and though established as much for the advancement as for the diffusion of knowledge, has had a most beneficial effect in promoting the appreciation of science by the public,—I mean the British Association. When that body was first instituted, both the theoretical advancement and the practical application of Mechanics, and the several branches of Physics, were allotted to a single section, called Section A. The business before that Section soon became so excessive in amount, and so multifarious in its character, that it was found necessary to institute Section G, for the purpose of considering the practical application of those branches of science to whose theoretical advancement Section A was now devoted; and notwithstanding this separation, those two Sections work harmoniously together for the promotion of kindred objects; and the same men are, in many instances, leading members of both. What Section G is to Section A in the British Association, this class of Engineering and Mechanics is to those of Physics and Mathematics in the University.

It being admitted, that Theoretical and Practical Mechanics are in harmony with each other, and depend on the same first principles, and that they differ only in the purposes to which those principles are applied, it now remains to be considered, in what manner that difference affects the mode of instruction to be followed in communicating those branches of science.

Mechanical knowledge may obviously be distinguished into three kinds: purely scientific knowledge,—purely practical knowledge—and that intermediate knowledge which relates to the application of scientific principles to practical purposes, and which arises from understanding the harmony of theory and practice.

The objects of instruction in purely scientific mechanics and physics are, first, to produce in the student that improvement of the understanding which results from the cultivation of natural knowledge, and that elevation of mind which flows from the contemplation of the order of the universe; and secondly, if possible, to qualify him to become a scientific discoverer. In this branch of study exactness is an essential feature; and mathematical difficulties must not be shrunk from when the nature of the subject leads to them. The ascertainment and illustration of truth are the objects; and structures and machines are looked upon merely as natural bodies are:—namely, as furnishing experimental data for the ascertaining of principles, and examples for their illustration.

Instruction in purely practical knowledge is that which the student acquires by his own experience and observation of the transaction of business. It enables him to judge of the quality of materials and workmanship, and of questions of convenience and commercial profit, to direct the operations of workmen, to imitate existing structures and machines, to follow established practical rules, and to transact the commercial business which is connected with mechanical pursuits.

The third and intermediate kind of instruction, which connects the first two, and for the promotion of which this Chair was established, relates to the application of scientific principles to practical purposes. It qualifies the student to plan a structure or a machine for a given purpose, without the necessity of copying some existing example, and to adapt his designs to situations to which no existing example affords a parallel. It enables him to compute the theoretical limit of the strength or stability of a structure, or the efficiency of a machine of a particular kind,—to ascertain how far an actual structure or machine fails to attain that limit,—to discover the causes of such shortcomings,—and to devise improvements for obviating such causes; and it enables him to judge how far an established practical rule is founded on reason, how far on mere custom, and how far on error.

There are certain characteristics in the mode of treating the subjects, by which this practical-scientific instruction ought to be distinguished from instruction for purely scientific purposes.

In the first place it will be universally admitted, that as far as is possible, mathematical intricacy ought to be avoided.

In the original discovery of a proposition of practical utility, by deduction from general principles and from experimental data, a complex algebraical investigation is often not merely useful, but indispensable; but in expounding such a proposition as a part of practical science, and applying it to practical purposes, simplicity is of the first importance:—and, in fact, the more thoroughly a scientific man has studied the higher mathematics, the more fully does he become aware of this truth,—and, I may add, the better qualified does he become to free the exposition and application of scientific principles from mathematical intricacy. I cannot better support this view than by referring to Sir John Herschel's *Outlines of*

Astronomy—a work in which one of the most profound mathematicians in the world has succeeded admirably in divesting of all mathematical intricacy the explanation of the principles of that natural science which employs the higher mathematics most.

In fact, the symbols of algebra, when employed in abstruse and complex theoretical investigations, constitute a sort of thought-saving machine, by whose aid a person skilled in its use can solve problems respecting quantities, and dispense with the mental labour of thinking of the quantities denoted by the symbols, except at the beginning and end of the operation. In treating of the practical application of scientific principles, an algebraical formula should only be employed when its shortness and simplicity are such as to render it a clearer expression of a proposition or rule than common language would be, and when there is no difficulty in keeping the thing represented by each symbol constantly before the mind.

Another characteristic by which instruction in practical science should be distinguished from purely scientific instruction, is one which appears to me to possess the advantage of calling into operation a mental faculty distinct from those which are exercised by theoretical science. It is of the following kind:—

In theoretical science, the question is—*What are we to think?* and when a doubtful point arises, for the solution of which either experimental data are wanting, or mathematical methods are not sufficiently advanced, it is the duty of philosophic minds not to dispute about the probability of conflicting suppositions, but to labour for the advancement of experimental inquiry and of mathematics, and await patiently the time when these shall be adequate to solve the question.

But in practical science the question is—*What are we to do?*—a question which involves the necessity for the immediate adoption of some rule of working. In doubtful cases, we cannot allow our machines and our works of improvement to wait for the advancement of science; and if existing data are insufficient to give an exact solution of the question, that approximate solution must be acted upon which the best data attainable show to be the most probable. A prompt and sound judgment in cases of this kind is one of the characteristics of a PRACTICAL MAN, in the right sense of that term.

In conclusion, I will now observe, that the cultivation of the Harmony between Theory and Practice in Mechanics—of the application of Science to the Mechanical Arts—besides all the benefits which it confers on us, by promoting the comfort and prosperity of individuals, and augmenting the wealth and power of the nation—confers on us also the more important benefit of raising the character of the mechanical arts, and of those who practise them. A great mechanical philosopher, the late Dr. Robison of

Edinburgh, after stating that the principles of Carpentry depend on two branches of the science of Statics, remarks—"It is this which makes Carpentry a liberal art."

So also is Masonry a liberal art,—so is the art of working in Iron, so is every art, when guided by scientific principles. Every structure or machine, whose design evinces the guidance of science, is to be regarded not merely as an instrument for promoting convenience and profit, but as a monument and testimony that those who planned and made it had studied the laws of nature; and this renders it an object of interest and value, how small soever its bulk, how common soever its material.

For a century there has stood, in a room in this College, a small, rude, and plain model, of appearance so uncouth, that when an artist lately introduced its likeness into a historical painting, those who saw the likeness, and knew nothing of the original, wondered what the artist meant by painting an object so unattractive.

But the artist was right; for ninety-one years ago a man took that model, applied to it his knowledge of natural laws, and made it into the first of those steam engines that now cover the land and the sea; and ever since, in Reason's eye, that small and uncouth mass of wood and metal shines with imperishable beauty, as the earliest embodiment of the genius of James Watt.

Thus it is that the commonest objects are by science rendered precious; and in like manner the engineer or the mechanic, who plans and works with understanding of the natural laws that regulate the results of his operations, rises to the dignity of a Sage.

INTRODUCTION.

DEFINITION OF GENERAL TERMS AND DIVISION OF THE SUBJECT.

ART. 1. **Mechanics** is the science of rest, motion, and force.

The *laws*, or *first principles* of mechanics, are the same for all bodies, celestial and terrestrial, natural and artificial.

The *methods of applying* the principles of mechanics to particular cases are more or less different, according to the circumstances of the case. Hence arise branches in the science of mechanics.

2. **Applied Mechanics.**—The branch to which the term "APPLIED MECHANICS" has been restricted by custom, consists of those consequences of the laws of mechanics which relate to works of human art.

A treatise on applied mechanics must commence by setting forth those first principles which are common to all branches of mechanics; but it must contain only such consequences of those principles as are applicable to purposes of art.

3. **Matter** (considered mechanically) is that which fills space.

4. **Bodies** are limited portions of matter. Bodies exist in three conditions—the solid, the liquid, and the gaseous. Solid bodies tend to preserve a definite size and shape. Liquid bodies tend to preserve a definite size only. Gaseous bodies tend to expand indefinitely. Bodies also exist in conditions intermediate between the solid and liquid, and possibly also between liquid and gaseous.

5. A **Material** or **Physical Volume** is the space occupied by a body or by a part of a body.

6. A **Material** or **Physical Surface** is the boundary of a body, or between two parts of a body.

7. **Line, Point, Physical Point, Measure of Length.**—In mechanics, as in geometry, a LINE is the boundary of a surface, or between two

parts of a surface; and a POINT is the boundary of a line, or between two parts of a line; but the term "*Physical Point*" is sometimes used by mechanical writers to denote an *immeasurably small body*—a sense inconsistent with the strict meaning of the word "point;" but still not leading to error, so long as it is rightly understood.

In *measuring* the dimensions of bodies, the standard British unit of length is the *yard*, being the length at the temperature of 62° Fahrenheit, and at the mean atmospheric pressure, between the two ends of a certain bar which is kept in the office of the Exchequer, at Westminster.

In computations respecting motion and force, and in expressing the dimensions of large structures, the unit of length commonly employed in Britain is the *foot*, being one-third of the yard.

In expressing the dimensions of machinery, the unit of length commonly employed in Britain is the *inch*, being one-thirty-sixth part of the yard. Fractions of an inch are very commonly stated by mechanics and other artificers in halves, quarters, eighths, sixteenths, and thirty-second parts; but according to a resolution of the Institution of Mechanical Engineers, passed at the meeting held at Manchester in June, 1857, the practice has been introduced of expressing fractions of an inch in decimals.

The French unit of length is the mètre, being about $\frac{1}{40000000}$ of the earth's circumference, measured round the poles. (See table at the end of the volume.)

8. **Rest** is the relation between two points, when the straight line joining them does not change in length nor in direction.

A body is at rest relatively to a point, when every point in the body is at rest relatively to the first mentioned point.

9. **Motion** is the relation between two points when the straight line joining them changes in length, or in direction, or in both.

A body moves relatively to a point when any point in the body moves relatively to the first mentioned point.

10. **Fixed Point.**—When a single point is spoken of as having motion or rest, some other point, either actual or ideal, is always either expressed or understood, *relatively* to which the motion or rest of the first point takes place. Such a point is called a *fixed point*.

So far as the phenomena of motion alone indicate, the choice of a fixed point with which to compare the positions of other points appears to be arbitrary, and a matter of convenience alone; but when the laws of force, as affecting motion, come to be considered, it will be seen that there are reasons for calling certain points fixed, in preference to others.

In the mechanics of the solar system, the fixed point is what is called the *common centre of gravity* of the bodies composing that system. In applied mechanics, the fixed point is either a point which is at rest relatively to the earth, or (if the structure or machine under consideration be moveable from place to place on the earth), a point which is at rest relatively to the structure, or to the frame of the machine, as the case may be.

Points, lines, surfaces, and volumes, which are at rest relatively to a fixed point, are fixed.

11. **Cinematics.**—The comparison of motions with each other, without reference to their causes, is the subject of a branch of geometry called "*Cinematics.*"

12. **Force** is an action between two bodies, either causing or tending to cause change in their relative rest or motion.

The notion of force is first obtained directly by sensation; for the forces exerted by the voluntary muscles can be felt. The existence of forces other than muscular tension is inferred from their effects.

13. **Equilibrium** or **Balance** is the condition of two or more forces which are so opposed that their combined action on a body produces no change in its rest or motion.

The notion of balance is first obtained by sensation; for the forces exerted by voluntary muscles can be felt to balance sometimes each other, and sometimes external pressures.

14. **Statics and Dynamics.**—Forces may take effect, either by balancing other forces, or by producing change of motion. The former of those effects is the subject of *Statics;* the latter that of *Dynamics;* these, together with *Cinematics,* already defined, form the three great divisions of pure, abstract, or general mechanics.

15. **Structures and Machines.**—The works of human art to which the science of applied mechanics relates, are divided into two classes, according as the parts of which they consist are intended to rest or to move relatively to each other. In the former case they are called *Structures;* in the latter, *Machines.* Structures are subjects of Statics alone; Machines, when the motions of their parts are considered alone, are subjects of Cinematics; when the forces acting on and between their parts are also considered, machines are subjects of Statics and Dynamics.

16. **General Arrangement of the Subject.**—The subject of the present treatise will be arranged as follows:—

I. FIRST PRINCIPLES OF STATICS.
II. THEORY OF STRUCTURES.
III. FIRST PRINCIPLES OF CINEMATICS.
IV. THEORY OF MECHANISM.
V. FIRST PRINCIPLES OF DYNAMICS.
VI. THEORY OF MACHINES.

PART I.

PRINCIPLES OF STATICS.

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19

CHAPTER I.

BALANCE AND MEASUREMENT OF FORCES ACTING IN ONE STRAIGHT LINE.

17. **Forces how Determined.**—Although every force (as has been stated in Art. 12) is an action between two bodies, still it is conducive to simplicity to consider in the first place the condition of one of those two bodies alone.

The nature of a force, as respects one of the two bodies between which it acts, is determined, or made known, when the following three things are known respecting it:—first, the *place*, or part of the body to which it is applied; secondly, the *direction* of its action; thirdly, its *magnitude*.

18. **Place of Application—Point of Application.**—The place of the application of a force to a body may be the whole or part of its internal mass; in which case the force is an attraction or a repulsion, according as it tends to move the bodies between which it acts towards or from each other; or the place of application may be the surface at which two bodies touch each other, or the bounding surface between two parts of the same body, in which case the force is a tension or pull, a thrust or push, or a lateral stress, according to circumstances.

Thus every force has its action distributed over a certain space, either a volume or a surface; and a force concentrated at a single point has no real existence. Nevertheless it is necessary, in treating of the principles of statics, to begin by demonstrating the properties of such ideal forces, conceived to be concentrated at single points. It will afterwards be shown how the conclusions so arrived at respecting *single forces* (as they may be called), are made applicable to the distributed forces which really act in nature.

In illustrating the principles of statics experimentally, a force concentrated at a single point may be represented with any required degree of accuracy by a force distributed over a very small space, if that space be made small enough.

19. **Supposition of Perfect Rigidity.**—In reasoning respecting forces concentrated at single points, they are assumed to be applied to solid bodies which are *perfectly rigid*, or incapable of alteration of figure under any forces which can be applied to them. This also is a supposition not realized in nature. It will afterwards be shown how its consequences are applied to actual bodies.

20. **Direction—Line of Action.**—The DIRECTION of a force is that of the motion which it tends to produce. A straight line drawn through the point of application of a single force, and along its direction, is the LINE OF ACTION of that force.

21. **Magnitude—Unit of Force.**—The magnitudes of two forces are equal, when being applied to the same body in opposite directions along the same line of action, they balance each other.

The magnitude of a force is expressed arithmetically by stating in numbers its ratio to a certain *unit* or *standard* of force, which is usually the *weight* (or attraction towards the earth), at a certain latitude, and at a certain level, of a known mass of a certain material. Thus the British unit of force is the *standard pound avoirdupois;* which is the weight in the latitude of London of a certain piece of platinum kept in the Exchequer office (See the Act 18 and 19 Vict., cap. 72; also a paper by Professor W. H. Miller, in the *Philosophical Transactions* for 1856).

For the sake of convenience or of compliance with custom, other units of force are occasionally employed in Britain, bearing certain ratios to the standard pound; such as—

The grain $= \frac{1}{7000}$ of a pound avoirdupois.
The troy pound $= 5{,}760$ grains $= 0{\cdot}82\dot{2}8571\dot{4}$ pound avoirdupois.
The hundredweight $= 112$ pounds avoirdupois.
The ton $= 2{,}240$ pounds avoirdupois.

The French standard unit of force is the *gramme*, which is the weight, in the latitude of Paris, of a cubic centimetre of pure water, measured at the temperature at which the density of water is greatest, viz., 4°·1 centigrade, or 39°·4 Fahrenheit, and under the pressure which supports a barometric column of 760 millimetres of mercury.

A comparison of French and British measures of force and of size is given in a table at the end of this volume.

22. **Resultant of Forces Acting in One Straight Line.**—The RESULTANT of any number of given forces applied to one body, is a single force capable of balancing that single force which balances the given forces; that is to say, the resultant of the given forces is equal and directly opposed to the force which balances the given forces; and is *equivalent* to the given forces so far as the balance of

the body is concerned. The given forces are called *components* of their resultant.

The resultant of any number of forces acting on one body in the same straight line of action, acts along that line, and is equal in magnitude to the sum of the component forces; it being understood, that when some of the component forces are opposed to the others, the word "*sum*" is to be taken in the algebraical sense; that is to say, that forces acting in the same direction are to be added to, and forces acting in opposite directions subtracted from each other.

23. **Representation of Forces by Lines.**—A single force may be represented in a drawing by a straight line; an extremity of the line indicating the point of application of the force,—the direction of the line, the direction of the force,—and the length of the line, the magnitude of the force, according to an arbitrary scale.

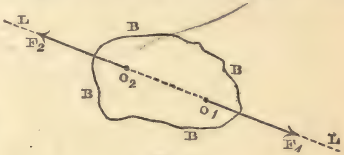


Fig. 1.

For example, in fig. 1, the fact that the body B B B B is acted upon at the point O_1 by a given force, may be expressed by drawing from O_1 a straight line $\overline{O_1 F_1}$ in the direction of the force, and of a length representing the magnitude of the force.

If the force represented by $\overline{O_1 F_1}$ is balanced by a force applied either at the same point, or at another point O_2 (which must be in the line of action L L of the force to be balanced), then the second force will be represented by a straight line $\overline{O_2 F_2}$, opposite in direction, and equal in length to $\overline{O_1 F_1}$, and lying in the same line of action L L.

If the body B B B B (fig. 2), be balanced by several forces acting in the same straight line L L, applied at points O_1 O_2, &c., and represented by lines $\overline{O_1 F_1}$, $\overline{O_2 F_2}$, &c.; then either direction in the line L L (such as the direction towards + L) is to be considered as positive, and the opposite direction (such as the direction towards − L) as negative; and if the sum of all the lines representing forces which point positively be equal to the sum of all those which point negatively, the algebraical sum of all the forces is nothing, and the body is balanced.

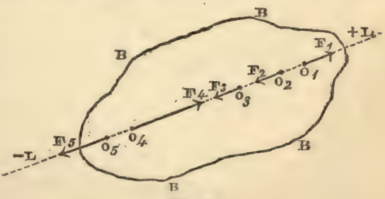


Fig. 2.

24. **Pressure.**—Most writers on mechanics, in treating of the first principles of statics, use the word "*pressure*" to denote *any balanced force.*

In the popular sense, which is also the sense generally employed in applied mechanics, the word *pressure* is used to denote a force, of the nature of a thrust, distributed over a surface; in other words, the kind of force with which a body tends to expand, or resists an effort to compress it.

In this treatise care will be taken so to employ the word "pressure" that the context shall show in what sense it is used.

CHAPTER II.

9/

THEORY OF COUPLES AND OF THE BALANCE OF PARALLEL FORCES.

SECTION 1.—*On Couples with the Same Axis.*

25. **Couples.**—Two forces of equal magnitude applied to the same body in parallel and opposite directions, but not in the same line of action, constitute what is called a "*couple.*"

26. **Force of a Couple—Arm or Leverage.**—The *force* of a couple is the common magnitude of the two equal forces; the *arm* or *leverage* of a couple is the perpendicular distance between the lines of action of the two equal forces.

27. **Tendency of a Couple—Plane of a Couple—Right-handed and Left-handed Couples.**—The tendency of a couple is to turn the body to which it is applied in the plane of the couple—that is, the plane which contains the lines of action of the two forces. (The plane in which a body turns, is any plane parallel to those planes in the body whose position is not altered by the turning). The *axis* of a couple is any line perpendicular to its plane. The turning of a body is said to be *right-handed* when it appears to a spectator to take place in the same direction with that of the hands of a watch, and *left-handed* when in the opposite direction; and couples are designated as right-handed or left-handed according to the direction of the turning which they tend to produce.

Thus in fig. 3, the equal and opposite forces $\overline{O_1 F_1}$, $\overline{O_2 F_2}$, whose leverage is $\overline{L_1 L_2}$, form a right-handed couple; and the equal and opposite forces $\overline{O_3 F_3}$, $\overline{O_4 F_4}$, form a left-handed couple.

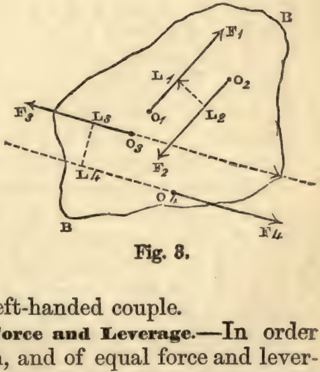


Fig. 3.

28. **Equivalent Couples of Equal Force and Leverage.**—In order that two couples similar in direction, and of equal force and leverage, may be exactly alike or *equivalent* in their tendency to turn the body, it is necessary and sufficient that their planes should be either identical or parallel.

Two couples applied to the same body in the same plane, or in parallel planes, of equal force and leverage, but opposite in direction, balance each other; and if for either of the two an equivalent couple be substituted, the equilibrium will not be disturbed.

29. **Moment of a Couple.**—The *moment* of a couple means the product of the magnitude of its force by the length of its arm. If the force be a certain number of pounds, and the arm a certain number of feet, the product of those two numbers is called the moment in *foot-pounds*, and similarly for other measures.

30. **Addition of Couples of Equal Force.**—LEMMA. *Two couples of equal force acting in the same direction, with the same axis, are equivalent to a couple whose moment is the sum of their moments.* §2 Let the two couples be denoted by A and B; let $F_A = F_B$ be their equal forces; let L_A and L_B be their respective arms; then $F_A L_A$ and $F_B L_B$ are their moments, which, as their forces are equal, are proportional to the arms. In fig. 4, let the forces F_A constituting A be applied in lines passing through a and c, $a c$ or L_A being perpendicular to the lines of action of the forces; and if the forces constituting B be not already applied as shown in the figure, substitute for B an equivalent couple of equal force and arm, having its forces F_B applied in lines parallel to the lines of action of the forces F_A, and passing one through the point c and the other through b, so that the arm $c\, b$ or L_B shall be in the same straight line with $a\, c$ or L_A. Then the equal and opposite forces F_A, F_B, applied at c, balance each other, and there remain only the equal and opposite forces F_A, F_B, applied at a and b, which form a couple whose force is $F_A = F_B$, and its arm $\overline{a\, b} = L_A + L_B$, being the sum of the arms of the couples A and B; so that its moment is the sum of their moments; and this couple is equivalent to the two couples A and B.

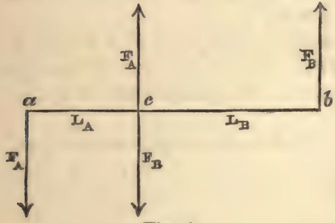


Fig. 4.

31. **Equivalent Couples of Equal Moment.**—THEOREM. *If the moments of two couples acting in the same direction and with the same axis are equal, those couples are equivalent.* Let one of the couples be called A, and let its force, arm, and moment be respectively F_A, L_A, and $F_A L_A$; let the other couple be called B, and let its force, arm, and moment be respectively F_B, L_B, and $F_B L_B$. The equality of the moments of those couples is expressed by the equation

$$F_A L_A = F_B L_B.$$

If the forces and arms of the two couples be commensurable, so that

$$F_A : F_B :: L_B : L_A :: m : n$$

(m and n being two whole numbers),

let $$f = \frac{F_A}{m} = \frac{F_B}{n},$$

and $$l = \frac{L_B}{m} = \frac{L_A}{n}.$$

Then the couple A is equivalent to $m n$ couples of the moment $f\,l$; and so also is the couple B; therefore the couples A and B are equivalent to each other.

If the forces and arms are incommensurable, it is always possible to find forces and arms which shall be commensurable, and shall differ from the given forces and arms by differences less than any given quantity; so that if the theorem were in error for incommensurable forces and arms, it would also be in error for certain commensurable forces and arms; but this is impossible; therefore the theorem is true for incommensurable as well as for commensurable forces and arms.

32. **Resultant of Couples with the Same Axis.**—COROLLARY. *A combination of any number of couples having the same axis is equivalent to a couple whose moment is the algebraical sum of the moments of the combined couples.*

33. **Equilibrium of Couples having the Same Axis.**—Two opposite couples of equal moment, having the same axis, balance each other. Any number of couples, having the same axis, balance each other when the moments of the right-handed couples are together equal to the moments of the left-handed couples; in other words, when the resultant moment is nothing.

34. **Representation of Couples by Lines.**—The nature and amount of the tendency of a couple to turn a body are completely known when the moment and direction of the couple, and the position of its axis, are known. These circumstances are expressed by means of a line in the following manner.

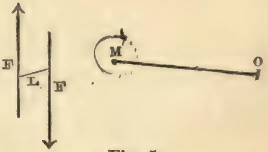


Fig. 5.

In fig. 5, from any point O draw a straight line O M, parallel to the axis (that is, perpendicular to the plane) of the couple to be represented, and in such a direction, that to an observer looking from O towards M the couple shall seem right-handed; and make the length of the line $\overline{OM}$ represent the moment of the couple, according to any assigned scale.

SECTION 2.—*On Couples with Different Axes.*

35. **Resultant of Two Couples with Different Axes.**—THEOREM. *If the two sides of a parallelogram represent the positions of the axes, and the directions and moments, of two couples acting on the same body, the diagonal of the parallelogram will in like manner represent the position of the axis, the direction and the moment of the resultant couple, which is equivalent to those two.*

In fig. 6, let the plane of the paper represent a plane which contains the axes of the two couples, and is therefore perpendicular to both their planes. Let $a\,c$, $c\,b$ be parts of the lines in which the planes of the couples A, B, respectively intersect the plane of the paper. If the couples are not already of equal force, reduce them to equivalent couples of equal force; let F denote the common magnitude of their forces, and let L_A, L_B denote the respective arms of the couples. From c, the intersection of the three planes already mentioned, take $c\,a = L_A$, $c\,b = L_B$, and join $a\,b$. Conceive the couple A (or an equivalent couple) to consist of the force + F acting forwards at a, and the equal and opposite force − F acting backwards at c; also conceive the couple B (or an equivalent couple) to consist of the force + F acting forwards at c, and the equal and opposite force − F acting backwards at b. The forces + F, − F, at c balance each other; and there are left the equal and opposite forces + F at a, and − F at b, forming the *resultant couple*, which is equivalent to the two couples A and B, and has for its arm the third side $\overline{a\,b} = L_C$ of the triangle $a\,b\,c$.

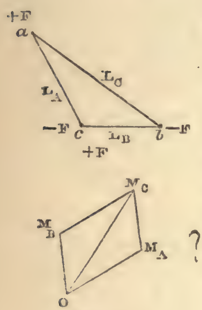


Fig. 6.

Now from any point O draw $\overline{O\,M_A}$ perpendicular to $a\,c$, and $\overline{O\,M_B}$ perpendicular to $b\,c$, and representing the axes, directions, and moments of the couples A and B: complete the parallelogram of which those lines are the sides, and draw its diagonal $\overline{O\,M_C}$. This diagonal will be perpendicular to $a\,b$, and will therefore represent the axis and direction of the resultant couple; and because of the similarity of the triangles $a\,b\,c$, $O\,M_C\,M_B$, the following proportions will exist:—

$$\overline{O\,M_A} : \overline{O\,M_B} : O\,M_C,$$
$$:: L_A : L_B : L_C;$$

and consequently $\overline{O\,M_C}$ will also represent the moment of the resultant couples.—Q. E. D.

36. **Equilibrium of Three Couples with Different Axes in the Same Plane.**—COROLLARY. *A couple equal and opposite to that represented by the diagonal* $\overline{OM_C}$ *balances the couples represented by the sides* $\overline{OM_A}$, $\overline{OM_B}$. *In other words, three couples represented by the three sides of a triangle balance each other.*

37. **Equilibrium of any Number of Couples.**—COROLLARY. *If a number of couples acting on the same body be represented by a series of lines joined end to end, so as to form sides of a polygon, and if the polygon is closed, these couples balance each other.* To fix the ideas let there be five couples, whose moments are respectively M_1, M_2, M_3, M_4, M_5; and let them be represented by the sides of the polygon in fig. 7 in such a manner that

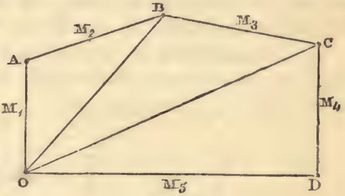


Fig. 7.

M_1	is represented by	$\overline{OA}$,	and seems	right-handed looking	from A towards O.
M_2	—	$\overline{AB}$,	—	—	from B towards A.
M_3	—	$\overline{BC}$,	—	—	from C towards B.
M_4	—	$\overline{CD}$,	—	—	from D towards C.
M_5	—	$\overline{DO}$,	—	—	from O towards D.

Then by the theorem of Article 35, the resultant of M_1 and M_2 is $\overline{OB}$; the resultant of this and M_3 is $\overline{OC}$; the resultant of this and M_4 is $\overline{OD}$, right-handed in looking from D towards O, and consequently equal and opposite to M_5, which last couple balances it, and reduces the final resultant to nothing.—Q. E. D.

This proposition evidently holds for any number of couples, and whether the closed polygon be plane or *gauche* (that is to say, not plane).

The resultant of the couples represented by all the sides of the polygon, except one, is equal and opposite to the couple represented by the excepted side.

SECTION 3.—*On Parallel Forces.*

38. **Balanced Parallel Forces in General.**—A balanced system of parallel forces consists either of pairs of directly opposed equal forces, or of couples of equal forces, or of combinations of such pairs and couples.

Hence the following propositions as to the relations amongst the *magnitudes* of systems of parallel forces are obvious :—

I. In a balanced system of parallel forces, the sums of the forces acting in opposite directions are equal; in other words, the algebraical sum of the magnitudes of all the forces taken with their proper signs is nothing.

II. The magnitude of the resultant of any combination of parallel forces is the algebraical sum of the magnitudes of the forces.

The relations amongst the *positions* of the lines of action of balanced parallel forces remain to be investigated; and in this inquiry, all pairs of directly opposed equal forces may be left out of consideration; for each such pair is independently balanced whatsoever its position may be; so that the question in each case is to be solved by means of the theory of couples.

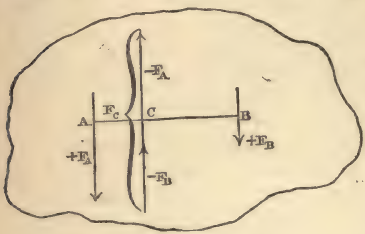


Fig. 8.

39. **Equilibrium of Three Parallel Forces in One Plane. Principle of the Lever.**—THEOREM. *If three parallel forces applied to one body balance each other, they must be in one plane; the two extreme forces must act in the same direction; the middle force must act in the opposite direction; and the magnitude of each force must be proportional to the distance between the lines of action of the other two.* Let a body (fig. 8) be maintained in equilibrio by two opposite couples having the same axis, and of equal moments,

$$F_A L_A = F_B L_B,$$

according to the notation already used; and let those couples be so applied to the body that the lines of action of two of these forces, $-F_A$, $-F_B$, which act in the same direction, shall coincide. Then those two forces are equivalent to the single middle force $F_C = -(F_A + F_B)$, equal and opposite to the sum of the extreme forces $+F_A$, $+F_B$, and in the same plane with them; and if the straight line A C B be drawn perpendicular to the lines of action of the forces, then

$$\overline{AC} = L_A;\ \overline{CB} = L_B;\ \overline{AB} = L_A + L_B;$$

and consequently

$$F_A : F_B : F_C :: \overline{CB} : \overline{AC} : \overline{AB};$$

so that each of the three forces is proportional to the distance between the lines of action of the other two; and if any three parallel forces balance each other, they must be equivalent to two couples, as shown in the figure.

40. **Resultant of Two Parallel Forces.**—The resultant of any two of the three forces F_A, F_B, F_C, is equal and opposite to the third.

Hence the resultant of two parallel forces is parallel to them, and in the same plane; if they act in the same direction, then their resultant is their sum, acts in the same direction, and lies between them; if they act in opposite directions, their resultant is their difference, acts in the direction of, and lies beyond, the preponderating force; and the distance between the lines of action of any two of those three forces—the resultant and its two components—is proportional to the third force.

In order that two opposite parallel forces may have a single resultant, it is necessary that they should be unequal, the resultant being their difference. Should they be equal, they constitute a couple, which has no single resultant.

41. **Resultant of a Couple and a Single Force in Parallel Planes.**—Let M denote the moment of a couple applied to a body (fig. 9); and at a point O let a single force F be applied, in a plane parallel to that of the couple. For the given couple substitute an equivalent couple, consisting of a force $-F$ equal and directly opposed to F at O, and a force F applied at A, the arm $\overline{AO}$ being $= \frac{M}{F}$, and of course parallel to the plane of the couple M. Then the forces at O balance each other, and F applied at A is the resultant of the single force F applied at O, and the couple M; that is to say, that if to a single force F there be added a couple M whose plane is parallel to the force, the effect of that addition is to shift the line of action of the force parallel to itself through a distance $OA = \frac{M}{F}$;—to the left if M is right-handed—to the right if M is left-handed.

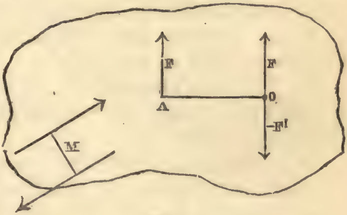


Fig. 9.

42. **Moment of a Force with respect to an Axis.**—Let the straight line F represent a force applied to a body. Let O X be any straight line perpendicular in direction to the line of action of the force, and not intersecting it, and let A B be the common perpendicular of those two lines. At B conceive a pair of equal and directly opposed forces to be applied in a line of action parallel to F, viz.: $F' = F$, and $-F' = -F$. The supposed application of such a pair of balanced forces does not alter the statical condition of the body. Then the original single force F, applied in a line tra-

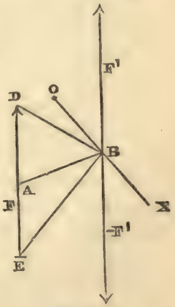


Fig. 10.

versing A, is equivalent to the force F′ applied in a line traversing B, the point in O X which is nearest to A, combined with the couple composed of F and − F′, whose moment is F · AB. This is called the *moment of the force* F *relatively to the axis* O X, and sometimes also, the *moment of the force* F *relatively to the plane* which contains O X, and is parallel to the line of action of the force.

If from the point B there be drawn two straight lines B D and B E, to the extremities of the line F representing the force, the area of the triangle B D E being $= \frac{1}{2} \mathrm{F} \cdot \overline{\mathrm{AB}}$, represents one-half of the moment of F relatively to O X.

43. **Equilibrium of any System of Parallel Forces in One Plane.**—In order that any system of parallel forces whose lines of action are in one plane may balance each other, it is necessary and sufficient that the following conditions should be fulfilled :—

I. (As already stated in Art. 38) that the algebraical sum of the forces shall be nothing :—

II. That the algebraical sum of the moments of the forces relatively to any axis perpendicular to the plane in which they act shall be nothing :—

two conditions which are expressed symbolically as follows :— let F denote any one of the forces, considered as positive or negative, according to the direction in which it acts ; let y be the perpendicular distance of the line of action of this force from an arbitrarily assumed axis O X, y also being considered as positive or negative, according to its direction ; then,

$$\text{Sum of forces,} \quad \Sigma \cdot \mathrm{F} = 0 ;$$
$$\text{Sum of moments,} \quad \Sigma \cdot y \mathrm{F} = 0.$$

For, by the last Article, each force F is equivalent to an equal and parallel force F′ applied directly to O X, combined with a couple y F ; and the system of forces F′, and the system of couples y F, must each be in equilibrio, because when combined they are equivalent to the balanced system of forces F.

In summing moments, right-handed couples are usually considered as positive, and left-handed couples as negative.

44. **Resultant of any Number of Parallel Forces in One Plane.**—The resultant of any number of parallel forces in one plane is a force in the same plane, whose magnitude is the algebraical sum of the magnitudes of the component forces, and whose position is such, that its moment relatively to any axis perpendicular to the plane in which it acts is the algebraical sum of the moments of the component forces. Hence let F_r denote the resultant of any number of parallel forces in one plane, and y_r the distance of the line of

action of that resultant from the assumed axis O X to which the positions of forces are referred: then

$$F_r = \Sigma \cdot \quad F;$$

$$y_r = \frac{\Sigma \cdot y F}{\Sigma \cdot \quad F}.$$

In some cases, the forces may have no single resultant, $\Sigma \cdot F$ being $= 0$; and then, unless the forces balance each other completely, their resultant is a couple of the moment $\Sigma \cdot y F$.

45. **Moments of a Force with respect to a Pair of Rectangular Axes** —In fig. 11, let F be any single force; O an arbitrarily-assumed point, called the "origin of co-ordinates;" $-X \; O + X$, $-Y \; O + Y$, a pair of axes traversing O, at right angles to each other and to the line of action of F. Let $A B = y$, be the common perpendicular of F and OX; let $AC = x$, be the common perpendicular of F and OY. x and y are the "rectangular co-ordinates" of the line of action of F relatively to the axes $-X O + X$, $-Y O + Y$, respectively. According to the arrangement of the axes in the figure, x is to be considered as positive to the right, and negative to the left, of $-Y O + Y$; and y is to be considered as positive to the left, and negative to the right, of $-X O + X$; right and left referring to the spectator's right and left hand. In the particular case represented, x and y are both positive. Forces, in the figure, are considered as positive upwards, and negative downwards; and in the particular case represented, F is positive.

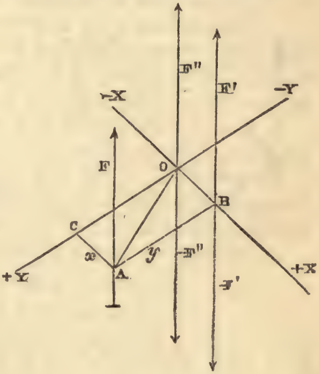


Fig. 11.

At B conceive a pair of equal and opposite forces, F′ and − F′, to be applied; F′ being equal and parallel to F, and in the same direction. Then, as in Article 42, F is equivalent to the single force F′ = F applied at B, combined with the couple constituted by F and − F′ with the arm y, whose moment is y F; being positive in the case represented, because the couple is right-handed. Next, at the origin O, conceive a pair of equal and opposite forces, F″ and − F″, to be applied, F″ being equal and parallel to F and F′, and in the same direction. Then the single force F′ is equivalent to the single force F″ = F′ = F applied at O, combined with the couple constituted by F′ and − F″ with the arm $OB = x$, whose moment is

$-xF$; being negative in the case represented, because the couple is left-handed.

Hence it appears finally, that a force F acting in a line whose co-ordinates with respect to a pair of rectangular axes perpendicular to that line are x and y, is equivalent to an equal and parallel force acting through the origin, combined with two couples whose moments are,

y F relatively to the axis O X, and $-x$ F relatively to the axis O Y; right-handed couples being considered positive; and + Y lying to the left of + X, as viewed by a spectator looking from + X towards O, with his head in the direction of positive forces.

46. **Equilibrium of any System of Parallel Forces.**—In order that any system of parallel forces, whether in one plane or not, may balance each other, it is necessary and sufficient that the three following conditions should be fulfilled:—

I. (As already stated in Art. 38), that the algebraical sum of the forces shall be nothing:—

II. and III. That the algebraical sums of the moments of the forces, relatively to a pair of axes at right angles to each other, and to the lines of action of the forces, shall each be nothing:—

conditions which are expressed symbolically as follows:—

$$\Sigma \cdot F = 0;\ \Sigma \cdot y F = 0;\ \Sigma \cdot x F = 0;$$

for by the last Article, each force F is equivalent to an equal and parallel force F″ applied directly to O, combined with two couples, y F with the axis O X, and $-x$ F with the axis O Y; and the system of forces F″, and the two systems of couples y F and $-x$ F, must each be in equilibrio, because when combined they are equivalent to the balanced system of forces F.

47. **Resultant of any Number of Parallel Forces.**—The resultant of any number of parallel forces, whether in one plane or not, is a force whose magnitude is the algebraical sum of the magnitudes of the component forces, and whose moments relatively to a pair of axes perpendicular to each other and to the lines of action of the forces, are respectively equal to the algebraical sums of the moments of the component forces relatively to the same axes. Hence let F_r denote the resultant, and x_r and y_r the co-ordinates of its line of action, then

$$F_r = \Sigma \cdot F,$$
$$x_r = \frac{\Sigma \cdot x F}{\Sigma \cdot F},$$
$$y_r = \frac{\Sigma \cdot y F}{\Sigma \cdot F}.$$

In some cases, the forces may have no single resultant, $\Sigma \cdot F$

being $= 0$; and then, unless the forces balance each other completely, their resultant is a couple, whose axis, direction, and moment are found as follows :—

Let $$M_x = \Sigma . y F; \quad M_y = - \Sigma . x F;$$

be the moments of the pair of partial resultant couples relatively to the axes O X and O Y respectively. From O, along those axes, set off two lines representing respectively M_x and M_y according to the rule of Art. 34; that is to say, proportional to those moments in length, and pointing in the direction from which those couples must respectively be viewed in order that they may appear right-handed. Complete the rectangle whose sides are those lines; its diagonal (as shown in Art. 35) will represent the axis, direction, and moment of the final resultant couple. Let M_r be the moment of this couple; then

$$M_r = \sqrt{\left\{ M_x^2 + M_y^2 \right\}};$$

and if θ be the angle which its axis makes with O X,

$$\cos \theta = \frac{M_x}{M_r}.$$

Section 4.—*On Centres of Parallel Forces.*

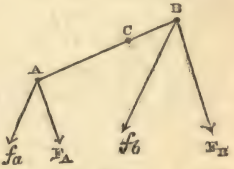


Fig. 12.

48. **Centre of a Pair of Parallel Forces.**—In fig. 12, let A and B represent a pair of points, to which a pair of parallel forces, F_A and F_B, of any given magnitudes, are applied. In the straight line joining A and B take the point C such, that its distances from A and B respectively shall be inversely proportional to the forces applied at those points. Then from the principle of Art. 40 it is obvious that the resultant of F_A and F_B traverses C. It is also obvious that the position of the point C depends solely on the proportionate magnitude of the parallel forces F_A and F_B, and not on their absolute magnitude, nor on the angular position of their lines of action; so that if for those forces there be substituted another pair of parallel forces, f_a, f_b, in any other angular position, and if those new forces bear to each other the same proportion with the original forces, viz. :—

$$f_a : f_b :: F_A : F_B :: \overline{BC} : \overline{AC},$$

the point C where the resultant cuts A B will still be the same.

This point is called the *Centre of Parallel Forces*, for a pair of forces applied at A and B respectively, and having the given ratio $\overline{BC} : \overline{AC}$.

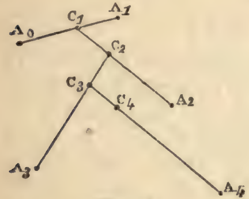


Fig. 13.

49. **Centre of any System of Parallel Forces.**—Let parallel forces, F_0, F_1, be applied at the points A_0 A_1 (fig. 13.), Draw the straight line A_0 A_1, in which take C_1, so that

$$F_0 : F_1 : : \overline{C_1 A_1} : \overline{C_1 A_0};$$

then will C_1 be the centre of a pair of parallel forces applied at A_0 and A_1, and having the proportion $F_0 : F_1$. At a third point, A_2, let a third parallel force, F_2, be applied. Then, because the forces F_0, F_1, are together equivalent to a parallel force, $F_0 + F_1$, applied at C_1, draw the straight line C_1 A_2, in which take C_2, so that

$$F_0 + F_1 : F_2 : : \overline{C_2 A_2} : \overline{C_2 C_1};$$

then will C_2 be the centre of three parallel forces applied at A_0, A_1, A_2, and having the proportions $F_0 : F_1 : F_2$. At a fourth point, A_3, let a fourth parallel force, F_3, be applied. Then, because the forces F_0, F_1, F_2, are together equivalent to a parallel force, $F_0 + F_1 + F_2$, applied at C_2, draw the straight line C_2, A_3, in which take C_3, so that

$$F_0 + F_1 + F_2 : F_3 : : \overline{C_3 A_3} : \overline{C_3 C_2};$$

then will C_3 be the centre of four parallel forces applied at A_0, A_1, A_2, A_3, and having the proportion $F_0 : F_1 : F_2 : F_3$. By continuing this process the centre of any system of parallel forces, how numerous soever, may be found; and hence results the following

THEOREM. *If there be given a system of points, and the mutual ratios of a system of parallel forces applied to those points, then there is one point, and one only, which is traversed by the line of action of the resultant of every system of parallel forces having the given mutual ratios and applied to the given system of points, whatsoever may be the absolute magnitudes of those forces, and the angular position of their lines of action.*

50. **Co-ordinates of Centre of Parallel Forces.**—The method of finding centres of parallel forces described in the preceding Article, though suitable for the demonstration of the theorem just stated, is tedious and inconvenient when the number of forces is great, in which case the best method is to find the rectangular co-ordinates of that point relatively to three fixed axes, as follows:—

Let O be any convenient point, taken as the origin of co-ordinates, and OX, OY, OZ, three axes of co-ordinates at right angles to each other.

Let A be any one of the points to which the system of parallel forces in question are applied. From A draw x parallel to O X, and perpendicular to the plane Y Z, y parallel to O Y, and perpendicular to the plane Z X, and z parallel to O Z, and perpendicular to the plane X Y. x, y, and z are the rectangular co-ordinates of A, which, being known, the position of A is determined. Let F denote either the magnitude of the force applied at A, or any magnitude proportional to that magnitude. x, y, z, and F are supposed to be known for every point of the given system of points.

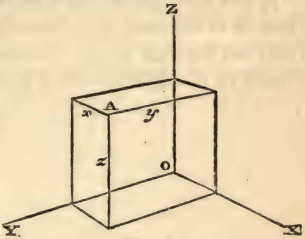


Fig. 14.

Then first, conceive all the parallel forces to act in lines parallel to the plane Y Z. Then the sum of their moments relatively to an axis in that plane is

$$\Sigma \cdot x \mathrm{F};$$

and consequently the distance of their resultant, and also of the centre of parallel forces from that plane is given (as in Articles 44 and 47), by the equation

$$x_r = \frac{\Sigma \cdot x \mathrm{F}}{\Sigma \cdot \mathrm{F}}.$$

Secondly, conceive all the parallel forces to act in lines parallel to the plane Z X. Then the sum of their moments relatively to an axis in that plane becomes

$$\Sigma \cdot y \mathrm{F};$$

and consequently the distance of their resultant, and also of the centre of parallel forces from that plane is given by the equation

$$y_r = \frac{\Sigma \cdot y \mathrm{F}}{\Sigma \cdot \mathrm{F}}.$$

Thirdly, conceive all the parallel forces to act in lines parallel to the plane X Y. Then the sum of their moments relatively to an axis in that plane becomes

$$\Sigma \cdot z \mathrm{F};$$

and consequently the distance of their resultant, and also of the centre of parallel forces from that plane is given by the equation

$$z_r = \frac{\Sigma \cdot z \mathrm{F}}{\Sigma \cdot \mathrm{F}}.$$

Thus are found x_r, y_r, z_r, the three rectangular co-ordinates of the centre of parallel forces, for a system of forces applied to any given system of points, and having any given mutual ratios.

If the parallel forces applied to a system of points are all equal, then it is obvious that the distance of the centre of parallel forces from any given plane is simply the mean of the distances of the points of the system from that plane.

CHAPTER III.

BALANCE OF INCLINED FORCES.

SECTION 1.—*Inclined Forces applied at One Point.*

51. **Parallelogram of Forces.**—THEOREM. *If two forces whose lines of action traverse one point be represented in direction and magnitude by the sides of a parallelogram, their resultant is represented by the diagonal.*

First Demonstration.—Through the point O (fig. 15), let two forces act, represented in direction and magnitude by $\overline{OA}$ and $\overline{OB}$. The resultant or equivalent single force of those two forces must be a force such, that its moment relatively to any axis whatsoever perpendicular to the plane of O A and O B, is the sum of the moments of $\overline{OA}$ and $\overline{OB}$ relatively to the same axis.

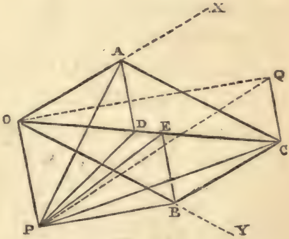


Fig. 15.

Now, *first*, the force represented in direction and magnitude by the diagonal $\overline{OC}$ of the parallelogram A B fulfils this condition. For let P be any point in the plane of O A and O B, and let an axis perpendicular to that plane traverse P. Join P A, P B, P C, P O. Then, as already shown in Art. 42, the moments of the forces $\overline{OA}$, $\overline{OB}$, $\overline{OC}$, relatively to the axis P, are represented respectively by the doubles of the triangles P O A, P O B, P O C. Draw $AD \parallel BE \parallel OP$, and join P D, P E. Then $\triangle POD = \triangle POA$, and $\triangle POE = \triangle POB$; but because $\overline{OD} + \overline{OE} = \overline{OC}$, $\therefore \triangle POC = \triangle POD + \triangle POE = \triangle POA + \triangle POB$; and the moment of $\overline{OC}$ relatively to P is equal to the sum of the moments of $\overline{OA}$ and $\overline{OB}$; and that whatsoever the position of P may be.

Secondly. The force represented by $\overline{OC}$ is the only force which fulfils this condition. For let $\overline{OQ}$ represent a force whose moment relatively to P is equal to the sum of the moments of $\overline{OA}$ and $\overline{OB}$. Join P Q. Then $\triangle OPQ = \triangle POC$, and $\therefore CQ \parallel PO$; so that

$\overline{OQ}$ fulfils the required condition for those axes only which are situated in a line $OP \parallel CQ$, and not for any other axis.

Therefore the diagonal $\overline{OC}$ of the parallelogram A B represents the resultant, and the only resultant, of the forces represented by $\overline{OA}$ and $\overline{OB}$.—Q. E. D.

Second Demonstration.—Suppose a perpendicular to be erected to the plane O A B at the point O, of any length whatsoever; call the other extremity of that perpendicular R; and at R conceive two forces to be applied, respectively equal, parallel, and opposite to $\overline{OA}$ and $\overline{OB}$. Then O R is the arm common to two couples whose axes and moments are represented (in the manner described in Art. 34) by lines perpendicular and proportional respectively to O A and O B. On the lines so representing the couples, construct a parallelogram; then, as shown in Art. 35, the diagonal of that parallelogram represents the resultant couple constituted by the resultant of $\overline{OA}$ and $\overline{OB}$ acting at O, and an equal and opposite force at R; and as the parallelogram of couples has its sides perpendicular and proportional to $\overline{OA}$ and $\overline{OB}$, its diagonal must be perpendicular and proportional to $\overline{OC}$, which consequently represents the resultant of $\overline{OA}$ and $\overline{OB}$.—Q. E. D.

[There are various other modes of demonstrating the theorem of the parallelogram of forces, all of which may be studied with advantage: especially those given by Dr. Whewell in his *Elementary Treatise on Mechanics*, and by Mr. Moseley in his *Mechanics of Engineering and Architecture.*]

52. **Equilibrium of Three Forces acting through One Point in One Plane.**—To balance the forces $\overline{OA}$ and $\overline{OB}$, a force is required equal and directly opposed to their resultant $\overline{OC}$. This may be otherwise expressed by saying, that if the directions and magnitudes of three forces be represented by the three sides of a triangle, (such as $\overline{OA}$, $\overline{AC}$, $\overline{CO}$), then those three forces, acting through one point, balance each other.

53. **Equilibrium of any System of Forces acting through One Point.**—COROLLARY. *If a number of forces acting through the same point be represented by lines equal and parallel to the sides of a closed polygon, those forces balance each other.* To fix the ideas, let there be five forces acting through the point O (fig. 16), and represented in direction and magnitude by the lines F_1, F_2, F_3, F_4, F_5, which are equal and parallel to the sides of the closed polygon O A B C D O · viz. :—

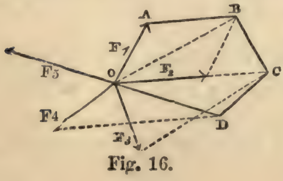


Fig. 16.

$$F_1 = \text{and} \parallel OA;$$
$$F_2 = \text{and} \parallel AB;$$
$$F_3 = \text{and} \parallel BC;$$
$$F_4 = \text{and} \parallel CD;$$
$$F_5 = \text{and} \parallel DO.$$

Then by the theorem of Art. 52, the resultant of F_1 and F_2 is O B; the resultant of F_1, F , and F_3 is O C; the resultant of F_1, F_2, F_3, and F_4 is O D, equal and opposite to F_5, so that the final resultant is nothing.

The closed polygon may be either plane or gauche.

54. **Parallelopiped of Forces.**—The simplest gauche polygon is one of four sides. Let O A B C E F G H (fig. 17), be a parallelopiped whose diagonal is O H. Then any three successive edges so placed as to begin at O and end at H, form, together with the diagonal H O, a closed quadrilateral; consequently if three forces F_1, F_2, F_3, acting through O, be represented by the three edges $\overline{OA}$, $\overline{OB}$, $\overline{OC}$, of a parallelopiped, the diagonal $\overline{OH}$ represents their resultant, and a fourth force F_4 equal and opposite to $\overline{OH}$ balances them.

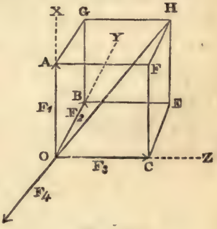


Fig. 17.

55. **Resolution of a Force into Two Components.**—From the theorem of Art. 51, it is evident that in order that a given single force may be resolvable into two components acting in given lines inclined to each other, it is necessary, *first*, that the lines of action of those components should intersect the line of action of the given force in one point; and *secondly*, that those three lines of action should be in one plane.

Returning, then, to fig. 15, let $\overline{OC}$ represent the given force, which it is required to resolve into two component forces, acting in the lines O X, O Y, which lie in one plane with O C, and intersect it in one point O.

Through C draw C A $\parallel$ O Y, cutting O X in A, and C B $\parallel$ O X, cutting O Y in B. Then will O A and O B represent the component forces required.

Two forces respectively equal to and directly opposed to $\overline{OA}$ and $\overline{OB}$ will balance $\overline{OC}$.

56. **Resolution of a Force into Three Components.**—In order that a given single force may be resolvable into three components acting in given lines inclined to each other, it is only necessary that the lines of action of the components should intersect the line of action of the given force in one point.

Returning to fig. 17, let $\overline{\mathrm{O\,H}}$ represent the given force which it is required to resolve into three component forces, acting in the lines O X, O Y, O Z, which intersect O H in one point O.

Through H draw three planes parallel respectively to the planes Y O Z, Z O X, X O Y, and cutting respectively O X in A, O Y in B, O Z in C. Then will $\overline{\mathrm{O\,A}}$, $\overline{\mathrm{O\,B}}$, $\overline{\mathrm{O\,C}}$, represent the component forces required.

Three forces respectively equal to, and directly opposed to $\overline{\mathrm{O\,A}}$, $\overline{\mathrm{O\,B}}$, and $\overline{\mathrm{O\,C}}$, will balance $\overline{\mathrm{O\,H}}$.

57. **Rectangular Components.**—The rectangular components of a force are those into which it is resolved when the directions of their lines of action are at right angles to each other.

For example, in fig. 17, suppose O X, O Y, O Z, to be three axes of co-ordinates at right angles to each other. Then $\overline{\mathrm{O\,H}}$ is resolved into three rectangular components simply by letting fall from H perpendiculars on O X, O Y, O Z, cutting them at A, B, C, respectively.

To express this case algebraically, let $\mathrm{F} = \overline{\mathrm{O\,H}}$ denote the force to be resolved. Let

$$\alpha = \angle \mathrm{X\,O\,H},\ \beta = \angle \mathrm{Y\,O\,H},\ \gamma = \angle \mathrm{Z\,O\,H},$$

be the angles which its line of action makes with the three rectangular axes. Then, as is well known, those three angles are connected by the equation

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1, \ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

Let

$$\mathrm{F}_1 = \overline{\mathrm{O\,A}},\ \mathrm{F}_2 = \overline{\mathrm{O\,B}},\ \mathrm{F}_3 = \overline{\mathrm{O\,C}},$$

be the three rectangular components of F; then

$$\left.\begin{array}{l} \mathrm{F}_1 = \mathrm{F} \cdot \cos \alpha \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \mathrm{F}_2 = \mathrm{F} \cdot \cos \beta \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \mathrm{F}_3 = \mathrm{F} \cdot \cos \gamma \ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\} (2.)$$

In order to distinguish properly the direction of the resultant F as compared with the directions of the axes, it is to be borne in mind that

$$\text{the cosine of an} \left\{ \begin{array}{c} \text{acute} \\ \text{obtuse} \end{array} \right\} \text{angle is} \left\{ \begin{array}{c} \text{positive.} \\ \text{negative.} \end{array} \right\}$$

From a well known property of right-angled triangles (also embodied in equation 1), it follows that

$$\mathrm{F}^2 = \mathrm{F}_1^2 + \mathrm{F}_2^2 + \mathrm{F}_3^2. \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.)$$

To express algebraically the case in which a force is resolved into two rectangular components in one plane with it, let the plane in question be that of O X and O Y. Then the angles are subject to the following equations :—

$$\gamma = \text{a right angle}; \ \alpha + \beta = \text{a right angle};$$

$$\cos\gamma = 0\ ;\ \cos\beta = \sin\alpha;\ \cos\alpha = \sin\beta.\dots\dots\dots(4.)$$

and consequently the equations 2 and 3 are reduced to the following :—

$$\left.\begin{array}{l} F_1 = F\cdot\cos\alpha = F\cdot\sin\beta\,;\dots\dots\dots\dots\dots \\ F_2 = F\cdot\sin\alpha = F\cdot\cos\beta\,;\dots\dots\dots\dots\dots \\ F_3 = 0; \qquad F^2 = F_1^2 + F_2^2.\dots\dots\dots\dots\dots \end{array}\right\}(5.)$$

In using these equations, the rule respecting the positive and negative signs of cosines is to be observed ; and it is also to be borne in mind, that the angle α is reckoned from O X in the direction towards Y, and the angle β from O Y in the reverse direction, that is, towards X, and that

$$\text{the sines of angles from} \left\{\begin{array}{c} 0^\circ \text{ to } 180^\circ \\ 180^\circ \text{ to } 360^\circ \end{array}\right\} \text{are} \left\{\begin{array}{c} \text{positive.} \\ \text{negative.} \end{array}\right\}$$

If a system of forces acting through one point balance each other, their resultant is nothing ; and therefore the rectangular components of their resultant, which are the resultants of their parallel systems of rectangular components, are each equal to nothing ; a case represented as follows :—

$$\Sigma\cdot F_1 = 0\ ;\ \Sigma\cdot F_2 = 0\ ;\ \Sigma\cdot F_3 = 0.\dots\dots\dots\dots(6.)$$

SECTION 2.—*Inclined Forces Applied to a System of Points.*

58. **Forces acting in One Plane.—Graphic Solution.** — Let any system of forces whose lines of action are in one plane, act together on a rigid body, and let it be required to find their resultant.

Assume an axis perpendicular to the plane of action of the forces at any point, and let it be called O Z. According to the principle of Art. 42, let each force be resolved into an equal and parallel force acting through O, and a couple tending to produce rotation about O Z ; so that if a force F be applied along a line whose perpendicular distance from O is L, that force shall be resolved into

$$F' = \text{and} \parallel F$$

acting through O, and a couple whose moment is

$$M = L F,$$

and which is right or left-handed according as O lies to the right or left of the direction of F.

The *magnitude* and *direction* of the resultant are to be found by forming a polygon with lines equal and parallel to those representing the forces, as in Art. 53, when, if the polygon is closed, the forces have no single resultant; but if not, then the resultant is equal, parallel, and opposite to that represented by the line which is required in order to close the polygon. Let R be its magnitude if any.

The *position* of the line of action of the resultant is found as follows:—

Let $\Sigma \cdot M$ be the resultant of the moments of all the couples M, distinguishing right-handed from left-handed, as in Arts. 27 and 32. If $\Sigma \cdot M = 0$, and also $R = 0$, then the couples and forces balance completely, and there is no resultant. If $\Sigma \cdot M = 0$, while R has magnitude, then the resultant acts through O. If $\Sigma \cdot M$ and R both have magnitude, then the line of action of the resultant R is at the perpendicular distance from O given by the equation

$$L_r = \frac{\Sigma \cdot M}{R},$$

and the direction of that perpendicular is indicated by the sign of $\Sigma \cdot M$. If $R = 0$, while $\Sigma \cdot M$ has magnitude, the only resultant of the given system of forces is the couple $\Sigma \cdot M$.

59. **Forces acting in One Plane.—Solution by Rectangular Co-ordinates.**—Through the point O as origin of co-ordinates, let any two axes be assumed, O X and O Y, perpendicular to each other and to O Z, and in the plane of action of the forces; and in looking from Z towards O, let Y lie to the right of X, so that rotation from X towards Y shall be right-handed. Let F, as before, denote any one of the forces; let α be the angle which its line of action makes to the right of O X; and let x and y be the co-ordinates of its point of application, or of any point in its line of action, relatively to the assumed origin and axes. Resolve each force F into its rectangular components as in Art. 57,

$$F_1 = F \cdot \cos \alpha; \ F_2 = F \cdot \sin \alpha;$$

then the rectangular components of the resultant are respectively

$$\left.\begin{array}{l} \text{parallel to O X, } \Sigma (F \cos \alpha) = R_1, \\ \text{parallel to O Y, } \Sigma (F \cdot \sin \alpha) = R_2, \end{array}\right\} \ldots\ldots\ldots\ldots (1.)$$

its magnitude is given by the equation

$$R^2 = R_1^2 + R_2^2; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.)$$

and the angle α_r which it makes to the right of O X is found by the equations

$$\cos \alpha_r = \frac{R_1}{R}; \ \sin \alpha_r = \frac{R_2}{R} \ldots\ldots\ldots\ldots\ldots\ldots (3.)$$

The quadrant in which the direction of R lies is indicated by the algebraical signs of R_1 and R_2, as already stated in Art. 57.

The perpendicular distance from O of the line of action of any force F is

$$L = x \cdot \sin \alpha - y \cdot \cos \alpha$$

which is positive or negative according as O lies to the right or to the left of that line of action; and hence the resultant moment of the system of forces relatively to the axis O Z is

$$\begin{aligned} \Sigma \cdot F L &= \Sigma \cdot F \, (x \sin \alpha - y \cos \alpha) \\ &= \Sigma \, (x F_2 - y F_1) \text{.......................(4.)} \end{aligned}$$

whence it follows, that the perpendicular distance of the resultant force from O is

$$L_r = \frac{\Sigma (x F_2 - y F_1)}{R} \text{....................(5.)}$$

Let x_r and y_r be the co-ordinates of any point in the line of action of the resultant; then the equation of that line is,

$$\left. \begin{array}{l} x_r R_2 - y_r R_1 = R L_r \\ \text{which is equivalent to} \\ x_r \sin \alpha_r - y_r \cos \alpha_r = L_r \end{array} \right\} \text{...............(6.)}$$

As in Art. 58, if $\Sigma \cdot F L = 0$, the resultant acts through the origin O; if $\Sigma \cdot F L$ has magnitude, and $R = 0$ (in which case $R_1 = 0$, $R_2 = 0$) the resultant is a couple. The conditions of equilibrium of the system of forces are

$$\left. \begin{array}{l} R_1 = 0; \; R_2 = 0; \; \Sigma \cdot F L = 0; \\ \text{or in other symbols} \\ \Sigma \cdot F_1 = 0; \; \Sigma \cdot F_2 = 0; \; \Sigma (x F_2 - y F_1) = 0. \end{array} \right\} \text{....(7.)}$$

The moment of the resultant relatively to the axis O Z can also be arrived at by considering the moment F L of each force as the resultant of $x F_2$, which is right-handed when x and F_2 are both positive, and of $- y F_1$, which is left-handed when y and F_1 are both positive.

60. **Any System of Forces.**—To find the resultant and the conditions of equilibrium of any system of forces acting through any system of points, the forces and points are to be referred to three rectangular axes of co-ordinates.

As in Art. 57, let O denote the origin of co-ordinates, and O X, O Y, O Z, the three rectangular axes; and let them be arranged (as in fig. 17), so that in looking from

$$\left. \begin{array}{c} X \\ Y \\ Z \end{array} \right\} \text{ towards O, rotation from } \left\{ \begin{array}{l} Y \text{ towards } Z \\ Z \text{ towards } X \\ X \text{ towards } Y \end{array} \right\}$$

shall appear right-handed.

Let F denote any one of the forces; x, y, z, the co-ordinates of a point in its line of action; and α, β, γ, the angles which its direction makes with the axes respectively. Then the three rectangular components of F being as in Art. 57,

$$\left.\begin{array}{l} F_1 = F \cdot \cos\alpha \text{ along } OX, \\ F_2 = F \cdot \cos\beta \text{ along } OY, \\ F_3 = F \cdot \cos\gamma \text{ along } OZ, \end{array}\right\} \dots\dots\dots\dots\dots\dots (1.)$$

it can be shown by reasoning similar to that of Art. 59, that the total moments of these components relatively to the three axes are respectively

$$\left.\begin{array}{l} y F_3 - z F_2 = F \cdot (y \cos\gamma - z \cos\beta) \text{ relatively to } OX, \\ z F_1 - x F_3 = F \ (z \cos\alpha - x \cos\gamma) \text{ relatively to } OY, \\ x F_2 - y F_1 = F \ (x \cos\beta - y \cos\alpha) \text{ relatively to } OZ; \end{array}\right\} (2.)$$

so that the force F is equivalent to the three forces of the formulæ 1 acting through O along the three axes, and the three couples of the formulæ 2 acting round the three axes.

Taking the algebraical sums of all the forces which act along the same axes, and of all the couples which act round the same axes, the six following quantities are found, which compose the resultant of the given system of forces;—

Forces.

$$\left.\begin{array}{l} \text{along } OX;\ R_1 = \Sigma \cdot F \cos\alpha, \\ \quad ,,\ \ OY;\ R_2 = \Sigma \cdot F \cos\beta, \\ \quad ,,\ \ OZ;\ R_3 = \Sigma \cdot F \cos\gamma, \end{array}\right\} \dots\dots\dots\dots (3.)$$

Couples.

$$\left.\begin{array}{l} \text{round } OX;\ M_1 = \Sigma\{F\ (y \cos\gamma - z \cos\beta)\}, \\ \quad ,,\ \ OY;\ M_2 = \Sigma\{F\ (z \cos\alpha - x \cos\gamma)\}, \\ \quad ,,\ \ OZ;\ M_3 = \Sigma\{F\ (x \cos\beta - y \cos\alpha)\}, \end{array}\right\} \dots\dots (4.)$$

The three forces R_1, R_2, R_3, are equivalent to a single force

$$R = \sqrt{(R_1^2 + R_2^2 + R_3^2)}, \dots\dots\dots\dots\dots\dots (5.)$$

acting through O in a line which makes with the axes the angles given by the equations

$$\cos\alpha_r = \frac{R_1}{R};\ \cos\beta_r = \frac{R_2}{R};\ \cos\ \gamma_r = \frac{R_3}{R} \dots\dots\dots\dots (6.)$$

The three couples M_1, M_2, M_3, according to Article 37, are equivalent to one couple, whose magnitude is given by the equation

$$M = \sqrt{(M_1^2 + M_2^2 + M_3^2)} \dots\dots\dots\dots\dots\dots (7.)$$

and whose axis makes with the axes of co-ordinates the angles given by the equations

$$\cos\lambda = \frac{M_1}{M}; \quad \cos\mu = \frac{M_2}{M}; \quad \cos\nu = \frac{M_3}{M} \dots\dots\dots\dots\dots (8.)$$

in which $\left\{\begin{matrix}\lambda\\ \mu\\ \nu\end{matrix}\right\}$ denote respectively the angles made by the axis of M with $\left\{\begin{matrix}OX\\ OY\\ OZ\end{matrix}\right\}$.

The **Conditions of Equilibrium** of the system of forces may be expressed in either of the two following forms :—

$$R_1 = 0; \; R_2 = 0; \; R_3 = 0: \; M_1 = 0; \; M_2 = 0; \; M_3 = 0 \dots (9.)$$

or

$$R = 0; \; M = 0 \dots\dots\dots\dots\dots\dots\dots\dots (10.)$$

When the system is not balanced, its resultant may fall under one or other of the following cases :—

Case I.—*When* M = 0, the resultant is the single force R acting through O.

Case II.—*When the axis of* M *is at right angles to the direction of* R,—a case expressed by either of the two following equations :—

$$\left.\begin{matrix}\cos\alpha_r \cos\lambda + \cos\beta_r \cos\mu + \cos\gamma_r \cos\nu = 0;\\ \text{or} \quad R_1 M_1 + R_2 M_2 + R_3 M_3 = 0;\end{matrix}\right\} \dots (11.)$$

the resultant of M and R is a single force equal and parallel to R, acting in a plane perpendicular to the axis of M, and at a perpendicular distance from O given by the equation

$$L = \frac{M}{R} \dots\dots\dots\dots\dots\dots\dots\dots\dots (12).$$

Case III.—*When* R = 0, there is no single resultant; and the only resultant is the couple M.

Case IV.—*When the axis of* M *is parallel to the line of action of* R, that is, when either

$$\lambda = \alpha_r; \; \mu = \beta_r; \; \nu = \gamma_r, \dots\dots\dots\dots\dots\dots (13).$$

or

$$\lambda = -\alpha_r; \; \mu = -\beta_r; \; \nu = -\gamma_r; \dots\dots\dots\dots (14).$$

there is no single resultant; and the system of forces is equivalent to the force R and the couple M, being incapable of being farther simplified.

Case V.—*When the axis of* M *is oblique to the direction of* R, making with it the angle given by the equation

$$\cos\theta = \cos\lambda \cos\alpha_r + \cos\mu \cos\beta_r + \cos\nu \cos\gamma_r, \dots (15).$$

the couple M is to be resolved into two rectangular components, viz. :—

$$\left.\begin{array}{l}\text{M}\sin\theta\text{ round an axis perpendicular to R, and in}\\ \quad\text{the plane containing the direction of R and of}\\ \quad\text{the axis of M;}\\ \qquad\qquad\text{M}\cos\theta\text{ round an axis parallel to R.}\end{array}\right\}\quad(16.)$$

The force R and the couple M sin θ are equivalent, as in Case II., to a single force equal and parallel to R, whose line of action is in a plane perpendicular to that containing R and the axis of M, and whose perpendicular distance from O is

$$L = \frac{M \sin \theta}{R} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (17.)$$

The couple M cos θ, whose axis is parallel to the line of action of R, is incapable of further combination.

Hence it appears finally, that every system of forces which is not self-balanced, is equivalent either, (A); to a single force, as in Cases I. and II. (B); to a couple, as in Case III. (C); to a force, combined with a couple whose axis is parallel to the line of action of the force, as in Cases IV. and V. This can occur with inclined forces only, it having been shown in Article 47, that the resultant of any number of parallel forces is either a single force or a couple.

CHAPTER IV.

ON PARALLEL PROJECTIONS IN STATICS.

61. **Parallel Projection of a Figure defined.**—If two figures be so related, that for each point in one there is a corresponding point in the other, and that to each pair of equal and parallel lines in the one there corresponds a pair of equal and parallel lines in the other, those figures are said to be PARALLEL PROJECTIONS of each other.

The relation between such a pair of figures may be otherwise expressed as follows:—Let any figure be referred to axes of co-ordinates, whether rectangular or oblique; let x, y, z, denote the co-ordinates of any point in it, which may be denoted by A: let a second figure be constructed from a second set of axes of co-ordinates, either agreeing with, or differing from, the first set as to rectangularity or obliquity; let x', y', z', be the co-ordinates in the second figure, of the point A′ which corresponds to any point A in the first figure, and let those co-ordinates be so related to the co-ordinates of A, that for each pair of corresponding points, A, A′, in the two figures, the three pairs of corresponding co-ordinates shall bear to each other three constant ratios, such as

$$\frac{x'}{x} = a; \ \frac{y'}{y} = b; \ \frac{z'}{z} = c;$$

then are these two figures parallel projections of each other.

62. **Geometrical Properties of Parallel Projections.**—The following are the geometrical properties of parallel projections which are of most importance in statics. Being purely geometrical propositions, they are not here demonstrated.

I.—A parallel projection of a system of three points, lying in one straight line and dividing it in a given proportion, is also a system of three points, lying in one straight line and dividing it in the same proportion.

II.—A parallel projection of a system of parallel lines whose lengths bear given ratios to each other, is also a system of parallel lines whose lengths bear the same ratios to each other.

III.—A parallel projection of a closed polygon is a closed polygon.

IV.—A parallel projection of a parallelogram is a parallelogram.

V.—A parallel projection of a parallelopiped is a parallelopiped.

VI.—A parallel projection of a pair of parallel plane surfaces, whose areas are in a given ratio, is also a pair of parallel plane surfaces, whose areas are in the same ratio.

VII.—A parallel projection of a pair of volumes having a given ratio, is a pair of volumes having the same ratio.

63. **Application to Parallel Forces.**—It has been shown in Chap. II., Sect. 3, that the equilibrium of any system of parallel forces depends on the mutual proportions of the forces and on those of the distances of their lines of action from given planes. By considering this in connection with the principles I. and II. of Article 62, it is evident, that if a balanced system of parallel forces be represented by a system of lines, then any system of lines which is a parallel projection of the first system, will also represent a balanced system of parallel forces ; and also, that if there be two systems of parallel forces represented by systems of lines which are parallel projections of each other, then are the respective resultants of those systems of forces, whether single forces or couples, represented by lines which are parallel projections of each other related in the same manner with the other pairs of corresponding lines in the two systems. In applying this principle to *couples*, it is to be observed, that they are *not* to be represented by single lines, as in Art. 34, but by pairs of equal and opposite lines, as in the previous articles, or by areas, as in Articles 42 and 51.

64. **Application to Centres of Parallel Forces.**—If two systems of points be parallel projections of each other ; and if to each of those systems there be applied a system of parallel forces bearing to each other the same system of ratios, then, by considering the principles I. and II. of Article 62 in conjunction with those of Chap. II., Sect. 4, it is evident that the centres of parallel forces for those two systems of points will be parallel projections of each other, mutually related in the same manner with the other pairs of corresponding points in the two systems.

65. **Application to Inclined Forces acting through One Point.**—From principles III., IV., and V., of Article 62, taken in conjunction with the principles of Chap. III., Sect. 1, it follows, that if a given system of lines represents a balanced system of forces acting through one point, then will any parallel projection of that system of lines also represent a balanced system of forces acting through one point ; and also, that if two systems of forces, each acting through one point, be represented by two systems of lines which are parallel projections of each other, then will the respective resultants of those two systems of forces be represented by a pair of lines which are

parallel projections of each other, mutually related in the same manner with other pairs of corresponding lines.

66. **Application to any System of Forces.**—As every system of forces applied to any system of points can be reduced, as in Art. 60, to a system of forces acting through one point, and certain systems of parallel forces, it follows that if a balanced system of forces acting through any system of points be represented by a system of lines, then will any parallel projection of that system of lines represent a balanced system of forces; and that if any two systems of forces be represented by lines which are parallel projections of each other, the lines, or sets of lines, representing their resultants, will be corresponding parallel projections of each other:—it being still observed, as in Article 63, that couples are to be represented by pairs of lines, as pairs of opposite forces, or by areas, and not by single lines along their axes.

CHAPTER V.

ON DISTRIBUTED FORCES.

67. **Restriction of the Subject.**—In Article 18 it has already been explained, that the action of every real force is distributed throughout some volume, or over some surface. It is always possible, however, to find either a *single resultant*, or a *resultant couple*, or a *combination of a single force with a couple* (like that described in Art. 60), to which a given distributed force is equivalent, so far as it affects the equilibrium of the body, or part of a body, to which it is applied.

In the application of Mechanics to Astronomy, Electricity, and Magnetism, it is often necessary to find the resultant of a distributed attraction or repulsion, whose direction is sensibly different at different points of the body to which it is applied; and problems thus arise of great difficulty and complexity. But in the application of Mechanics to Structures and Machines, the only force distributed throughout the volume of a body which it is necessary to consider, is its *weight*, or attraction towards the earth; and the bodies considered are in every instance so small as compared with the earth, that this attraction may, without appreciable error, be held to act in parallel directions at each point in each body. Moreover, the forces distributed over surfaces, which have to be considered in applied mechanics, are either parallel at each point of their surfaces of application, or capable of being resolved into sets of parallel forces. Hence, in applied mechanics, *parallel distributed forces* have alone to be considered; every such force is statically equivalent either to a single resultant, or to a resultant couple; and the problem of finding such resultant is comparatively simple.

68. **The Intensity of a Distributed Force** is the ratio which the magnitude of that force, expressed in units of force, bears to the space over which it is distributed, expressed in units of volume, or in units of surface, as the case may be. An *unit of Intensity* is an unit of force distributed over an unit of volume or of surface, as the case may be; so that there are two kinds of units of intensity. For example, *one pound per cubic foot* is an unit of intensity for a force distributed throughout a volume, such as weight; and *one*

pound per square foot is an unit of intensity for a force distributed over a surface, such as pressure or friction.

The intensity of a force acting at a single point would be infinite, if such a force were possible.

SECTION 1.—*Of Weight, and Centres of Gravity.*

69. The **Specific Gravity** of a body is a number proportional to the weight of an unit of its volume; for example, the weight in pounds, of a cubic foot of the volume of the body. The *pound per cubic foot* is the most convenient unit of specific gravity for practical purposes; but in tables of specific gravity, a special unit is usually employed, viz., the weight, at a fixed temperature, of unity of volume of water. In Britain, that fixed temperature is usually 62° Fahrenheit; in France, and on the continent of Europe generally, it is the temperature at which water is most dense, viz., 3°·95 centigrade, or 39°·1 Fahrenheit.

In a table at the end of this volume are given the specific gravities of such materials as most commonly occur in structures and machines. So far as this and similar tables relate to solid materials, they must be regarded as approximate only; for the specific gravity of the same solid substance varies not only in different specimens, but frequently even in different parts of the same specimen; still the approximate values are sufficiently near the truth for practical purposes in the art of construction.

70. The **Centre of Gravity** of a body, or of a system of bodies, is the point always traversed by the resultant of the weight of the body or system of bodies,—in other words, the *centre of parallel forces* for the weight of the body or system of bodies.

To *support* a body, that is, to balance its weight, the resultant of the supporting force must act through the centre of gravity.

71. **Centre of Gravity of a Homogeneous Body having a Centre of Figure.**—Let a body be *homogeneous*, or of equal specific gravity throughout; let it also be so far *symmetrical*, as to have a *centre of figure;* that is, a point within the body, which bisects every diameter of the body drawn through it; then it is self-evident, that the centre of figure of the body must also be its centre of gravity.

Amongst the bodies which answer this description are, the sphere, the ellipsoid, the circular cylinder, the elliptic cylinder, prisms whose bases have centres of figure, and parallelopipeds, whether right or oblique.

16 72. **Bodies having Planes or Axes of Symmetry.**—If a homogeneous body be of a figure which is symmetrical on either side of a given plane, the centre of gravity must be in that plane. If two or more such *planes of symmetry* intersect in one line, or *axis of*

symmetry, the centre of gravity must be in that axis. If three or more planes of symmetry intersect each other in a point, that point must be the centre of gravity.

The following are examples :—

I. In fig. 18, let A B C be an equilateral triangle, the base of a *right equilateral triangular prism.* This prism has one plane of symmetry parallel to its bases at the middle of its length. It has also three planes of symmetry, A a, B b, C c, each traversing one edge of the prism and bisecting the opposite side, and those three planes intersect in an axis G, whose perpendicular distance from any edge is two-thirds of the distance from that edge to the opposite side, that is,

$$\frac{\overline{GA}}{\overline{Aa}} = \frac{\overline{GB}}{\overline{Bb}} = \frac{\overline{GC}}{\overline{Cc}} = \frac{2}{3}.$$

The centre of gravity of the prism is at the middle of this axis.

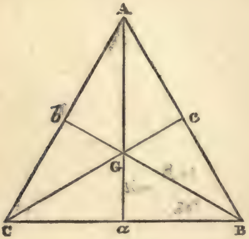


Fig. 18.

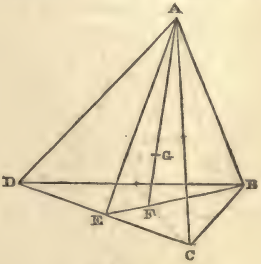


Fig. 19.

II. In fig. 19, let A B C D be a *regular tetraedron*, or triangular pyramid, bounded by four equilateral triangles. Bisect any edge D C in E; then the plane A B E drawn through the point of bisection and the opposite edge is a plane of symmetry. There are six such planes, and they intersect each other in one point G, which is therefore the centre of gravity of the tetraedron.

It may be shown by geometry, that the point G can be found in the following manner. From any summit, such as B, draw B E, bisecting one of the opposite edges, such as D C. In B E take $\overline{BF} = \frac{2}{3} \overline{BE}$. Join A F, in which take $\overline{AG} = \frac{3}{4} \overline{AF}$; then is G the centre of gravity sought.

73. **System of Symmetrical Bodies.**—Let a connected system of bodies whose absolute or proportional weights are known, and whose centres of gravity are also known by reason of the symmetry

and homogeneity of each body, be arranged in any manner; then the *common centre of gravity* of the whole system of bodies is the same with the *centre of parallel forces* for a system of forces equal or proportional to the weights of the bodies, and acting through their respective centres of gravity.

Consequently, applying to this case the principles of Chap. II., Section 4, Article 50, the centre of gravity is found in the following manner. Let $\overline{yz}$ denote any fixed plane, x the perpendicular distance of the centre of gravity of any one of the bodies from that plane, and W the weight of that body, so that Wx is the moment of the weight of the body in question with respect to any axis in the plane $\overline{yz}$.

Let x_0 denote the perpendicular distance of the common centre of gravity from the plane $\overline{yz}$. Then we have, total moment of the system relatively to any axis in the plane $\overline{yz}$,

$$x_0 \cdot \Sigma \cdot W = \Sigma \cdot W x;$$

and consequently,

$$x_0 = \frac{\Sigma \cdot W x}{\Sigma \cdot W}.$$

By proceeding in a similar manner, the distances of the common centre of gravity of the system of bodies from two other fixed planes, either perpendicular or oblique to $\overline{yz}$ and to each other, are found so as to determine its position completely.

The same process is applicable to any body whose figure is capable of being divided into symmetrical figures.

74. **Homogeneous Body of any Figure.**—Let w be the specific gravity of a homogeneous body of any figure, V its volume, and $W = wV$ its weight. Conceive three fixed co-ordinate planes, $\overline{yz}$, $\overline{zx}$, and $\overline{xy}$, perpendicular to each other, and let x_0, y_0, z_0, be the co-ordinates of the centre of gravity, which it is required to find; so that wVx_0, wVy_0, wVz_0, are the moments of the body relatively to the three co-ordinate planes respectively. Conceive the space in and near the body to be divided by three series of equidistant planes parallel to the co-ordinate planes respectively, into equal and similar small rectangular molecules, whose dimensions, parallel to x, y, and z, respectively, are

$$\Delta x, \Delta y, \Delta z.$$

Let x, y, z, be the co-ordinates of the centre of one of these molecules. Then its volume is

$$\Delta x \Delta y \Delta z;$$

its weight

$$w \Delta x \Delta y \Delta z,$$

and its moments relatively to the three co-ordinate planes respectively,

$$x w \,\Delta x \,\Delta y \,\Delta z; \; y w \,\Delta x \,\Delta y \,\Delta z; \; z w \,\Delta x \,\Delta y \,\Delta z.$$

Whatsoever may be the figure of the body whose centre of gravity is sought, a figure *approximating* to it may be *built* by putting together a proper number of suitably arranged rectangular molecules; so that

$$\left.\begin{array}{c} \mathrm{V} = \Sigma \cdot \Delta x \,\Delta y \,\Delta z \textit{ nearly}; \\ \mathrm{W} = w \mathrm{V} = w \cdot \Sigma \cdot \Delta x \,\Delta y \,\Delta z \textit{ nearly}; \\ w \mathrm{V} x_0 = w \cdot \Sigma \cdot x \,\Delta x \,\Delta y \,\Delta z \textit{ nearly}; \\ \text{therefore omitting the common and constant factor } w, \\ x_0 = \dfrac{\Sigma \cdot x \,\Delta x \,\Delta y \,\Delta z}{\Sigma \cdot \Delta x \,\Delta y \,\Delta z} \textit{ nearly}; \\ \text{and similar approximate formulæ for } y_0 \text{ and } z_0. \end{array}\right\} \ldots\ldots(1.)$$

Now, it is evident, that the smaller the dimensions Δx, Δy, Δz, of each rectangular molecule,—or in other words, the more minute the subdivision of the space in and near the body into small rectangles, the more nearly will the approximate figure, built up of rectangular molecules, agree with the exact figure of the body, and, consequently, the more nearly will the results of the approximate formulæ (1.) agree with the true results; which, therefore, are the *limits* towards which the results of these formulæ continually approach nearer and nearer, as the dimensions Δx, Δy, Δz, are diminished. Such limits are found by the process called *integration*,* and are expressed in the following manner:—

$$\left.\begin{array}{ll} \text{volume} & \mathrm{V} = \iiint dx\,dy\,dz; \\ \text{weight} & \mathrm{W} = w\mathrm{V} = w \iiint dx\,dy\,dz; \end{array}\right\} \ldots\ldots(2.)$$

$$\text{moments} \left\{ \begin{array}{l} \mathrm{W} x_0 = w \iiint x\,dx\,dy\,dz; \\ \mathrm{W} y_0 = w \iiint y\,dx\,dy\,dz; \\ \mathrm{W} z_0 = w \iiint z\,dx\,dy\,dz; \end{array} \right\} \ldots\ldots(3.)$$

---

* For further elucidation of the meaning of symbols of integration, and for explanations of processes of approximately computing the values of integrals, see Art. 81 i the sequel.

$$\begin{array}{c}\text{co-ordinates}\\\text{of the}\\\text{centre of}\\\text{gravity}\end{array}\left\{\begin{aligned} x_0 &= \frac{\iiint x\,dx\,dy\,dz}{\iiint dx\,dy\,dz}; \\ y_0 &= \frac{\iiint y\,dx\,dy\,dz}{\iiint dx\,dy\,dz}; \\ z_0 &= \frac{\iiint z\,dx\,dy\,dz}{\iiint dx\,dy\,dz}; \end{aligned}\right\}\text{.........(4.)}$$

Such are the general formulæ for finding the centre of gravity of a homogeneous body, of any form whatsoever.

75. **Centre of Gravity found by Addition.**—When the figure of a body consists of parts, whose respective centres of gravity are known, the centre of gravity of the whole is to be found as in Article 73.

76. **Centre of Gravity found by Subtraction.**—When the figure of a homogeneous body, whose centre of gravity is sought, can be made by taking away a figure whose centre of gravity is known from a larger figure whose centre of gravity is known also, the following method may be used.

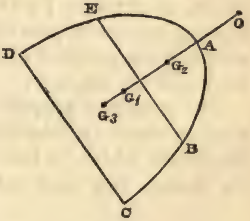


Fig. 20.

Let A C D be the larger figure, G_1 its known centre of gravity, W_1 its weight. Let A B E be the smaller figure, whose centre of gravity G_2 is known, W_2 its weight. Let E B C D be the figure whose centre of gravity G_3 is sought, made by taking away A B E from A C D, so that its weight is

$$W_3 = W_1 - W_2.$$

Join $G_1 G_2$; G_3 will be in the prolongation of that straight line beyond G_1. In the same straight line produced, take any point O as origin of co-ordinates, and an axis at O perpendicular to $O G_2 G_1$ as axis of moments. Make $\overline{O G_1} = x_1$; $\overline{O G_2} = x_2$, $\overline{O G_3}$ (the unknown quantity) $= x_3$.

Then the moment of W_3 relatively to the axis at O is

$$x_3 W_3 = x_1 W_1 - x_2 W_2,$$

and therefore

$$x_3 = \frac{x_1 W_1 - x_2 W_2}{W_1 - W_2}.$$

77. **Centre of Gravity Altered by Transposition.**—In fig. 21, let A B C D be a body of the weight W_0, whose centre of gravity G_0 is known. Let the figure of this body be altered, by transposing a part whose weight is W_1, from the position E C F to the position F D H, so that the new figure of the body is A B H E. Let G_1 be the original, and G_2 the new position of the centre of gravity of the transposed part. Then the moment of the body relatively to any axis in a plane perpendicular to $G_1 G_2$ will be altered by the amount $W_1 \cdot \overline{G_1 G_2}$; and the centre of gravity of the whole body will be shifted to G_3, in a direction G_0 G_3 parallel to G_1 G_2, and through a distance given by the formula

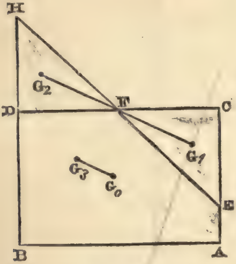


Fig. 21.

$$\overline{G_0 G_3} = \overline{G_1 G_2} \frac{W_1}{W_0}.$$

78. **Centres of Gravity of Prisms and Flat Plates.**—The general formulæ of Article 74 are intended not so much for direct use in finding centres of gravity, as for the deduction of formulæ of a more simple form adapted to particular classes of cases. Of such the following is an example.

The centre of gravity of a right prism with parallel ends lies in a plane midway between its ends; that of a flat plate of uniform thickness, which in fact is a short prism, in a plane midway between its faces. Let such middle plane be taken for that of xy; any point in it O (fig. 22), for the origin, and two rectangular axes in it, O X and O Y, for axes of co-ordinates, to which A B, the transverse section of the plate, is referred. Conceive the figure A B to be divided into narrow bands, by equi-distant lines parallel to one of the axes of co-ordinates O Y, and at the distance Δx apart. Let x be the distance of the middle line of one of these bands from O Y, and y_1, y_2, the distances of the two extremities of that middle line from O X. Then the band is approximately equal to a rectangular band of the length $y_2 - y_1$, and breadth Δx, the co-ordinates of whose centre are x, and $\frac{y_2 + y_1}{2}$. Consequently, if z be the uniform thick

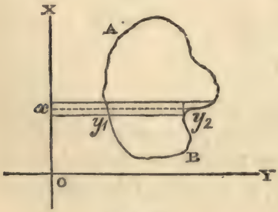


Fig. 22.

ness of the plate, and w its specific gravity, we have for a single band,

$$\begin{aligned} \text{area} &= (y_2 - y_1) \Delta x \textit{ nearly};\\ \text{volume} &= z\,(y_2 - y_1) \Delta x \textit{ nearly};\\ \text{weight} &= w\,z\,(y_2 - y_1) \Delta x \textit{ nearly}; \end{aligned}$$

moment relatively to O Y,

$$= w\,z\,x\,(y_2 - y_1) \Delta x \textit{ nearly};$$

moment relatively to O X,

$$= w\,z\,\frac{y_2^2 - y_1^2}{2} \Delta x \textit{ nearly};$$

and for the whole plate

$$\begin{aligned} \text{area} &= \Sigma \cdot (y_2 - y_1) \Delta x \textit{ nearly};\\ \text{volume } \mathrm{V} &= z \cdot \Sigma\,(y_2 - y_1) \Delta x \textit{ nearly};\\ \text{weight } \mathrm{W} &= w\,z \cdot \Sigma\,(y_2 - y_1) \Delta x \textit{ nearly}; \end{aligned}$$

moment relatively to O Y,

$$x_o \mathrm{W} = w\,z \cdot \Sigma\, x\,(y_2 - y_1) \Delta x \textit{ nearly};$$

moment relatively to O X,

$$y_o \mathrm{W} = w\,z \cdot \Sigma\,\frac{y_2^2 - y_1^2}{2} \Delta x \textit{ nearly}; \qquad \ldots\ldots\ldots(1.)$$

consequently, the co-ordinates of the centre of gravity of the plate (omitting the common factors $w\,z$), are

$$x_o = \frac{\Sigma \cdot x\,(y_2 - y_1) \Delta x}{\Sigma \cdot (y_2 - y_1) \Delta x} \textit{ nearly};$$

$$y_o = \frac{\Sigma \cdot (y_2^2 - y_1^2) \Delta x}{2\,\Sigma \cdot (y_2 - y_1) \Delta x} \textit{ nearly}.$$

The more minutely the cross-section A B is subdivided into bands, the more nearly do these approximate formulæ agree with the truth; so that the true results are the limits to which the results of the approximate formulæ (1.) approach continually as Δx becomes smaller; that is to say, in the notation of the integral calculus,

$$\left.\begin{aligned} \text{area} &= \int (y_2 - y_1)\,dx;\\ \text{volume } \mathrm{V} &= z \int (y_2 - y_1)\,dx;\\ \text{weight } w\,\mathrm{V} &= w\,z \int (y_2 - y_1)\,dx; \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots(2.)$$

$$\text{moments} \quad \left\{ \begin{array}{l} x_0\,\mathrm{W} = w\,z \int x\,(y_2 - y_1)\,d\,x\,; \\ y_0\,\mathrm{W} = \dfrac{w\,z}{2} \int (y_2^2 - y_1^2)\,d\,x\,; \end{array} \right\} \ldots\ldots\ldots\ldots\ldots (3.)$$

$$\begin{array}{c}\text{co-ordinates} \\ \text{of the} \\ \text{centre of gravity}\end{array} \left\{ \begin{array}{l} x_0 = \dfrac{\int x\,(y_2 - y_1)\,d\,x}{\int (y_2 - y_1)\,d\,x}\,; \\[2ex] y_0 = \dfrac{\int (y_2^2 - y_1^2)\,d\,x}{2\int (y_2 - y_1)\,d\,x}. \end{array} \right\} \ldots\ldots\ldots\ldots\ldots (4.)$$

The foregoing process is what is usually called by writers on mechanics, "*finding the centre of gravity of a plane surface;*" but this phrase ought always to be understood to signify "*finding the centre of gravity of a homogeneous plate of uniform thickness, the faces of which are plane surfaces of a given figure.*"

79. **Body with Similar Cross-sections.**—Let all the cross-sections of a body made by planes parallel to a given plane (say that of $\overline{x\,y}$), be similar figures, but of different sizes. The areas of the different cross-sections are to each other as the squares of their corresponding linear dimensions. Let ς denote some definite linear dimension of a cross-section whose distance from the plane $\overline{x\,y}$ is z, so that its area shall be

$$a\,\varsigma^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

a being a constant. Let x_1, y_1, z, be the co-ordinates of the centre of gravity of a flat plate having its middle plane coincident with the given cross-section. Then, by reasoning similar to that of Articles 74 and 78, we find the following results for the whole body:—

$$\left. \begin{array}{ll} \text{volume} & \mathrm{V} = \quad a \int \varsigma^2\,d\,z\,; \\ \text{weight} & \mathrm{W} = w\,a \int \varsigma^2\,d\,z\,; \end{array} \right\} \ldots\ldots\ldots\ldots\ldots (2.)$$

$$\text{moments} \quad \left\{ \begin{array}{l} x_0\,\mathrm{W} = w\,a \int x_1\,\varsigma^2\,d\,z\,; \\ y_0\,\mathrm{W} = w\,a \int y_1\,\varsigma^2\,d\,z\,; \\ z_0\,\mathrm{W} = w\,a \int z\,\varsigma^2\,d\,z\,; \end{array} \right\} \ldots\ldots\ldots\ldots\ldots\ldots (3.)$$

$$\text{co-ordinates of centre of gravity}\left\{\begin{aligned} x_o &= \frac{\int x_1\, \varsigma^2\, d\, z}{\int \varsigma^2\, d\, z}; \\ y_o &= \frac{\int y_1\, \varsigma^2\, d\, z}{\int \varsigma^2\, d\, z}; \\ z_o &= \frac{\int z\, \varsigma^2\, d\, z}{\int \varsigma^2\, d\, z}. \end{aligned}\right\} \cdots\cdots\cdots\cdots (4.)$$

When the centres of all the cross-sections lie in one straight line, as in pyramids, cones, conoids, and solids of revolution generally, the centre of gravity lies in that line, which may be taken as the axis of z, making $x_o = 0$, $y_o = 0$; so that z_o is the only co-ordinate which requires to be determined.

80. **Curved Rod.**—In fig. 23, let R R represent a curved rod so slender, that its diameter may, without sensible error, be neglected in comparison with its radius of curvature at any point; let a denote its sectional area, uniform throughout, and w, as usual, its specific gravity; so that the weight of an unit of length of the rod is wa. Let OX, OY, OZ be rectangular axes of co-ordinates. Suppose the rod to be divided into arcs, so short as to be nearly straight; let the length of any one of these arcs be denoted by $\Delta\, \varsigma$; let S S represent it in the figure, and let M be the middle of its length. Then M is *nearly* the centre of gravity of $\Delta\, \varsigma$. Let $\mathrm{M\,P} = x$ be the perpendicular distance from M to the plane of $\overline{y\, z}$. Then for the short arc S S we have,

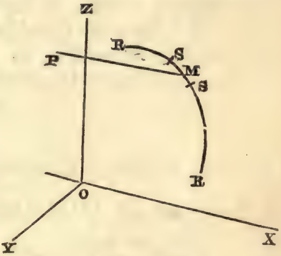


Fig. 23.

$$\text{weight} \qquad = w\, a\, \Delta\, \varsigma;$$

moment with respect to an axis in the plane $\overline{y\, z}$,

$$= w\, a\, x\, \Delta\, \varsigma \text{ nearly};$$

and for the entire rod,

$$\mathrm{W} = w\, a\, \Sigma \cdot \Delta\, \varsigma;$$

$$\left.\begin{aligned} &\text{moment} \quad x_o \mathrm{W} = w\, a\, \Sigma \cdot x\, \Delta\, \varsigma \text{ nearly}; \\ &\left.\begin{array}{l}\text{co-ordinate of} \\ \text{centre of gravity}\end{array}\right\} x_o = \frac{\Sigma \cdot x\, \Delta\, \varsigma}{\Sigma \cdot \Delta\, \varsigma} \text{ nearly}; \end{aligned}\right\} \cdots\cdots (1.)$$

and similar equations for y_o and z_o. Proceeding by the method of limits as before, we obtain as the exact formulæ—

$$\left.\begin{aligned} \mathrm{W} &= w\,a \int d\,s; \\ x_o\,\mathrm{W} &= w\,a \int x\,d\,s; \\ x_o &= \frac{\int x\,d\,s}{\int d\,s}. \end{aligned}\right\} \text{.................(2.)}$$

and similar equations for y_o and z_o. The foregoing process is what is often called by writers on Mechanics, "*finding the centre of gravity of a curved line;*" but what ought more properly to be called, "*finding the centre of gravity of a slender curved rod of uniform thickness.*"

81. **Approximate Computation of Integrals.**—Frequent reference having been made to the process of *integration*, as being essential to the solution of most problems connected with distributed force, the present article is intended to afford to those who have not made that branch of mathematics a special study, some elementary information respecting it.

The meaning of the symbol of an integral, viz.:—

$$\int u\,d\,x,$$

is of the following kind:—

In fig. 24, let A C D B be a plane area, of which one boundary, A B, is a portion of an axis of abscissæ O X,—the opposite boundary, C D, a curve of any figure,—and the remaining boundaries A C, B D, ordinates perpendicular to O X, whose respective abscissæ, or distances from the origin O, are

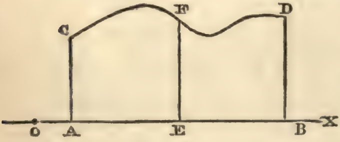


Fig. 24.

$$\overline{\mathrm{O\,A}} = a;\ \overline{\mathrm{O\,B}} = b.$$

Let $\overline{\mathrm{E\,F}} = u$ be any ordinate whatsoever of the curve C D, and O E = x the corresponding abscissa. Then the integral denoted by the symbol,

$$\int_a^b u\,d\,x,$$

means, *the area of the figure* A C D B. The abscissæ a and b which are the least and greatest values of x, and which indicate the longitudinal extent of the area, are called the *limits of integration;* but when the extent of the area is otherwise indicated, the symbols of those limits are sometimes omitted, as in the preceding Articles.

When the relation between u and x is expressed by any ordinary algebraical equation, the value of the integral for a given pair of values of its limits can generally be found by means of formulæ which are contained in works on the Integral Calculus, or by means of mathematical tables.

Cases may arise, however, in which u cannot be so expressed in terms of x; and then approximate methods must be employed. Those approximate methods, of which two are here described, are founded upon the division of the area to be measured into bands by parallel and equi-distant ordinates, the approximate computation of the areas of those bands, and the adding of them together; and the more minute that division is, the more near is the result to the truth.

First Approximation.

Divide the area A C D B, as in fig. 25, into any convenient number of bands by parallel ordinates, whose uniform distance apart is Δx; so that if n be the number of bands, $n + 1$ will be the number of ordinates, and

$$b - a = n \Delta x,$$

the length of the figure.

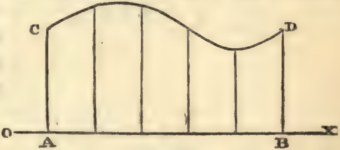


Fig. 25.

Let u', u'', denote the two ordinates which bound one of the bands; then the area of that band is

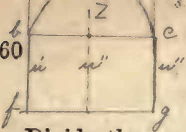
$$\frac{u' + u''}{2} \cdot \Delta x, \textit{ nearly};$$

and consequently, adding together the approximate areas of all the bands,—denoting the extreme ordinates as follows,—

$$\overline{\mathrm{A\,C}} = u_a; \ \overline{\mathrm{B\,D}} = u_b;$$

and the intermediate ordinates by u_i, we find for the approximate value of the integral—

$$\int_a^b u\, d\, x = \left(\frac{u_a}{2} + \frac{u_b}{2} + \Sigma \cdot u_i \right) \Delta \cdot x, \ldots\ldots\ldots\ldots\ldots (1.)$$



[illegible] (4.) $fbcg = \left(\frac{u'+u'''}{2}\right)2dx = (u'+$

Adding 3+4, $fbhcg = \left(\frac{u'+4u''+u'''}{3}\right)dx$

Second Approximation.

Divide the area A C D B, as in fig. 26, into an *even* number of bands, by parallel ordinates, whose uniform distance apart is Δx. The ordinates are marked alternately by plain lines and by dotted lines, so as to arrange the bands in pairs. Considering any one pair of bands, such as E F H G, and assuming that the curve F H is nearly a parabola, it appears, from the properties of that curve, that the area of that pair of bands is

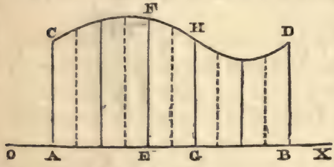


Fig. 26.

$$\frac{(u' + 4\,u'' + u''')\,\Delta\,x}{3}, \textit{ nearly;}$$

in which u' and u''' denote the plain ordinates $\overline{\mathrm{E\,F}}$ and $\overline{\mathrm{G\,H}}$, and u'' the intermediate dotted ordinate; and consequently, adding together the approximate areas of all the pairs of bands, we find, for the approximate value of the integral—

$$\int_a^b u\,d\,x = \Big(u_a + u_b + 2\,\Sigma\cdot u_i\text{ (plain)}$$

$$+ 4\,\Sigma\cdot u_i\text{ (dotted)}\Big)\frac{\Delta\,x}{3}, \ldots\ldots\ldots\ldots\ldots\ldots(2.)$$

It is obvious, that if the values of the ordinates u required in these computations can be calculated, it is unnecessary to draw the figure to a scale, although a sketch of it may be useful to assist the memory.

When the symbol of integration is repeated, so as to make a *double integral*, such as

$$\int\!\!\int u\cdot d\,x\,d\,y,$$

or a *triple integral*, such as

$$\int\!\!\int\!\!\int u\cdot d\,x\,d\,y\,d\,z,$$

it is to be understood as follows:—

Let

$$v = \int u\cdot d\,x$$

be the value of this single integral for a given value of y. Construct a curve whose abscissæ are the various values of y within the prescribed limits, and its ordinates the corresponding values of v. Then the area of that curve is denoted by

$$\int v\cdot d\,y = \int\!\!\int u\cdot d\,x\,d\,y.$$

Next, let
$$t = \int v \cdot dy$$

be the value of this double integral for a given value of z. Construct a curve whose abscissæ are the various values of z within the prescribed limits, and its ordinates the corresponding values of t. Then the area of that curve is denoted by

$$\int t \, . \, dz = \int\int v \cdot dy \, dz = \int\int\int u \cdot dx \, dy \, dz;$$

and so on for any number of successive integrations.

82. **Centre of Gravity found by Projection.**—According to the geometrical properties of parallel projections, as stated in Chap. IV., Article 62, a parallel projection of a pair of volumes having a given ratio is a pair of volumes having the same ratio; and hence, if a body of any figure be divided by a system of plane or other surfaces into parts or molecules, either equal, or bearing any given system of proportions to each other, and if a second body, whose figure is a parallel projection of that of the first body, be divided in the same manner by a system of plane or other surfaces which are the corresponding projections of the first system of plane or other surfaces, the parts or molecules of the second body will bear to each other the same system of ratios, of equality or otherwise, which the parts of the first body do.

Also, the centres of gravity of the parts of the second body will be the parallel projections of the centres of gravity of the parts of the first body.

And hence it follows (according to Article 64), that *if the figures of two bodies are parallel projections of each other, the centres of gravity of these two bodies are corresponding points in these parallel projections.*

To express this symbolically,—as in Article 61, let x, y, z, be the co-ordinates, rectangular or oblique, of any point in the figure of the first body; x', y', z', those of the corresponding point in the second body; x_o, y_o, z_o, the co-ordinates of the centre of gravity of the first body; x'_o, y'_o, z'_o, those of the centre of gravity of the second body; then

$$\frac{x'_o}{x_o} = \frac{x'}{x}; \quad \frac{y'_o}{y_o} = \frac{y'}{y}; \quad \frac{z'_o}{z_o} = \frac{z'}{z}.$$

This theorem facilitates much the finding of the centres of gravity of figures which are parallel projections of more simple or more symmetrical figures.

For example:—it appears, from symmetry, as in Art. 72, that the centre of gravity of an equilateral triangular prism is at the point of intersection of the lines joining the three angles of the middle section of the prism with the middle points of the opposite sides of that section. But all triangular prisms are parallel projections of each other; hence the above described point of intersection is the centre of gravity of any triangular prism.

Also, as in Art. 72, the centre of gravity of a regular tetraedron is at the point of intersection of the planes joining each of the edges with the middle point of the opposite edge. But all tetraedrons are parallel projections of each other; hence that point of intersection is the centre of gravity in any tetraedron.

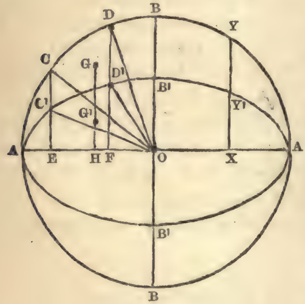


Fig. 27.

As a third example, let it be supposed that a formula is known (which will be given in the sequel) for finding the centre of gravity of a sector of a circular disc, and let it be required to find the centre of gravity of a sector of an elliptic disc. In fig. 27, let A B′ A B′ be the ellipse, A O A $= 2\,a$, and B′ O B′ $= 2\,b$, its axes, and C′ O D′ the sector whose centre of gravity is required. One of the parallel projections of the ellipse is a circle, ABAB, whose radius is the semi-axis major a. The ellipse and the circle being both referred to rectangular co-ordinates, with their centre as origin, x and y denoting the co-ordinates parallel to O A and O B respectively of a point in the circle, and x' and y' those of the corresponding point in the ellipse, those co-ordinates are thus related :—

$$\frac{x'}{x} = 1; \ \frac{y'}{y} = \frac{b}{a}.$$

Through C′ and D′ respectively draw E C′ C and F D′ D, parallel to O B, and cutting the circle in C and D respectively; the circular sector C O D is the parallel projection of the elliptic sector C′ O D′. Let G be the centre of gravity of the sector of the circular disc, its co-ordinates being

$$\overline{\mathrm{O\,H}} = x_o; \ \overline{\mathrm{H\,G}} = y_o.$$

Then the co-ordinates of the centre of gravity G′ of the sector of the elliptic disc are

$$\overline{\mathrm{O\,H}} = x'_o = x_o;$$

$$\overline{\mathrm{H\,G'}} = y'_o = \frac{b\,y_o}{a}.$$

Further examples of the results of this process will be found in the next Article.

83. **Examples of Centres of Gravity.**—The following examples consist of formulæ for the weight, the moment with respect to some specified axis, and the position of the centre of gravity, of homogeneous bodies of those forms which most commonly occur in practice. In each case, as in the formulæ of the preceding Articles, w denotes the specific gravity of the body, W, its weight, and x_o, &c., the co-ordinates of its centre of gravity, which in the diagrams is marked G, the origin of co-ordinates being marked O.

A.—Prisms and Cylinders with Parallel Bases.

The word *cylinder* is here to be taken in its most general meaning, as comprehending all solids traced by the motion of a plane curvilinear figure parallel to itself.

The examples here given apply, of course, to flat plates of uniform thickness.

In the formulæ for weights and moments, the length or thickness is supposed to be *unity*.

The centre of gravity, in each case, is at the middle of the length (or thickness); and the formulæ give its situation in the plane figure which represents the cross-section of the prism or cylinder, and which is specified at the commencement of each example.

I. *Triangle.* (Fig. 28) O, any angle. Bisect opposite side B C in D. Join O D.

$h = OD \sin ODC$

$$x_o = \overline{OG} = \frac{2}{3}\overline{OD}.$$

$$W = w \cdot \frac{\overline{OD} \cdot \overline{BC} \cdot \sin. \angle ODC}{2}.$$

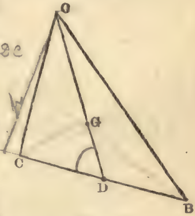


Fig. 28.

II. *Polygon.* Divide it into triangles; find the centre of gravity of each; then find their common centre of gravity as in Art. 75.

III. *Trapezoid.* (Fig. 29.) x_o of $\triangle ACL = \frac{1}{3}CY = YG_2$

A B || C E. x_o of ▱ $CEBL = \frac{1}{2}DE = SG_1$

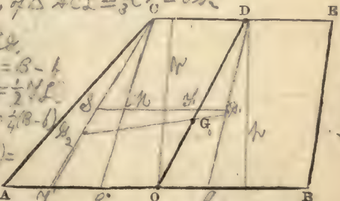
Greatest breadth, A B = B. $AL = B - b$

Least " C E = b. $SN = \frac{1}{2}NL$

Bisect A B in O, C E in D; $SM = \frac{1}{4}(B-b)$

join O D. $SG_1 = \frac{1}{2}b + \frac{1}{4}(B-b) = \frac{1}{4}(B+b)$

$$x_o = \overline{OG} = \frac{\overline{OD}}{2}\left(1 - \frac{1}{3} \cdot \frac{B-b}{B+b}\right)$$



$SG_2 = \frac{2}{3}CY - \frac{1}{2}CY$ Fig. 29. $= \frac{1}{6}CY = \frac{1}{6}DO$

$$W = w \cdot \overline{OD} \cdot \frac{B+b}{2} \cdot \sin \angle ODE.$$

$SG_2 = SM = \frac{1}{4}(B-b)$ $SG_1 : SG_2 :: GG_2 : GG_1$. i.e.

$M_2 = \frac{1}{2} w \frac{2}{3} x_1^2 AB \sin OBC$. $M = \frac{1}{2}$ [illegible] $\frac{2}{3}(ABx_1^2 - CEx_2^2)$

$= \frac{M}{W} = \frac{2(ABx_1^2 - CEx_2^2)}{3(ABx_1 - CEx_2)}$

$x_1 : AB :: x_2 : CE$

$AB = \frac{CEx_1}{x_2}$

$x_0 = \frac{2}{3} \frac{x_1^3 - x_2^3}{x_1^2 - x_2^2}$

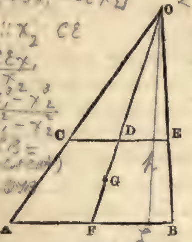


Fig. 30.

27 IV. *Trapezoid.* (Second solution.) (Fig. 30.) O, point where inclined sides meet. Let $\overline{OF} = x_1$, $\overline{OD} = x_2$, $\overline{OG} = x_o$.

$$x_o = \frac{2}{3} \cdot \frac{x_1^3 - x_2^3}{x_1^2 - x_2^2}$$

$$W = w \cdot \frac{x_1^2 - x_2^2}{2} \cdot \sin^2 \angle OFB.$$

$$(\cot \angle OAB + \cot \angle OBA).$$

— OAB (cot OAB + cot OBA)

$$x_o W = w \cdot \frac{x_1^3 - x_2^3}{3} \cdot \sin^2 \angle OFB.$$

$$(\cot \angle OAB + \cot \angle OBA).$$

[illegible] area of Trap = dif of [illegible] areas of Δs [illegible] are readily found.) 28

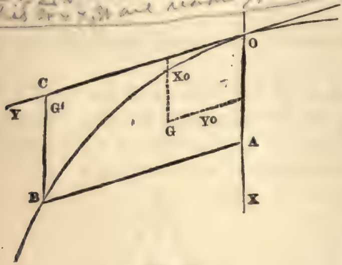


Fig. 31.

V. *Parabolic Half-Segment.* (O A B, fig. 31.) O, vertex of diameter O X; $\overline{OA} = x_1$; $\overline{AB} = y_1$, ordinate ‖ tangent O C Y.

$$x_o = \frac{3}{5} x_1.$$

$$y_o = \frac{3}{8} y_1.$$

$$W = \frac{2}{3} w \cdot x_1 y_1 \cdot \sin \angle XOY.$$

29 VI. *Parabolic Spandril.* (O B C, fig. 31.) G′, centre of gravity,

$$x_o = \frac{3}{10} x_1; \ y_o = \frac{3}{4} y_1.$$

$$W = \frac{1}{3} w \cdot x_1 y_1 \cdot \sin \angle XOY.$$

VII. *Circular Sector.* (O A C, fig. 32.) Let O X bisect the angle A O C; O Y ⊥ O X.

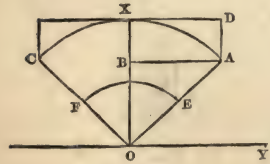


Fig. 32.

Radius $\overline{OA} = r$

Half-arc, to radius unity, $\frac{AC}{2\,AO} = \theta$.

$$x_o = \frac{2}{3} r \frac{\sin \theta}{\theta}; \ y_o = 0.$$

$$W = w r^2 \theta$$

VIII. *Circular Half-Segment.* (A B X, fig. 32.)

$$x_0 = \frac{2}{3} r \cdot \frac{\sin^3 \theta}{\theta - \sin \theta \cos \theta}.$$

$$y_0 = r \cdot \frac{4 \sin^2 \frac{\theta}{2} - \sin^2 \theta \cos \theta}{3 (\theta - \cos \theta \sin \theta)}.$$

$$W = \frac{1}{2} w r^2 (\theta - \cos \theta \sin \theta).$$

IX. *Circular Spandril.* (A D X, fig. 32.)

$$x_0 = \frac{1}{3} r \cdot \frac{\sin^3 \theta}{2 \sin \theta - \sin \theta \cos \theta - \theta}.$$

$$y_0 = \frac{1}{3} r \cdot \frac{3 \sin^2 \theta - 2 \sin^2 \theta \cos \theta - 4 \sin^2 \frac{\theta}{2}}{2 \sin \theta - \sin \theta \cos \theta - \theta}.$$

$$W = w r^2 \cdot \left(\sin \theta - \frac{1}{2} \sin \theta \cos \theta - \frac{\theta}{2} \right).$$

X. *Sector of Ring.* (A C F E, fig. 32.) $\overline{OA} = r$; $\overline{OE} = r'$.

$$x_0 = \frac{2}{3} \cdot \frac{r^3 - r'^3}{r^2 - r'^2} \cdot \frac{\sin \theta}{\theta}.$$

$$y_0 = 0.$$

$$W = w (r^2 - r'^2) \theta.$$

XI. *Elliptic Sector, Half-Segment, or Spandril.* Centre of gravity to be found by projection from that of corresponding circular figure, as in Article 82.

B.—Wedges.

A *Wedge* is a solid bounded by two planes which meet in an edge, and by a cylindrical or prismatic surface (*cylindrical*, as before, being used in the most general sense).

XII. *General Formulæ for Wedges.* (Fig. 33.) All wedges may be divided into parts such as the figure here represented. O A Y, O X Y, planes meeting in the edge O Y; A X Y, cylindrical (or prismatic) surface perpendicular to the plane O X Y; O X A, plane triangle perpendicular to the edge O Y; O Z, axis perpendicular to X O Y. Let O X $= x_1$; X A $= z_1$. Then $z = \frac{z_1 x}{x_1}$;

$$W = w \cdot \frac{z_1}{x_1} \int x y \cdot d x$$

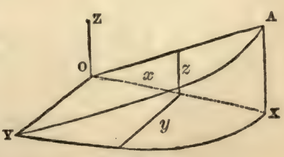


Fig. 33.

$$x_0 = \frac{\int x^2 y \cdot d x}{\int x y \cdot d x};$$

$$y_0 = \frac{\int x y^2 \cdot d x}{2 \int x y \cdot d x};$$

$z_0 = \frac{z_1 x_0}{2 x_1}$. (This last equation denoting that G is in the plane which traverses O Y and bisects A X.)

In a symmetrical wedge, if O be taken at the middle of the edge, $y_0 = 0$. Such is the case in the following examples, in each of which, length of edge $= 2 y_1$.

XIII. *Rectangular Wedge.* (= Triangular Prism.) (Fig. 34.)

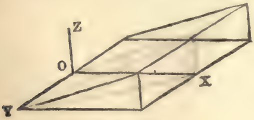


Fig. 34.

$$y = y_1.$$

$$W = w \cdot x_1 y_1 z_1.$$

$$x_0 = \frac{2}{3} x_1.$$

XIV. *Triangular Wedge.* (= Triangular Pyramid.)

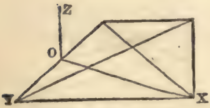


Fig. 35.

$$y = y_1 \left(1 - \frac{x}{x_1}\right)$$

$$W = \frac{1}{3} w \cdot x_1 y_1 z_1.$$

$$x_0 = \frac{1}{2} x_1.$$

XV. *Semicircular Wedge.* (Fig. 36.)

$$\text{Radius } \overline{OX} = \overline{OY} = r.$$

$$y = \sqrt{r^2 - x^2}.$$

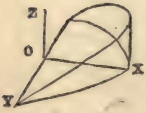


Fig. 36.

$$W = \frac{2}{3} w \cdot r^2 z_1.$$

$$x_0 = \frac{3 \pi}{16} r.$$

$$(\pi = 3 \cdot 1416 \textit{ nearly}).$$

37 XVI. *Annular, or Hollow Semicircular Wedge.* (Fig. 37.)

External radius, r; internal, r'.

$$W = \frac{2}{3}\, w \cdot \left(r^3 - r'^3\right) \frac{z_1}{r}.$$

$$x_0 = \frac{3\,\pi}{16} \cdot \frac{r^4 - r'^4}{r^3 - r'^3}.$$

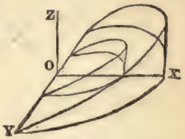


Fig. 37.

C.—Cones and Pyramids.

Let O denote the apex of the cone or pyramid, taken as the origin, and X the centre of gravity of a supposed prism whose middle section coincides with the base of the cone, or pyramid. The centre of gravity will lie in the axis OX.

Denote the area of the base by A, and the angle which it makes with the axis by θ.

38 XVII. *Complete Cone or Pyramid.* Let the height $\overline{OX} = h$;

$$x_0 = \frac{3}{4}\, h.$$

$$W = \frac{1}{3}\, w \cdot A\, h \sin \theta.$$

39 XVIII. *Truncated Cone or Pyramid.* Height of portion truncated $= h'$.

$$x_0 = \frac{3}{4} \cdot \frac{h^4 - h'^4}{h^3 - h'^3}.$$

$$W = \frac{1}{3}\, w A h \cdot \left(1 - \frac{h'^3}{h^3}\right) \sin \theta.$$

40

D.—Portions of a Sphere.

XIX. *Zone or Ring of a Spherical Shell,* bounded by two conical surfaces having their common apex at the centre O of the sphere (fig. 38).

OX, axis of cones and zone.
r, external radius } of shell.
r', internal radius } of shell.
$\angle XOA = \alpha$, half-angle of less } cone.
$\angle XOB = \beta$, ,, greater } cone.

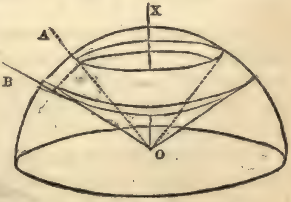


Fig. 38.

$$x_0 = \frac{3}{4} \cdot \frac{r^4 - r'^4}{r^3 - r'^3} \cdot \frac{\cos \alpha + \cos \beta}{2}.$$

$$W = \frac{2 \pi w}{3} \left(r^3 - r'^3\right) . \left(\cos \beta - \cos \alpha\right).$$

XX. *Sector of a Hemispherical Shell.* (C X D, fig. 39.) O Y bisects angle D O C; $\frac{1}{2}$ D O C $= \theta$.

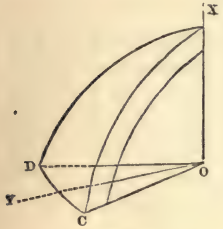


Fig. 39.

$$x_0 = \frac{3}{8} \cdot \frac{r^4 - r'^4}{r^3 - r'^3}.$$

$$y_0 = \frac{3 \pi}{16} \cdot \frac{r^4 - r'^4}{r^3 - r'^3} \cdot \frac{\sin \theta}{\theta}.$$

$$W = \frac{2 \theta w}{3} \left(r^3 - r'^3\right).$$

84. **Heterogeneous Body.**—If a body consists of parts of definite figure and extent, whose specific gravities are different, although each individual part is homogeneous, the centres of gravity of the parts are to be found as in Article 74 and the subsequent Articles, and the common centre of gravity of the whole as in Article 73.

85. **Centre of Gravity found Experimentally.**—The centre of gravity of a body of moderate size may be found approximately by experiment, by hanging it up successively by a single cord in two different positions, and finding the single point in the body which in both positions is intersected by the axis of the cord. For the resistance of the cord is equivalent sensibly to a single force acting along its axis; and as that force balances the weight of the body when hung by the cord, its line of action must, in all positions of the body, traverse the centre of gravity of the body.

SECTION 2.—*Of Stress, and its Resultants and Centres.*

86. **Stress, its Nature and Intensity.**—The word STRESS has been adopted as a general term to comprehend various forces which are exerted between contiguous bodies, or parts of bodies, and which are distributed over the surface of contact of the masses between which they act.

The INTENSITY of a stress is its amount in units of force, divided by the extent of the surface over which it acts, in units of area. The French and British units of intensity of stress are compared in a table annexed to this volume. The following table shows a comparison between different British units of intensity of stress :—

	Pounds on the square foot.	Pounds on the square inch.
One pound on the square inch,	144	1
One pound on the square foot,	1	$\frac{1}{144}$
One inch of mercury (that is, weight of a column of mercury at 32° Fahr., one inch high),	70·73	0·4912
One foot of water (at 39°·4 Fahr.),	62·425	0·4335
One inch of water,	5·2021	0·036125
One atmosphere, of 29·922 inches of mercury,	2116·4	14·7

87. **Classes of Stress.**—Stress may be classed as follows :—

I. *Thrust,* or *Pressure,* is the force which acts between two contiguous bodies, or parts of a body, when each pushes the other from itself, and which tends to *compress* or shorten each body on which it acts, in the direction of its action. It is the kind of force which is exerted by a fluid tending to expand, against the bodies which surround it.

Thrust may be either *normal* or *oblique,* relative to the surface at which it acts.

II. *Pull,* or *Tension,* is the force which acts between two contiguous bodies, or parts of a body, when each draws the other towards itself, and which tends to lengthen each body on which it acts, in the direction of its action.

Pull, like thrust, may be either *normal* or *oblique,* relatively to the surface at which it acts.

III. *Shear,* or *Tangential Stress,* is the force which acts between two contiguous bodies or parts of a body, when each draws the other sideways, in a direction parallel to their surface of contact, and which tends to distort each body on which it acts.

In expressing a Thrust and a Pull in parallel directions algebraically, if one is treated as positive, the other must be treated as negative. The choice of the positive or negative sign for either is a matter of convenience. In treating of the general theory of stress, the more usual system is to call a *pull* positive, and a *thrust* negative: thus, let p denote the intensity of a stress, and n a certain number of pounds per square foot; $p = n$ will denote a *pull,* and $p = -n$ a *thrust* of the same intensity. But in treating of certain special applications of the theory, to cases in which thrust is the only or the predominant stress, it becomes more convenient to reverse this system, calling thrust positive, and pull negative.

The word "*Pressure,*" although, strictly speaking, equivalent to "*thrust,*" is sometimes applied to *stress* in general; and when this is the case, it is to be understood that thrust is treated as positive.

88. **Resultant of Stress: its Magnitude.**—If to a plane surface of any figure, whose area is S, there be applied a stress, either normal, oblique, or tangential, and parallel in direction at all points of the surface (according to the restriction stated in Art. 67), then if the intensity of the stress be uniform over all the surface, and denoted by p, the amount or magnitude of its resultant will be

$$\mathrm{P} = p\,\mathrm{S} \quad \text{(1.)}$$

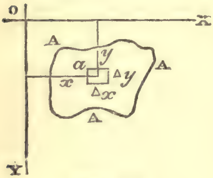


Fig. 40.

If the intensity of the stress is not uniform, that amount is to be found by integration. For example, in fig. 40, let A A A be the plane surface, and let it be referred to rectangular axes of co-ordinates in its own plane, O X, O Y. Conceive that plane to be divided into small rectangles by a network of lines parallel to O X and O Y respectively, and let Δx, Δy, be the dimensions of any one of these rectangles, such as that marked a in the figure. Conceive a figure approximating to that of the given plane surface to be composed of several of these small rectangles, so that

$$\mathrm{S} = \Sigma \cdot \Delta x \, \Delta y \textit{ nearly;} \quad \text{(2.)}$$

let p be the intensity of the stress at the centre of any particular rectangle, so that the stress on that rectangle is

$$p \, \Delta x \, \Delta y \textit{ nearly.}$$

Then the amount of the resultant stress is given approximately by the equation

$$\mathrm{P} = \Sigma \cdot p \, \Delta x \, \Delta y \textit{ nearly} \quad \text{(3.)}$$

Then passing, as in previous examples, to the integrals, or limits towards which the sums in the equations 2 and 3 approach as the minuteness of the subdivision into rectangles is indefinitely increased, we find, for the exact equations,

$$\left.\begin{aligned} \mathrm{S} &= \iint dx\,dy; \\ \mathrm{P} &= \iint p \cdot dx\,dy. \end{aligned}\right\} \quad \text{(4.)}$$

The *mean intensity* of the stress is given by the following equation:—

$$p_0 = \frac{\mathrm{P}}{\mathrm{S}} = \frac{\iint p\,dx\,dy}{\iint dx\,dy} \quad \text{(5.)}$$

A convenient mode of representing to the mind the foregoing process is as follows:—In fig. 41, let A A be the given plane surface; O X, O Y, the two axes of co-ordinates in its plane; O Z, a third axis perpendicular to that plane. Conceive a solid to exist, bounded at one end by the given plane surface A A, laterally by a cylindrical or prismatic surface generated by the motion of a straight line parallel to O Z round the outline of A A, and at the other end by a surface B B, of such a figure, that its ordinate z at any point shall be proportional to the intensity of the stress at the point of the surface A A from which that ordinate proceeds, as shown by the equation

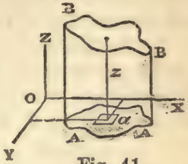


Fig. 41.

$$z = \frac{p}{w} \text{ ....(6.)}$$

The volume of this ideal solid will be

$$\mathrm{V} = \int\int z \cdot dx\,dy \text{ ....(7.)}$$

So that if it be conceived to consist of a material whose specific gravity is w, the amount of the stress will be equal to the weight of the solid, that is to say,

$$\mathrm{P} = w\,\mathrm{V} \text{ ....(8.)}$$

If the stress be of opposite signs at different points of the plane surface A A, the surface B B and the solid which it terminates will be partly at one side of A A and partly at the opposite side, as in fig. 42; and in this case, the two parts into which the solid A B A B is divided by the plane X O Y, are to be regarded as having opposite signs, and V is to be held to represent the *difference* of their volumes.

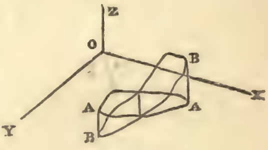


Fig. 42.

The *mean stress* of equation 5 is evidently

$$p_0 = w\,z_0 \text{ ....(9.)}$$

in which z_0 is the height of a parallel-ended prism or cylinder standing on the base A A A, and of volume equal to the solid A B A B.

89. **The Centre of Stress, or of Pressure,** in any surface, is the point traversed by the resultant of the whole stress, or in other words, the *Centre of Parallel Forces* for the whole stress. From the principles already proved in Chap. II., Section 4, it follows, that

the position of this point does *not* depend upon the direction of the stress, nor upon its absolute magnitude; but solely on the form of the surface at which the stress acts, and on the proportions between the intensities of the stress at different points.

As in Article 88, conceive a figure approximating to that of the given plane surface A A A (fig. 40), to be composed of several small rectangles; let α β denote the angles which the direction of the stress makes with O X, O Y respectively. Then the moments, relative to the co-ordinate planes, Z O X, Z O Y, of the components parallel to those planes of the stress on $\Delta x \Delta y$, are given by the approximate equations.

$$\left.\begin{array}{ll}\text{Moment relatively to } ZOX, & y p \,\Delta x\, \Delta y \quad \sin\beta \\ \quad ,, \qquad\qquad ,, \qquad ZOY, & -x p\, \Delta x\, \Delta y \cdot \sin\alpha\end{array}\right\} \textit{nearly.}$$

Summing all such moments, and passing to the integral or limit of the sum, as in former examples, we find the following expressions, in which x_0 and y_0 denote the co-ordinates of the centre of stress;

$$\left.\begin{array}{l} y_0 \mathrm{P} \cdot \sin\beta = \sin\beta \int\int yp \cdot dx\,dy \\ x_0 \mathrm{P} \cdot \sin\alpha = \sin\alpha \int\int xp \cdot dx\,dy \end{array}\right\} \dots\dots\dots \text{(1.)}$$

Consequently the co-ordinates of the centre of stress are

$$\left.\begin{array}{l} x_0 = \dfrac{\int\int xp \cdot dx\,dy}{\int\int p \cdot dx\,dy}; = \\ \\ y_0 = \dfrac{\int\int yp \cdot dx\,dy}{\int\int p \cdot dx\,dy}; \end{array}\right\} \dots\dots\dots\dots\dots \text{(2.)}$$

$$\frac{a_x}{h} \iint \frac{xp\,dx\,dy}{p\,dx\,dy} =$$

which are evidently the same with the co-ordinates, parallel to O X and O Y, of the *centre of gravity* of the ideal solid of fig. 41, whose ordinates z are proportional to the intensity of the pressure at the points on which they stand.

When the intensity of the stress is positive and negative at different points of the surface A A A, cases occur in which the positive and negative parts of the stress balance each other, so that the total stress is nothing, that is to say,

$$\int\int p\,dx\,dy = 0.$$

In such cases, the resultant of the stress (if any) is a *couple*, and there is no centre of stress. This case will be further considered in the sequel.

90. **Centre of Uniform Stress.**—If the intensity of the stress be uniform, the factor p in equation 2 of Article 89 becomes constant, and may be removed from both numerator and denominator of the expressions for x_0 and y_0, which then become simply the co-ordinates of the *centre of gravity of a flat plate* of the figure A A A.

This also appears from the consideration, that the surface B B in fig. 41 becomes a plane parallel to A A, and the solid A B A B, a parallel-ended prism or cylinder.

91. **Moment of Uniformly Varying Stress.**—By an *uniformly varying* stress is understood a stress whose intensity, at a given point of the surface to which it is applied, is proportional to the distance of that point from a given straight line. For example, let the given straight line be taken as the axis O Y; then the following equation

$$p = ax, \quad \dots\dots\dots\dots(1.)$$

a being a constant, represents the law of variation of the intensity of an uniformly varying stress.

The *amount* of an uniformly varying stress is given by the equation

$$\mathrm{P} = \int\int p \cdot dx\,dy = a \int\int x \cdot dx\,dy \dots\ \dots\dots(2.)$$

which, if the axis O Y traverses the *centre of gravity of a plate of the figure of the surface of action* A A A, becomes equal to *nothing*, the positive and negative values of p balancing each other. In this case, O Y is called a NEUTRAL AXIS of the surface A A A.

In fig. 43, let A A A represent the plane surface of action of a stress; let O be its centre of gravity (that is, the centre of gravity of a flat plate of which A A A is the figure); $-$Y O Y the neutral axis of the stress applied; $-$X O X perpendicular to $-$Y O Y, and in the plane of A A A; $-$Z O Z perpendicular to that plane. Conceive a plane B B inclined to A A A to traverse the neutral axis, and to form, with the plane A A A, a pair of wedges bounded by a cylindrical or prismatic surface parallel to $-$Z O Z. The ordinate z, drawn from any point of A A A to B B, will be proportional to the intensity of the stress at that point of A A A, and will indicate by its upward or downward direction whether that stress is positive or negative; and the nullity of the total stress will be indicated by

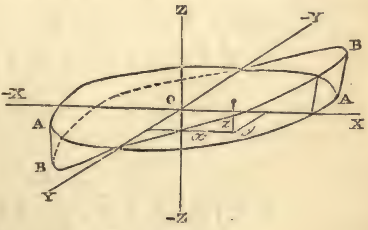


Fig. 43.

the equality of the positive wedge above A A A, and the negative wedge below A A A. The resultant of the whole stress is a couple, whose moment, and the position of its axis, are found in the following manner, by the application of the process of Chap. III., Sect. 2, Article 60.

Let α, β, γ, be the angles which the direction of the stress makes with OX, OY, OZ, respectively. Let $\Delta x \Delta y$ denote, as before, the area of a small rectangular portion of the surface, x, y, the co-ordinates of its centre (for which $z=0$), and $p=ax$, the intensity of the stress on it, so that

$$\Delta P = p \Delta x \Delta y = a x \Delta x \Delta y$$

is the force acting on this rectangle.

The moments of this force relatively to the three axes of co-ordinates, are found to be as follows, by making the proper substitutions in equation 2 of Article 60:—

$$\begin{aligned} &\text{round OX};\ \Delta P \cdot y \cos \gamma; \\ &\quad\text{,,}\quad \text{OY};\ -\Delta P \cdot x \cos \gamma; \\ &\quad\text{,,}\quad \text{OZ};\ \Delta P (x \cos \beta - y \cos \alpha). \end{aligned}$$

Summing and integrating those moments, the following are found to be the total moments:—

$$\left.\begin{aligned} &\text{round OX};\ M_1 = a \cdot \cos \gamma \iint x y \cdot dx\,dy \\ &\quad\text{,,}\quad \text{OY};\ M_2 = -a \cos \gamma \iint x^2 \cdot dx\,dy \\ &\quad\text{,,}\quad \text{OZ};\ M_3 = a \left\{ \cos \beta \iint x^2 \cdot dx\,dy - \cos \alpha \iint xy \cdot dx\,dy \right\} \end{aligned}\right\} (3.)$$

For the sake of brevity, let

$$\iint x^2 \cdot dx\,dy = \mathrm{I};\ \iint x y \cdot dx\,dy = \mathrm{K};\ \ldots\ldots(3\text{A}.)$$

then, as in equation 7 of Article 60, we find, for the moment of the resultant couple,

$$\begin{aligned} M &= \sqrt{(M_1^2 + M_2^2 + M_3^2)} \\ &= a \cdot \sqrt{\{(\mathrm{I}^2 + \mathrm{K}^2) \cos^2 \gamma + \mathrm{I}^2 \cdot \cos^2 \beta + \mathrm{K}^2 \cdot \cos^2 \alpha} \\ &\qquad - 2 \mathrm{I} \mathrm{K} \cdot \cos \alpha \cdot \cos \beta.\} \\ &= a \sqrt{(\mathrm{I}^2 \cdot \sin^2 \alpha + \mathrm{K}^2 \cdot \sin^2 \beta - 2 \mathrm{I} \mathrm{K} \cdot \cos \alpha \cos \beta)};\ldots(4.) \end{aligned}$$

and for the angles λ, μ, ν, made by the axis of that couple with the axes of co-ordinates, we find the angles whose cosines are as follows:—

$$\cos \lambda = \frac{M_1}{M};\ \cos \mu = \frac{M_2}{M};\ \cos \nu = \frac{M_3}{M} \ldots\ldots\ldots\ldots(5.)$$

The following equation is easily verified:—

$$\cos\alpha\cos\lambda + \cos\beta\cos\mu + \cos\gamma\cos\nu = 0 \ldots\ldots\ldots (5\text{A}).$$

This indicates what is of itself obvious; that the axis of the resultant couple M is perpendicular to the direction of the stress.

The following form is often the most convenient for the constant a. Let p_1 be the intensity of the stress at some fixed distance, x_1, from the neutral axis; then

$$a = \frac{p_1}{x_1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.)$$

92. **Moment of Bending Stress.**—If the uniformly varying stress be normal to the surface at which it acts; that is to say in symbols, if

$$\cos\alpha = 0;\ \cos\beta = 0;\ \cos\gamma = 1; \ldots\ldots\ldots\ldots (1.)$$

then it is evident that

$$\mathrm{M}_3 = 0;\ \cos\nu = 0; \ldots\ldots\ldots\ldots\ldots\ldots (2.)$$

or in words, that the axis of the resultant couple is in the plane of the surface AAA. Such a stress as this is called a *bending stress*, for reasons which will be explained in treating of the strength of materials. The equations of Article 91, when applied to this case, become as follows:—

$$\left.\begin{array}{c} \mathrm{M}_1 = a\,\mathrm{K};\ \mathrm{M}_2 = -a\,\mathrm{I}; \\ \mathrm{M} = a \cdot \sqrt{(\mathrm{I}^2 + \mathrm{K}^2)}; \\ \cos\lambda = \sin\mu = \dfrac{\mathrm{K}}{\sqrt{\mathrm{I}^2 + \mathrm{K}^2}}; \\ \cos\mu = \sin\lambda = \dfrac{-\mathrm{I}}{\sqrt{\mathrm{I}^2 + \mathrm{K}^2}}; \\ \therefore \tan\mu = -\dfrac{\mathrm{K}}{\mathrm{I}}; \end{array}\right\} \ldots\ldots\ldots (3.)$$

If the figure AAA is symmetrical on either side of the axis OX, then for every point at which y has a given positive value, there is a corresponding point for which y has a negative value of equal amount; so that for such a figure

$$\mathrm{K} = \iint xy \cdot dx\,dy = 0,$$

and the same equation may be fulfilled also for certain unsymmetrical figures. In this case we have

$$\mathrm{M}_1 = 0;\ \mathrm{M} = \mathrm{M}_2 = -a\,\mathrm{I};\ \mu = 0; \ldots\ldots\ldots\ldots (4.)$$

so that the axis of the couple coincides with the neutral axis.

93. **Moment of Twisting Stress.**—If the stress be tangential, its tendency is obviously to *twist* the surface A A A about the axis O Z. In this case we have

$$\left.\begin{array}{l}\cos\gamma = 0\,;\ \cos\alpha = \sin\beta\,;\ \cos\beta = \sin\alpha\,;\\ M_1 = 0\,;\ M_2 = 0\,;\\ M = M_3 = a\,(I\sin\alpha - K\cos\alpha)\,;\\ \cos\lambda = 0\,;\ \cos\mu = 0\,;\ \cos\nu = 1.\end{array}\right\}\text{......(1.)}$$

In the cases referred to in Article 92, for which $K = 0$, we find

$$M = a\,I\sin\alpha\,;\text{........................(2.)}$$

so that in these cases it is only the component of the stress parallel to the neutral axis which produces the twisting couple.

94. **Centre of Uniformly Varying Stress.**—When the amount of an uniformly varying stress has magnitude, that stress may be considered as made up of two parts, viz. :—

First, an uniform stress, whose intensity is the *mean* intensity of the entire stress, and whose centre is the centre of figure, O, of the surface of action. As in Article 88, equation 5, this mean intensity may be represented by

$$p_0 = \frac{P}{S} = \frac{\text{total stress}}{\text{area}}\text{..................(1.)}$$

Secondly, an uniformly-varying stress, whose neutral axis traverses O, whose amount is $= 0$, and whose intensity, p', at a given point, is the *deviation* of the intensity at that point from the mean; so that the intensity of the entire stress is given by the equation

$$p = p_0 + p' = p_0 + a\,x\ \text{..................(2.)}$$

Let M be the moment of this second part of the stress; its effect as has been already shown in Article 60, case 2, is to shift th resultant P parallel to itself through a distance

$$L = \frac{M}{P}\text{..............................(3.)}$$

to the opposite side to that whose name designates the tendency o the couple M; and the direction of the line L is perpendicular a once to that of the stress, and to that of the axis of the couple M.

The co-ordinates relatively to the point O of the centre of stres as thus shifted, being the point where the line of action of th shifted resultant cuts the plane of A A A, are most easily found b adapting the equation 2 of Art. 89 to the present case, as follows:—

$$\left.\begin{array}{l}\left.\begin{array}{c}\text{perpendicular}\\ \text{to the}\\ \text{neutral axis}\end{array}\right\} x_0 = \dfrac{\int\int x p' \cdot dx\,dy}{P} = \dfrac{a\int\int x^2 \cdot dx\,dy}{P} = \dfrac{a\mathrm{I}}{P};\\ \left.\begin{array}{c}\text{along the}\\ \text{neutral axis}\end{array}\right\} y_0 = \dfrac{\int\int y p' \cdot dx\,dy}{P} = \dfrac{a\int\int xy \cdot dx\,dy}{P} = \dfrac{a\mathrm{K}}{P}.\end{array}\right\} (4.)$$

The angle θ which the line joining O and the centre of stress makes with the neutral axis OY, is that whose cotangent is

$$\cot\text{an}\ \theta = \frac{y_0}{x_0} = \frac{\mathrm{K}}{\mathrm{I}} \ldots\ldots\ldots\ldots\ldots\ldots (5.)$$

This line will be called the axis *conjugate* to the neutral axis $-$Y O Y. When K = 0, it is perpendicular to the neutral axis.

95. **Moments of Inertia of a Surface.**—The integral $\mathrm{I} = \int\int x^2 \cdot dx\,dy$ is sometimes called the *moment of inertia* of the surface AAA relatively to the neutral axis — YOY. This is a term adopted from the science of Dynamics for reasons which will afterwards appear. The present Article is intended to point out certain relations which exist amongst the moments of inertia of a plane surface of a given figure relatively to different neutral axes; a knowledge of which relations is useful in the determination of the moment of a bending or twisting stress.

Let A A in fig. 44 represent a plane surface of any figure, O its centre of gravity, YOY, XOX, a pair of rectangular axes crossing each other at O, in any position.

Taking YOY as a neutral axis, let the moment of inertia relatively to it be

$$\left.\begin{array}{l} \mathrm{I} = \int\int x^2 \cdot dx\,dy;\\ \text{let the moment of inertia re-}\\ \text{latively to XOX as a neutral}\\ \text{axis be}\\ \mathrm{J} = \int\int y^2 \cdot dx\,dy;\\ \text{and let}\\ \mathrm{K} = \int\int xy \cdot dx\,dy.\end{array}\right\} (1.)$$

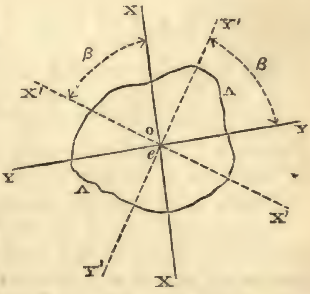


Fig. 44.

Now let Y′OY′, X′OX′, be a new pair of rectangular axes, in any position making the angle

$$\mathrm{YOY'} = \mathrm{XOX'} = \beta$$

with the original pair of axes; and let

$$\left.\begin{aligned} I' &= \iint x'^2 \cdot d\,x'\,d\,y'; \\ J' &= \iint y'^2 \cdot d\,x'\,d\,y'; \\ K' &= \iint x'\,y' \cdot d\,x'\,d\,y'. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(2.)$$

The following relations exist between the original co-ordinates, x, y, of a given point, and the new co-ordinates x', y', of the same point;

$$\left.\begin{aligned} x' &= x \cos \beta - y \sin \beta; \\ y' &= x \sin \beta + y \cos \beta; \\ x'^2 + y'^2 &= x^2 + y^2. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(3.)$$

(This last quantity, which is the square of the distance of the given point from O, is what is called an *Isotropic Function* of the co-ordinates; being of equal magnitude in whatsoever position the rectangular co-ordinates are placed.)

From the equations (3), the following relations are easily deduced between the original integrals I, J, K, and the new integrals I′, J′, K′:—

$$\left.\begin{aligned} I' &= I \cdot \cos^2 \beta + J \cdot \sin^2 \beta - 2\,K \cdot \cos \beta \sin \beta; \\ J' &= I \cdot \sin^2 \beta + J \cdot \cos^2 \beta + 2\,K \cdot \cos \beta \sin \beta; \\ K' &= (I - J) \cos \beta \cdot \sin \beta + K\,(\cos^2 \beta - \sin^2 \beta.) \end{aligned}\right\} \ldots(4.)$$

Also, the following functions of those integrals are found to be *isotropic;*

$$I + J = I' + J' = \iint (x^2 + y^2) \cdot d\,x\,d\,y \ldots\ldots(5.)$$

(called the *polar moment of inertia*);

$$I\,J - K^2 = I'\,J' - K'^2 \ldots\ldots\ldots\ldots\ldots\ldots(6.)$$

Equation 5 may be thus expressed in words:—

THEOREM I. *The sum of the moments of inertia of a surface relatively to a pair of rectangular neutral axes is isotropic.*

Equations 5 and 6 in conjunction lead to the following consequences. Because the sum I′ + J′ is constant, I′ must be a maximum and J′ a minimum for that position of the rectangular axes which makes the difference I′ — J′ a maximum. And because

$$(I' - J')^2 = (I' + J')^2 - 4\,I'\,J',$$

I′ — J′ must be a maximum for that position of the axis which makes I′ J′ a minimum. But by equation 6, I′ J′ — K′² is constant for all positions of the axes; therefore when $K' = 0$, $I'J'$ is a minimum, $I' - J'$ a maximum, I' a maximum, and J' a minimum.

Hence follows, in the first place,

THEOREM II. *In every plane surface there is a pair of rectangular neutral axes for one of which the moment of inertia is greater, and for the other less, than for any other neutral axis.*

These axes are called *Principal Axes.* Let I_1, J_1, be the maximum and minimum moments of inertia relatively to them, and let β_1 be the angle which their position makes with the originally-assumed axes; then because $K_1 = 0$, we have, from the third of the equations (4)

$$\tan 2\beta_1 = \frac{2\cos\beta_1 \sin\beta_1}{\cos^2\beta_1 - \sin^2\beta_1} = \frac{-2K}{I - J} \dots\dots\dots\dots (7.)$$

and because $I_1 + J_1 = I + J$, and $I_1 J_1 = IJ - K^2$, we have, by the solution of a quadratic equation,

$$\left.\begin{aligned} I_1 &= \frac{I + J}{2} + \sqrt{\left\{\frac{(I - J)^2}{4} + K^2\right\}} \dots\dots \\ J_1 &= \frac{I + J}{2} - \sqrt{\left\{\frac{(I - J)^2}{4} + K^2\right\}} \dots\dots \end{aligned}\right\} (8.)$$

The position of the principal axes, and the values of I_1, J_1, being once known, the integrals I', J', K', for any pair of axes which make the angle β' with the principal axes, are given by the equations

$$\left.\begin{aligned} I' &= I_1 \cos^2\beta' + J_1 \sin^2\beta'; \\ J' &= I_1 \sin^2\beta' + J_1 \cos^2\beta'; \\ K' &= (I_1 - J_1)\cos\beta' \sin\beta'. \end{aligned}\right\} \dots\dots\dots\dots\dots (9.)$$

If $I_1 = J_1$, then $I' = J' = I_1$, and $K' = 0$, for all axes whatsoever; and the given figure may be said to have its moment of inertia *completely isotropic.*

Next, as to *Conjugate Axes.* By equation 5, Article 94, we have for the angle which the axis conjugate to OY makes with OY

$$\text{cotan}\ \theta = \frac{K}{I}.$$

For the principal axes, $K = 0$, $\text{cotan}\ \theta = 0$, and θ is a right angle; from which follows—

THEOREM III. *The principal axes are conjugate to each other:*—that is, if either of them be taken for neutral axis, the other will be the conjugate axis.

Returning to equation 4 of the present Article, let us suppose, that the axis conjugate to the originally assumed neutral axis YOY, has been determined, and that its position is Y′OY′, so that

$$\beta = \theta.$$

Let this conjugate axis be assumed as a new neutral axis. Then the integrals I′, J′, K′, belonging to it are determined by substituting θ for β in the equation 4; that is, by substituting for cos β and sin β, the values of cos θ and sin θ in terms of K and I, viz.:—

$$\cos\theta = \frac{K}{\sqrt{I^2+K^2}};\ \sin\theta = \frac{I}{\sqrt{I^2+K^2}}.$$

which substitution having been made, we find

$$\left.\begin{aligned} I' &= \frac{I\,(I\,J - K^2)}{I^2+K^2} \\ K' &= \frac{-\,K\,(I\,J - K^2)}{I^2+K^2} \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (10.)$$

Now let it be required to find the angle θ', which the *new conjugate axis* makes with the *new neutral axis* Y′O Y′. This angle is given by the equation

$$\text{cotan}\,\theta' = \frac{K'}{I'} = -\,\frac{K}{I} = -\,\text{cotan}\,\theta,$$

whence

$$\theta' = -\,\theta, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11.)$$

or in words,

THEOREM IV. *If the axis conjugate to a given neutral axis be taken as a new neutral axis, the original neutral axis will be the new conjugate axis.*

The following mode of graphically representing the preceding theorems and relations depends on well known properties of the ellipse.

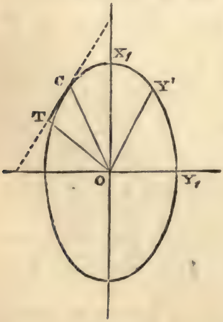


Fig. 45.

In fig. 45, let O X_1 O Y_1 perpendicular to each other, represent the principal axes of a surface. With the semi-axes,

$$\left.\begin{aligned} a &= \overline{O\,X_1} = \sqrt{I_1} \\ b &= \overline{O\,Y_1} = \sqrt{J_1} \end{aligned}\right\} \ldots (12.)$$

describe an ellipse, so that the square of each semi-axis shall represent the moment of inertia round the other.

Let the semidiameter O Y′ be drawn in the direction of any assumed neutral axis, and let $\angle\ Y_1 O\, Y' = \beta'$. Draw O C, the semidiameter conjugate to OY', so that the tangent CT shall be parallel to OY'. Let $\overline{CT} = t$, and let the normal $\overline{OT} = n$. Then it is well known that 9

and that

$$\left.\begin{aligned} n^2 &= a^2 \cdot \cos^2 \beta' + b^2 \sin^2 \beta'; \\ n\,t &= (a^2 - b^2) \cdot \cos \beta' \cdot \sin \beta'; \end{aligned}\right\} \text{............(13.)}$$

consequently, comparing this equation with the equation 9, we find,

$$\left.\begin{aligned} I' &= n^2; \\ K' &= n\,t; \\ \text{cotan}\ \theta = \frac{K}{I} &= \frac{t}{n} = \text{cotan}\ Y'OC; \end{aligned}\right\} \text{......(14.)}$$

so that the square of the normal OT represents the moment of inertia for the neutral axis OY', and the semidiameter $\overline{OC}$ conjugate to OY' is also the conjugate axis of the neutral axis OY'. and *vice versâ*. 10

In finding the moment of inertia of a surface of complex figure, it may sometimes be desirable to divide it into parts, each of more simple figure, find the moment of inertia of each, and add the results together.

In a case of this kind, the neutral axis of the whole surface will not necessarily traverse the centre of gravity of each of its parts, and it becomes necessary to use formulæ for finding the moment of inertia of a figure relatively to an axis not traversing its centre of gravity.

Let OY denote such an axis, x the distance of any point of the given figure from it, and x_0 the distance of the centre of gravity of the given figure from the axis OY. Through that centre of gravity conceive an axis $O'Y'$ to be drawn parallel to OY; the point which is at the distance x from OY, is at the distance

$$x' = x - x_0$$

from $O'Y'$.

The required moment of inertia is

$$I = \iint x^2\,dx\,dy;$$

but

$$x^2 = x_0^2 + 2\,x_0\,x' + x'^2;$$

therefore,

$$I = x_0^2\,S + 2\,x_0 \iint x' \cdot dx\,dy + \iint x'^2 \cdot dx\,dy;$$

and because $O'Y'$ traverses the centre of gravity of S,

$$\iint x' \cdot dx\,dy = 0;$$

so that the middle term of the expression for I vanishes, leaving

$$\mathrm{I} = x_0^2 \mathrm{S} + \int\int x'^2 \cdot d\,x\,d\,y\,; \ldots\ldots\ldots\ldots\ldots (15.)$$

or in words,—

THEOREM V. *The moment of inertia of a surface relatively to an axis not traversing its centre of gravity is greater than the moment of inertia round a parallel axis traversing its centre of gravity, by the product of the area of the surface into the square of the distance between those two axes.*

The following is a table of the *principal* (or maxima and minima) *moments of inertia* of surfaces-of-action of stress of those figures which most commonly occur in practice :—

Figure.	Maximum I_1 (neutral axis O Y).	Minimum J_1 (neutral axis O X).
I. RECTANGLE.—Length along O X, h; breadth along O Y, b..........	$\frac{h^3 b}{12}$	$\frac{h b^3}{12}$
II. SQUARE.—Side $= h$..................	$\frac{h^4}{12}$	$\frac{h^4}{12}$
III. ELLIPSE.—Longer axis, h...... Shorter axis, b......	$\frac{\pi h^3 b}{64}$	$\frac{\pi h b^3}{64}$
(N.B.—$\frac{\pi}{64} = \frac{1}{20 \cdot 4}$ *nearly*).		
IV. CIRCLE.—Diameter, h..............	$\frac{\pi h^4}{64}$	$\frac{\pi h^4}{64}$
V. Hollow symmetrical figures; subtract I or J for inner figure, from I or J for outer figure.		
VI. Symmetrical assemblage of rectangles; dimensions of any one $h \parallel x$, $b \parallel y$; distance of its centre from O Y, x_0; from O X, y_0......	$\Sigma \cdot \frac{h^3 b}{12} + \Sigma \cdot h\,b\,x_0^2$	$\Sigma \cdot \frac{h b^3}{12} + \Sigma \cdot h\,b\,y_0^2$.

SECTION 3.—*Of Internal Stress, its Composition and Resolution.*

96. **Internal Stress in General.**—If a body be conceived to be divided into two parts by an ideal plane traversing it in any direction, the force exerted between those two parts at the plane of division is an *internal stress.* The finding of the resultant, and of the centre of stress, for an internal stress, depend upon the principles relating to stress in general, which have been explained in the last section. The present section refers to a different class of problems, viz., the relations between the different stresses which can exist together in one body at one point.

A body may be divided into two parts by a plane traversing a given point, in an indefinite number of ways, by varying the angular position of the plane; and the stress which acts between the two parts may vary in direction, or intensity, or in both, as the position of the plane varies. The object of the present section is to show the laws of such variation ; and also the effect of applying different stresses simultaneously to one body.

The investigations in this section relate strictly to stress of *uniform intensity ;* but their results are made applicable to stress of variable intensity to any required degree of accuracy, by sufficiently contracting the space under consideration, so that the variations of the stress within its limits shall not exceed the assigned limits of deviation from uniformity.

97. **Simple Stress and its Normal Intensity.**—A simple stress is a pull or a thrust. In the following investigations a pull will be treated as positive, and a thrust as negative.

In fig. 46, let a prismatic solid body, or part of a solid body, whose sides are parallel to the axis O X, be kept in equilibrio by a pull applied in opposite directions to its two ends, of uniform intensity, and of the amount P.

Let an ideal plane A A, perpendicular to O X, be conceived to divide the body into two parts, and let the area of that plane of section be S. That each of these parts may be in equilibrio, it is necessary that they should act upon each other, at the plane of section A A, with a pull in the direction O X, of the amount P, and of the intensity

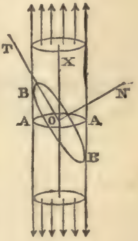


Fig. 46.

$$p_x = \frac{P}{S}.$$

This, which is the intensity of the stress as distributed over a plane normal to its direction, may be called its *normal intensity.*

98. **Reduction of Simple Stress to an Oblique Plane.**—Next, let the plane of section be conceived to have the position B B, oblique to O X ; let O N be a line normal to B B, and O T a line at the intersection of the planes B B and X O N. Let the *obliquity* of the plane of section be denoted by

$$\theta = \angle X O N = \angle T O A.$$

The two parts into which B B divides the body must exert on each other, as in the former case, a pull of the amount P, and in the direction O X ; but the area over which that pull is distributed is now

$$\text{area B B} = \frac{S}{\cos\theta};$$

consequently, the intensity of the stress, as *reduced to the oblique plane of section*, is

$$p_r = \frac{P\cos\theta}{S} = p_x \cdot \cos\theta.$$

99. **Resolution of Oblique Stress into Normal and Tangential Components.**—The oblique stress P on the plane of section B B may be resolved by the principles of Articles 55, 57, into two components, viz. :—

Normal component along O N, ...	$P\cos\theta$;
Tangential component along O T,............	$P\sin\theta$;

and the intensities of these components are,

$$\left.\begin{array}{ll} \text{Normal ;} & p_n = p_r \cos\theta = p_x \cdot \cos^2\theta\,; \\ \text{Tangential ;} & p_t = p_r \sin\theta = p_x \cdot \cos\theta \sin\theta \end{array}\right\} \ldots\ldots(1.)$$

Suppose another oblique plane of section to cut the body at right angles to B B, so that its obliquity is

$$\theta' = 90^0 - \theta;$$

and let the intensity of the stress on the new plane be denoted by accented letters ; then

$$\left.\begin{array}{l} p'_n = p_x \cdot \cos^2\theta' = p_x \cdot \sin^2\theta\,; \\ p'_t = p_t\,;\ p_n + p'_n = p_x\,; \end{array}\right\} \ldots\ldots\ldots\ldots\ldots(2.)$$

so that we obtain the following

THEOREM. *On a pair of planes of section whose obliquities are together equal to a right angle, the tangential components of a simple stress are of equal intensity, and the intensities of the normal components are together equal to the normal intensity of the stress.*

100. **Compound Stress** is that internal condition of a body which is made by the combined action of two or more simple stresses in different directions. A compound stress is known when the directions and the intensities, relatively to given planes, of the simple stresses composing it are known. The same compound stress may be analyzed (as the ensuing Articles will show) into groups of simple stresses, in different ways ; such groups of simple stresses are said to be *equivalent* to each other. The problems of finding of a group of stresses equivalent to another, and of determining the relations which must exist between co-existing stresses, are solved by considering the conditions of equilibrium of some internal part of the solid, of prismatic or pyramidal figure, bounded by ideal planes.

101. **Pair of Conjugate Stresses.**—THEOREM. *If the stress on a given plane in a body be in a given direction, the stress on any plane parallel to that direction must be in a direction parallel to the first-mentioned plane.*

In fig. 47, let Y O Y represent, in section, a given plane traversing a body, and let the stress on that plane be in the direction X O X. Consider the condition of a prismatic portion of the body represented in section by A B C D, bounded by a pair of planes A B, D C, parallel to the given plane, and a pair of planes A D, B C, parallel to each other and to the given direction X O X, and having for its axis a line in the plane Y O Y, cutting X O X in O.

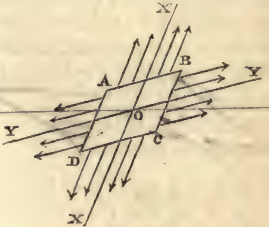


Fig. 47.

The equal resultant forces exerted by the other parts of the body on the faces A B and D C of this prism are directly opposed, their common line of action traversing the axis O; and they are therefore independently balanced. Therefore the forces exerted by the other parts of the body on the faces A D and B C of the prism must be independently balanced, and have their resultants directly opposed; which cannot be unless their direction is parallel to the plane Y O Y. Therefore, &c.—Q. E. D.

A pair of stresses, each acting on a plane parallel to the direction of the other, are said to be *conjugate*. In a rigid body, it is evident that their intensities are independent of each other, and that they may be of the same, or of opposite kinds:—a pair of pulls, a pair of thrusts, or a pull and a thrust.

In those cases (of frequent occurrence in practice) in which the planes of action of a pair of conjugate stresses are both perpendicular to the plane which contains their two directions, their obliquity is the same, being the complement of the angle which they make with each other.

102. **Three Conjugate Stresses** may act together in one body, the direction of each being parallel to the line of intersection of the planes of action of the other two; and in a rigid body, the kinds and intensities of those stresses are independent of each other. Thus, in fig. 47, if X O X and Y O Y represent the directions of two stresses, each acting on a plane which traverses the direction of the other, the intersection of those planes (which may make any angle with X O X and Y O Y), will give a third direction, being that of a third stress of either kind and of any intensity, which may act on the plane X O Y, and will be conjugate to each of the other two.

Three is the greatest number of a group of conjugate stresses; for it is evidently impossible to introduce a fourth stress which shall be conjugate at once to each of the other three.

The relations between the three angles which the directions of three conjugate stresses make with each other, the three *obliquities* of those stresses (being the angles which they make with the perpendiculars to their respective planes of action), and the three angles which those perpendiculars make with each other, as found by the ordinary rules of spherical trigonometry, are given by the following formulæ.

General Case. Let x, y, z, denote the directions of the three conjugate stresses;

$\widehat{yz}$, $\widehat{zx}$, $\widehat{xy}$, their inclinations to each other;

u, v, w, the directions of the perpendiculars to their planes of action, so that $u \perp$ plane yz, $v \perp$ plane zx, $w \perp$ plane xy;

$\widehat{vw}$, $\widehat{wu}$, $\widehat{uv}$, the inclinations of those perpendiculars to each other;

$\widehat{ux}$, $\widehat{vy}$, $\widehat{wz}$, the respective obliquities of the stresses.

Then those nine angles are related as follows:—

$$\text{Let } 1 - \cos^2 \widehat{yz} - \cos^2 \widehat{zx} - \cos^2 \widehat{xy} + 2 \cos \widehat{yz} \cos \widehat{zx} \cos \widehat{xy} = \mathrm{C}; \quad \text{......................(1.)}$$

Then

$$\left.\begin{aligned}
\sin \widehat{vw} &= \frac{\sqrt{\mathrm{C}}}{\sin \widehat{zx} \cdot \sin \widehat{xy}}; \quad \cos \widehat{vw} = \frac{\cos \widehat{zx} \cdot \cos \widehat{xy} - \cos \widehat{yz}}{\sin \widehat{zx} \cdot \sin \widehat{xy}};\\
\sin \widehat{wu} &= \frac{\sqrt{\mathrm{C}}}{\sin \widehat{xy} \cdot \sin \widehat{yz}}; \quad \cos \widehat{wu} = \frac{\cos \widehat{xy} \cdot \cos \widehat{yz} - \cos \widehat{zx}}{\sin \widehat{xy} \cdot \sin \widehat{yz}};\\
\sin \widehat{uv} &= \frac{\sqrt{\mathrm{C}}}{\sin \widehat{yz} \cdot \sin \widehat{zx}}; \quad \cos \widehat{uv} = \frac{\cos \widehat{yz} \cdot \cos \widehat{zx} - \cos \widehat{xy}}{\sin \widehat{yz} \cdot \sin \widehat{zx}}.
\end{aligned}\right\} \text{(2.)}$$

$$\cos \widehat{ux} = \frac{\sqrt{\mathrm{C}}}{\sin \widehat{yz}}; \quad \cos \widehat{vy} = \frac{\sqrt{\mathrm{C}}}{\sin \widehat{zx}}; \quad \cos \widehat{wz} = \frac{\sqrt{\mathrm{C}}}{\sin \widehat{xy}} \text{......(3.)}$$

Restricted Case I. Suppose two of the stresses, for example, those parallel to x and y, to be perpendicular to each other, and oblique to the third. Then

$$\left.\begin{aligned}
\cos \widehat{xy} = 0; \ \sin \widehat{xy} = 1;\\
\mathrm{C} = 1 - \cos^2 \widehat{yz} - \cos^2 \widehat{zx};
\end{aligned}\right\} \text{............(4.)}$$

$$\left.\begin{aligned}
\sin \widehat{vw} &= \frac{\sqrt{C}}{\sin \widehat{zx}}; \quad \cos \widehat{vw} = \frac{-\cos \widehat{yz}}{\sin \widehat{zx}}; \\
\sin \widehat{wu} &= \frac{\sqrt{C}}{\sin \widehat{yz}}; \quad \cos \widehat{wu} = \frac{-\cos \widehat{zx}}{\sin \widehat{yz}}; \\
\sin \widehat{uv} &= \frac{\sqrt{C}}{\sin \widehat{yz} \cdot \sin \widehat{zx}}; \quad \cos \widehat{uv} = \frac{\cos \widehat{yz} \cdot \cos \widehat{zx}}{\sin \widehat{yz} \cdot \sin \widehat{zx}};
\end{aligned}\right\} \dots\dots\dots (5.)$$

$$\cos \widehat{ux} = \frac{\sqrt{C}}{\sin \widehat{yz}}; \; \cos \widehat{vy} = \frac{\sqrt{C}}{\sin \widehat{zx}}; \; \cos \widehat{wz} = \sqrt{C} \dots (6.)$$

Restricted Case II. Suppose one of the stresses (such as z) to be perpendicular to the other two, which are oblique to each other. Then

$$\left.\begin{aligned}
\cos \widehat{yz} = 0; \; \cos \widehat{zx} = 0; \\
\sin \widehat{yz} = 1; \; \sin \widehat{zx} = 1; \\
C = \sin^2 \widehat{xy}.
\end{aligned}\right\} \dots\dots\dots\dots\dots\dots (7.)$$

$$\left.\begin{aligned}
&\sin \widehat{vw} = 1; \; \cos \widehat{vw} = 0; \; (\text{or } \widehat{vw} = 90^0); \\
&\sin \widehat{wu} = 1; \; \cos \widehat{wu} = 0; \; (\text{or } \widehat{wu} = 90^0); \\
&\sin \widehat{uv} = \sin \widehat{xy}; \; \cos \widehat{uv} = -\cos \widehat{xy}; \\
&\qquad (\text{or, } \widehat{uv} + \widehat{xy} = 180^0).
\end{aligned}\right\} \dots\dots\dots (8.)$$

$$\left.\begin{aligned}
&\cos \widehat{ux} = \sin \widehat{xy}; \; \cos \widehat{vy} = \sin \widehat{xy}; \; \cos \widehat{wz} = 1; \\
&\text{or } \widehat{ux} = \widehat{vy} = 90^0 - \widehat{xy}; \; \widehat{wz} = 0;
\end{aligned}\right\} \dots (9.)$$

results identical with those given at the end of Article 101.

Restricted Case III. All three stresses perpendicular to each other. In this case the normals to the three planes of action are perpendicular to each other, and coincide with the directions of the stresses.

103. **Planes of Equal Shear, or Tangential Stress.**—Theorem. *If the stresses on a given pair of planes be tangential to those planes, and parallel to a third plane which is perpendicular to the pair of planes, those stresses must be of equal intensity.*

Let the third plane be represented by the plane of the paper in fig. 48, and let the pair of planes on which the stresses are tangen-

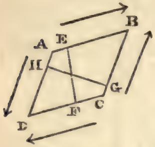


Fig. 48.

tial, and parallel to the plane of the paper, be parallel respectively to AB and AD. Consider the condition of a right prism of any length, represented in section by ABCD, and bounded by a pair of parallel planes, AB, CD, and a pair of parallel planes, AD, CB. Let p_t denote the intensity of the shear or tangential stress on AB, CD, and planes parallel to them, and p'_t the intensity of the shear, or tangential stress on AD, CB, and planes parallel to them. The forces exerted by the other parts of the body on the pair of faces AB, CD, form a couple (right-handed in the figure), of which the arm is the perpendicular distance $\overline{\mathrm{EF}}$, between AB and CD, and the moment,—

$$p_t \cdot \text{area AB} \cdot \overline{\mathrm{EF}}.$$

The forces exerted by the other parts of the body on the pair o faces AD, CB, form a couple (left-handed in the drawing), of which the arm is the perpendicular distance $\overline{\mathrm{GH}}$ between AD and CB, and the moment

$$p'_t \cdot \text{area AD} \cdot \overline{\mathrm{GH}}.$$

The equilibrium of the prism requires that these opposite moments shall be equal. But the products, area AB $\cdot\,\overline{\mathrm{EF}}$, and area AD $\cdot\,\overline{\mathrm{GH}}$ are equal, each of them being the volume of the prism; therefore the intensities of the tangential stresses

$$p_t = p'_t$$

are equal.—Q. E. D.

The above demonstration shows that a shear upon a given plane cannot exist alone as a solitary or simple stress, but must be combined with a shear of equal intensity on a different plane. The tendency of the action of the pair of shearing stresses represented in the figure on the prism ABCD is obviously to *distort* it, by lengthening the diagonal DB, and shortening the diagonal AC, so as to sharpen the angles D and B, and flatten the angles A and C.

104. **Stress on Three Rectangular Planes.**—THEOREM. *If there be oblique stress on three planes at right angles to each other, the tangential components of the stress on any two of those planes in directions parallel to the third plane must be of equal intensity.*

Let $\overline{yz}, \overline{zx}, \overline{xy}$, denote the three rectangular planes whose intersections are the rectangular axes of x, y, and z. Consider the condition of a rectangular portion of the body, having its three pairs of faces parallel respectively to the three planes, and its centre at the point of intersection of the three axes. Let ABCD (fig. 49), represent the section of that rectangular solid by the plane of $\overline{xy}$, the faces

AB, CD being parallel to the plane $\overline{yz}$, and the faces AD, CB, to the plane $\overline{zx}$. Let the equal and parallel lines $\overline{\mathrm{XR}}$ represent the intensities of the forces exerted by the other parts of the body on the pair of faces AB, CD; resolve each of these forces into a component $\overline{\mathrm{XN}}$, parallel to the plane $\overline{zx}$, and a tangential component, $\overline{\mathrm{XT}}$, parallel to the axis of y; the resultants of the components X N will act through the axis of z, and will produce no couple round that axis; the components $\overline{\mathrm{XT}}$ will form a couple acting round that axis. In the same manner the intensities of the forces exerted on the faces AD, CB, being represented by the equal and parallel lines, $\overline{\mathrm{Y}r}$, are resolved into the components, $\overline{\mathrm{Y}n}$, whose resultants act through the axis of z, and the components $\overline{\mathrm{Y}t}$, which form a couple acting round that axis, which, by the conditions of equilibrium of the rectangular solid ABCD, must be equal and opposite to the former couple; and by reasoning similar to that of Article 103, it is shown that the intensities of the tangential stresses constituting these couples,

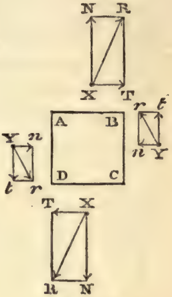


Fig. 49.

$$\overline{\mathrm{XT}} = \overline{\mathrm{Y}t},$$

must be equal; and similar demonstrations apply to the other planes and stresses.

To represent this symbolically:—let p, as before, denote the intensity of a stress; and let small letters affixed below p be used, the first small letter to denote the direction perpendicular to the plane on which the stress acts, and the second to denote the direction of the stress itself:—for example, let p_{yz} denote the intensity of the stress *on* the plane normal to y (that is, the plane $\overline{zx}$), *in* the direction of z. Then resolving the stress on each of the three rectangular planes into three rectangular components, we have the following notation:—

$$\begin{array}{llll} \text{Plane.} & & \text{Direction.} & \\ & x & y & z \\ \overline{yz} \;\ldots\ldots & p_{xx} \;\ldots\ldots & p_{xy} \;\ldots\ldots & p_{xz} \\ \overline{zx} \;\ldots\ldots & p_{yx} \;\ldots\ldots & p_{yy} \;\ldots\ldots & p_{yz} \\ \overline{xy} \;\ldots\ldots & p_{zx} \;\ldots\ldots & p_{zy} \;\ldots\ldots & p_{zz} \end{array} \left.\begin{array}{c} \\ \\ \\ \end{array}\right\} \text{intensities.}$$

Then, in virtue of the Theorems of Articles 101 and 102, we have the *normal stresses*, p_{xx}, p_{yy}, p_{zz}, conjugate and independent; and in virtue of the theorem of this Article, there are *three pairs of tangential stresses of equal intensity,*

$$p_{yz}=p_{zy};\ p_{zx}=p_{xz};\ p_{xy}=p_{yx}.$$

[The reader who wishes to confine his attention to the more simple class of problems may pass at once to Article 108, page 95.]

53 105. **Tetraedron of Stress.**—PROBLEM I. *The intensities of three conjugate stresses on three planes traversing a body being given, it is required to find the direction and intensity of the stress on a fourth plane, traversing the same body in any direction.*

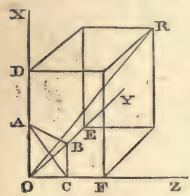


Fig. 50.

In fig. 50, let Y O Z, Z O X, X O Y, be the three planes, on which act conjugate stresses in the directions O X, O Y, O Z, of the intensities p_x, p_y, p_z. Draw a plane parallel to the fourth plane, cutting the three conjugate planes in the triangle A B C, so as to form with them the triangular pyramid or tetraedron O A B C. Then must the stresses on the four triangular faces of that tetraedron balance each other; and the total stress on A B C will be equal and opposite to the resultant of the total stresses on O B C, O C A, and O A B.

On O X, O Y, O Z, respectively take

$$\overline{OD} = \text{total stress on } OBC = p_x \cdot \text{area } OBC,$$

$$\overline{OE} = \text{total stress on } OCA = p_y \cdot \text{area } OCA,$$

$$\overline{OF} = \text{total stress on } OAB = p_z \cdot \text{area } OAB.$$

Complete the parallelopiped O D E F R; then will its diagonal $\overline{OR}$ represent the direction and amount of the total stress on an area of the fourth plane equal to that of A B C; and the intensity of that stress will be $\dfrac{\overline{OR}}{\text{area } ABC}$. Q. E. I.

Hence it appears, that if the stresses on three conjugate planes in a body be given, the stress on any other plane may be determined; from which it follows, *That every possible system of stresses which can co-exist in a body, is capable of being resolved into, or expressed by means of, a system of three conjugate stresses.*

PROBLEM II. *The directions and intensities of the stresses on three rectangular co-ordinate planes being given, it is required to find the direction and intensity of the stress on a fourth plane in any position.*

Let the planes Y O Z, Z O X, X O Y, in fig. 50, represent the rectangular co-ordinate planes, so that O X, O Y, O Z, are *now* at right angles to each other (instead of being, as in Problem I., in any directions). Reduce the three given stresses, as in Article 104, to rectangular components, with the notation already explained.

Let A B C, as in Problem I., be a triangle parallel to the fourth plane, enclosing, with three triangles in the co-ordinate planes, the tetraedron O A B C. The total stress on A B C will be equal and opposite to the resultant of all the rectangular components of the total stresses on O B C, O C A, and O A B.

Therefore, on O X, O Y, O Z, respectively, take

$$\overline{\mathrm{OD}} = p_{xx} \cdot \text{area OBC} + p_{xy} \cdot \text{area OCA} + p_{zx} \cdot \text{area OAB},$$

$$\overline{\mathrm{OE}} = p_{xy} \cdot \text{area OBC} + p_{yy} \cdot \text{area OCA} + p_{yz} \cdot \text{area OAB},$$

$$\overline{\mathrm{OF}} = p_{zx} \cdot \text{area OBC} + p_{yz} \cdot \text{area OCA} + p_{zz} \cdot \text{area OAB};$$

Complete the rectangle O D E F R; then will its diagonal $\overline{\mathrm{OR}}$ represent the direction and amount of the total stress on an area of the fourth plane equal to A B C, and the intensity of that stress will be $\dfrac{\overline{\mathrm{OR}}}{\text{area ABC}}$. Q. E. I.

To express this algebraically, let $\widehat{xn}$, $\widehat{yn}$, $\widehat{zn}$, denote the angles which a normal to the fourth plane makes with the three rectangular axes respectively; $\widehat{xr}$, $\widehat{yr}$, $\widehat{zr}$, the angles which the direction of the stress on that plane makes with the three rectangular axes respectively; and p_r the intensity of that stress. Then, it is well known that

$$\text{area OBC} = \text{area ABC} \cdot \cos \widehat{xn},$$

$$\text{area OCA} = \text{area ABC} \cdot \cos \widehat{yn},$$

$$\text{area OAB} = \text{area ABC} \cdot \cos \widehat{zn};$$

so that the rectangular components of the intensity p_r are

$$\left.\begin{array}{l} p_{nx} = p_{xx} \cdot \cos \widehat{xn} + p_{xy} \cdot \cos \widehat{yn} + p_{zx} \cdot \cos \widehat{zn} \\ p_{ny} = p_{xy} \cdot \cos \widehat{xn} + p_{yy} \cdot \cos \widehat{yn} + p_{yz} \cdot \cos \widehat{zn} \\ p_{nz} = p_{zx} \cdot \cos \widehat{xn} + p_{yz} \cdot \cos \widehat{yn} + p_{zz} \cdot \cos \widehat{zn} \end{array}\right\} \text{.........(1.)}$$

The resultant intensity of the stress required is given by the equation

$$p_r = \sqrt{(p_{nx}^2 + p_{ny}^2 + p_{nz}^2)} \text{.......................(2.)}$$

and its direction by the equation

$$\cos \widehat{xr} = \frac{p_{nx}}{p_r}; \quad \cos \widehat{yr} = \frac{p_{ny}}{p_r}; \quad \cos \widehat{zr} = \frac{p_{nz}}{p_r}. \text{.........(3.)}$$

Hence it appears, that if the rectangular components of the stress on three rectangular planes in a body be given, the stress on any fourth plane may be determined; from which it follows, *That every possible system of stresses which can co-exist in a body, is capable of being resolved into, or expressed by means of, the three normal stresses, and the six pairs of tangential stresses, on three rectangular co-ordinate planes.*

106. **Transformation of Stress.**—For the direction of the normal to the new plane of action, A B C, which direction is denoted by n in Problem II. of Article 105, let there be successively assumed the directions of *three new rectangular axes* x', y', z', and let it be required to express the rectangular components, $p_{x'x'}$, &c., of a given compound stress relatively to those new axes, in terms of the rectangular components, p_{xx}, &c., of the same compound stress relatively to the original rectangular axes, x, y, z.

To solve this question, let n be taken to denote *any one* of the three new axes. The three components, parallel to the original axes, of the stress on the plane normal to n, are given by equation 1 of Article 105. Each of these components being further resolved into its components parallel to the new axes, and the nine components so found collected into three sums of intensities parallel to the new axes, the following results are obtained:—

$$p_{nx'} = p_{nx} \cdot \cos \widehat{xx'} + p_{ny} \cdot \cos \widehat{yx'} + p_{nz} \cdot \cos \widehat{zx'};$$

$$p_{ny'} = p_{nx} \cdot \cos \widehat{xy'} + p_{ny} \cdot \cos \widehat{yy'} + p_{nz} \cdot \cos \widehat{zy'};$$

$$p_{nz'} = p_{nx} \cdot \cos \widehat{xz'} + p_{ny} \cdot \cos \widehat{yz'} + p_{nz} \cdot \cos \widehat{zz'}.$$

For n are now to be substituted successively, both in $p_{nx'}$, &c., and in the values of p_{nx}, &c., according to equation 1 of Article 105, the symbols x', y', z'; and thus are obtained finally the following *equations of transformation*:—

NORMAL STRESSES.—

$$p_{x'x'} = p_{xx} \cos^2 \widehat{xx'} + p_{yy} \cos^2 \widehat{yx'} + p_{zz} \cos^2 \widehat{zx'}$$
$$+ 2\, p_{yz} \cos \widehat{yx'} \cos \widehat{zx'} + 2\, p_{zx} \cos \widehat{zx'} \cos \widehat{xx'} + 2\, p_{xy} \cos \widehat{xx'} \cos \widehat{yx'};$$

$$p_{y'y'} = p_{xx} \cos^2 \widehat{xy'} + p_{yy} \cos^2 \widehat{yy'} + p_{zz} \cos^2 \widehat{zy'}$$
$$+ 2\, p_{yz} \cos \widehat{yy'} \cos \widehat{zy'} + 2\, p_{zx} \cos \widehat{zy'} \cos \widehat{xy'} + 2\, p_{xy} \cos \widehat{xy'} \cos \widehat{yy'};$$

$$p_{z'z'} = p_{xx} \cos^2 \widehat{xz'} + p_{yy} \cos^2 \widehat{yz'} + p_{zz} \cos^2 \widehat{zz'}$$
$$+ 2\, p_{yz} \cos \widehat{yz'} \cos \widehat{zz'} + 2\, p_{zx} \cos \widehat{zz'} \cos \widehat{xz'} + 2\, p_{xy} \cos \widehat{xz'} \cos \widehat{yz'};$$

TANGENTIAL STRESSES.—

$$p_{y'z'} = p_{xx} \cos\widehat{xy'} \cos\widehat{xz'} + p_{yy} \cos\widehat{yy'} \cos\widehat{yz'} + p_{zz} \cos\widehat{zy'} \cos\widehat{zz'}$$
$$+ p_{yz}(\cos\widehat{zy'} \cos\widehat{yz'} + \cos\widehat{yy'} \cos\widehat{zz'}) + p_{zx}(\cos\widehat{xy'} \cos\widehat{zz'} + \cos\widehat{zy'} \cos\widehat{xz'})$$
$$+ p_{xy}(\cos\widehat{yy'} \cos\widehat{xz'} + \cos\widehat{xy'} \cos\widehat{yz'});$$

$$p_{z'x'} = p_{xx} \cos\widehat{xz'} \cos\widehat{xx'} + p_{yy} \cos\widehat{yz'} \cos\widehat{yx'} + p_{zz} \cos\widehat{zz'} \cos\widehat{zx'}$$
$$+ p_{yz}(\cos\widehat{zz'} \cos\widehat{yx'} + \cos\widehat{yz'} \cos\widehat{zx'}) + p_{zx}(\cos\widehat{xz'} \cos\widehat{zx'} + \cos\widehat{zz'} \cos\widehat{xx'})$$
$$+ p_{xy}(\cos\widehat{yz'} \cos\widehat{xx'} + \cos\widehat{xz'} \cos\widehat{yx'});$$

$$p_{x'y'} = p_{xx} \cos\widehat{xx'} \cos\widehat{xy'} + p_{yy} \cos\widehat{yx'} \cos\widehat{yy'} + p_{zz} \cos\widehat{zx'} \cos\widehat{zy'}$$
$$+ p_{yz}(\cos\widehat{zx'} \cos\widehat{yy'} + \cos\widehat{yx'} \cos\widehat{zy'}) + p_{zx}(\cos\widehat{xx'} \cos\widehat{zy'} + \cos\widehat{zx'} \cos\widehat{xy'})$$
$$+ p_{xy}(\cos\widehat{yx'} \cos\widehat{xy'} + \cos\widehat{xx'} \cos\widehat{yy'}).$$

The two systems of component stresses, p_{xx}, &c., relative to the axes x, y, z, and $p_{x'x'}$, &c., relative to the axes x', y', z', which constitute the *same compound stress*, are said to be *equivalent* to each other.

107. **Principal Axes of Stress.** — THEOREM. *For every state of stress in a body, there is a system of three planes perpendicular to each other, on each of which the stress is wholly normal.*

Referring to the equation 3 of Article 105, it is evident that the condition, that the direction of stress on a plane shall coincide with the normal to that plane, is expressed by the equations

$$\cos\widehat{xr} = \frac{p_{nx}}{p_r} = \cos\widehat{xn}; \quad \cos\widehat{yr} = \frac{p_{ny}}{p_r} = \cos\widehat{yn};$$

$$\cos\widehat{zr} = \frac{p_{nz}}{p_r} = \cos\widehat{zn} \text{.....................} (1.)$$

Introducing these values into the equation 1 of Article 105, we obtain the following :—

$$\left.\begin{aligned} (p_{xx} - p_r)\cos\widehat{xn} + p_{xy}\cos\widehat{yn} + p_{zx}\cos\widehat{zn} = 0; \\ p_{xy}\cos\widehat{xn} + (p_{yy} - p_r)\cos\widehat{yn} + p_{yz}\cos\widehat{zn} = 0; \\ p_{zx}\cos\widehat{xn} + p_{yz}\cos\widehat{yn} + (p_{zz} - p_r)\cos\widehat{zn} = 0. \end{aligned}\right\} \text{.....} (2.)$$

From these equations, by elimination of the three cosines, is obtained the following cubic equation ;—

Let
$$\left.\begin{aligned} p_{xx}+p_{yy}+p_{zz}&=\mathrm{A}\,;\\ p_{yy}p_{zz}+p_{zz}p_{xx}+p_{xx}p_{yy}-p_{yz}^2-p_{zx}^2-p_{xy}^2&=\mathrm{B}\,;\\ p_{xx}p_{yy}p_{zz}+2\,p_{yz}p_{zx}p_{xy}-p_{xx}p_{yz}^2-p_{yy}p_{zx}^2-p_{zz}p_{xy}^2&=\mathrm{C}\,;\end{aligned}\right\}(3.)$$

Then
$$p_r^3-\mathrm{A}\,p_r^2+\mathrm{B}\,p_r-\mathrm{C}=0\ldots\ldots\ldots\ldots\ldots\ldots(4.)$$

The solution of this cubic equation gives *three roots*, or values of the stress p_r, which satisfy the condition of being normal to their planes of action; and according to the properties of conjugate stresses stated in Article 102, the directions of those three normal stresses must be perpendicular to each other.—Q. E. D.

The three conjugate normal stresses are called *principal stresses*, and their directions, *principal axes* of stress.

If p_r denote the intensity of one of those principal stresses, the angles which it makes with the originally assumed axes of x, y, z, are found by means of the following equations, deduced by elimination from the equation 2 of this Article :—

$$\cos \overset{\wedge}{x\,n}\,\{p_{zx}p_{xy}+(p_r-p_{xx})\,p_{yz}\}=\cos \overset{\wedge}{y\,n}\,\{p_{xy}p_{yz}+(p_r-p_{yy})\,p_{zx}\}$$
$$=\cos \overset{\wedge}{z\,n}\,\{p_{yz}p_{zx}+(p_r-p_{zz})\,p_{xy}\}\ldots\ldots\ldots\ldots(5.)$$

Let p_1, p_2, p_3, denote the three values of p_r, which satisfy equation 4. Then, from the well known properties of equations, it follows that the co-efficients of that equation have the following values :—

$$\left.\begin{aligned} \mathrm{A}&=p_1+p_2+p_3\,;\\ \mathrm{B}&=p_2p_3+p_3p_1+p_1p_2\,;\\ \mathrm{C}&=p_1p_2p_3.\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots(6.)$$

Hence it appears, that for a given state of stress, the three functions denoted by A, B, C, in the equations 3 and 6, are the same for all positions of the set of rectangular axes of x, y, z, or are *isotropic*, in the sense already explained in Article 95.

Let the principal axes of stress now be taken for axes of rectangular co-ordinates, and denoted by x, y, z; and let it be required to find the direction and the intensity p, of the stress on a plane whose normal makes the angles $\overset{\wedge}{x\,n}$, $\overset{\wedge}{y\,n}$, $\overset{\wedge}{z\,n}$, with those axes. For this purpose the equations 1, 2, and 3, of Article 105, are to be modified by making

$$p_{xx}=p_1\,;\ p_{yy}=p_2\,;\ p_{zz}=p_3\,;\ p_{yz}=p_{zx}=p_{xy}=0.$$

Thus we obtain

$$p \cos \widehat{xp} = p_1 \cos \widehat{xn}\,; \; p \cos \widehat{yp} = p_2 \cos \widehat{yn}\,;$$
$$p \cos \widehat{zp} = p_3 \cos \widehat{zn} \ldots\ldots\ldots\ldots (7.)$$

$$p = \surd \left\{ p_1^2 \cdot \cos^2 \widehat{xn} + p_2^2 \cdot \cos^2 \widehat{yn} + p_3^2 \cdot \cos^2 \widehat{zn} \right\} \ldots (8.)$$

The equations 7 are easily transformed into the following :—

$$\frac{\cos \widehat{xn}}{p} = \frac{\cos \widehat{xp}}{p_1}\,; \; \frac{\cos \widehat{yn}}{p} = \frac{\cos \widehat{yp}}{p_2}\,; \; \frac{\cos \widehat{zn}}{p} = \frac{\cos \widehat{zp}}{p_3} \ldots (9.)$$

Which equations being squared and added, and the square root of the sum extracted, give the following value for the *reciprocal* of the intensity required :—

$$\frac{1}{p} = \surd \left\{ \frac{\cos^2 \widehat{xp}}{p_1^2} + \frac{\cos^2 \widehat{yp}}{p_2^2} + \frac{\cos^2 \widehat{zp}}{p_3^2} \right\} \ldots\ldots\ldots (10.)$$

the well known equation of an *ellipsoid*, in which p_1, p_2, p_3, denote the three semi-axes, and p the semidiameter in any given direction.

The cosine of the *obliquity* of the stress p is given by the equation

$$\cos \widehat{np} = \cos \widehat{xn} \cos \widehat{xp} + \cos \widehat{yn} \cos \widehat{yp} + \cos \widehat{zn} \cos \widehat{zp}$$
$$= p \left\{ \frac{\cos^2 \widehat{xp}}{p_1} + \frac{\cos^2 \widehat{yp}}{p_2} + \frac{\cos^2 \widehat{zp}}{p_3} \right\}$$
$$= \frac{1}{p} (p_1 \cos^2 \widehat{xn} + p_2 \cos^2 \widehat{yn} + p_3 \cos^2 \widehat{zn})\,; \ldots\ldots (11.)$$

and this cosine, by being

$$\left.\begin{array}{l}\text{positive}\\ \text{nothing}\\ \text{negative}\end{array}\right\} \begin{array}{c}\text{indicates}\\ \text{that the}\\ \text{stress } p \text{ is}\end{array} \left\{\begin{array}{l}\text{a pull}\\ \text{a shear}\\ \text{a thrust}\end{array}\right\}.$$

108. **Stress Parallel to One Plane.**—In most practical questions respecting the stress in structures, the directions of the stresses chiefly to be considered are parallel to one plane, to which their planes of action are perpendicular, the remaining stress, if any, being a principal stress, and perpendicular to the plane to which the others are parallel.

The problems concerning the relations amongst stresses parallel to one plane, might be solved by considering them as particular cases of the more general problems respecting stresses in any direction, which have been treated of in Articles 105, 106, and 107; but the complexity of the investigations and results in those Articles, makes it preferable to demonstrate the principles relating to stresses parallel to one plane, independently.

PROBLEM I. *The intensities and directions of a pair of conjugate stresses, parallel to a plane which is perpendicular to their planes of action, being given, it is required to find the direction and intensity of the stress on a fourth plane, perpendicular also to the first mentioned plane.*

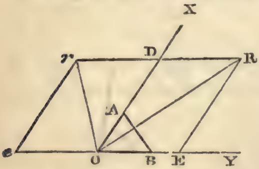


Fig. 51.

In fig. 51, let the plane of the paper represent the plane to which the stresses are parallel; let O X and O Y represent the directions of the pair of conjugate stresses, whose intensities are p_x and p_y; and let A B be the plane, the stress on which is sought. Consider the condition of a prism, O A B, bounded by the plane A B, and by planes parallel to O X and O Y respectively. The force exerted by the other parts of the body on the face O A of the prism, will be proportional to

$$p_y \cdot \overline{\mathrm{OA}};$$

on O Y take $\overline{\mathrm{OE}}$ to represent that force. The force exerted by the other parts of the body on the face O B of the prism, will be proportional to

$$p_x \cdot \overline{\mathrm{OB}};$$

on O X take $\overline{\mathrm{OD}}$ to represent this force. The force exerted by the other parts of the body on the face A B of the prism, must balance the forces exerted on O A and A B; therefore complete the parallelogram O D R E; its diagonal $\overline{\mathrm{OR}}$ will represent the direction and *amount* of the stress on A B, and the *intensity* of that stress will be

$$p_r = \frac{\overline{\mathrm{OR}}}{\overline{\mathrm{AB}}}$$

$$= \sqrt{\left\{\frac{p_x^2 \cdot \overline{\mathrm{OB}}^2 + p_y^2 \cdot \overline{\mathrm{OA}}^2 + 2 p_x p_y \cdot \overline{\mathrm{OB}} \cdot \overline{\mathrm{OA}} \cos \angle \mathrm{XOY}}{\overline{\mathrm{OB}}^2 + \overline{\mathrm{OA}}^2 - 2\, \overline{\mathrm{OB}} \cdot \overline{\mathrm{OA}} \cos \angle \mathrm{XOY}.}\right\}}$$

The parallelogram marked in the figure with the capital letters R, E, corresponds to the case in which p_x and p_y are of the *same* kind, both pulls, or both thrusts, in which case p_r is of the same kind also; the parallelogram marked with the small letters, *r*, *e*, corresponds to the case in which p_x and p_y are of *opposite* kinds, one being a pull and the other a thrust; in which case p_r agrees in kind with that one of the given conjugate stresses whose direction falls to the same side of A B with it. When Or is parallel to A B, p_r is a shear, or tangential stress.

PROBLEM II. *The intensities and directions of the stresses on a pair of planes perpendicular to each other and to a plane to which the stresses are parallel, being given, it is required to find the intensity and direction of the stress on a plane in any position perpendicular to that plane to which the stresses are parallel.*

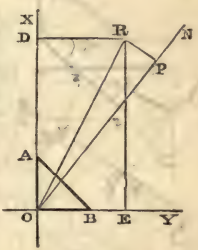


Fig. 52.

In fig. 52, let the plane of the paper represent the plane to which the stresses are parallel, and O X, O Y, the pair of rectangular planes on which the stresses are given. Let those stresses be resolved, as in Article 99, into rectangular normal and tangential components. Let p_{xx} denote the intensity of the normal stress on the plane O Y, which stress is parallel to O X; let p_{yy} denote the intensity of the normal stress on the plane O X, which stress is parallel to O Y.

In virtue of the Theorem of Article 103, the tangential stresses on those two planes must be of equal intensity; and they may therefore be denoted by one symbol, p_{xy}, which symbol may be read as meaning

$$\begin{array}{l}\text{the intensity of} \\ \text{the stress along}\end{array} \left\{\begin{array}{c} x \\ y \end{array}\right\} \begin{array}{l}\text{on a plane} \\ \text{normal to}\end{array} \left\{\begin{array}{c} y \\ x \end{array}\right\}.$$

Let O N be a line normal to the plane the stress on which is sought, making with O X the angle X O N $= \widehat{xn}$. Consider the condition of a prism O A B, of the length *unity*, bounded by the planes O A $\perp y$, O B $\perp x$, A B $\perp$ O N. The areas of the faces of that prism have the following proportions :—

$$\overline{OB} = \overline{AB} \cdot \cos \widehat{xn}; \ \overline{OA} = \overline{AB} \cdot \sin \widehat{xn}.$$

The forces exerted on the faces O A and O B, in a direction parallel to x, consist of the normal stress on O B, and the tangential stress on O A; that is to say,

$$p_{xx} \cdot \overline{OB} + p_{xy} \cdot \overline{OA} = \overline{AB} \cdot \left\{ p_{xx} \cdot \cos \widehat{xn} + p_{xy} \cdot \sin \widehat{xn} \right\}.$$

Let this be represented by $\overline{OD}$.

The forces exerted on the faces O A and O B, in a direction parallel to y, consist of the normal stress on O A, and the tangential stress on O B; that is to say,

$$p_{xy} \cdot \overline{OB} + p_{yy} \cdot \overline{OA} = \overline{AB} \cdot \left\{ p_{xy} \cdot \cos \widehat{xn} + p_{yy} \cdot \sin \widehat{xn} \right\}.$$

Let this be represented by $\overline{OE}$.

Complete the rectangle O D R E; the amount and direction of the stress on A B will be represented by its diagonal,

$$\overline{\mathrm{OR}} = \sqrt{(\overline{\mathrm{OD}}^2 + \overline{\mathrm{OE}}^2)}$$

and the *intensity* of that stress by

$$p_r = \frac{\overline{\mathrm{OR}}}{\overline{\mathrm{AB}}} = \sqrt{\left\{ p_{xx}{}^2 \cdot \cos^2 \widehat{xn} + p_{yy}{}^2 \cdot \sin^2 \widehat{xn} + p_{xy}{}^2 + 2 p_{xy} (p_{xx} + p_{yy}) \cos \widehat{xn} \cdot \sin \widehat{xn} \right\}} \ldots\ldots\ldots\ldots\ldots (1.)$$

From R draw R P perpendicular to the normal O N; then the *normal* and the *tangential components* of the total stress on A B will be represented respectively by

$$\overline{\mathrm{OP}} = \overline{\mathrm{OD}} \cdot \cos \widehat{xn} + \overline{\mathrm{OE}} \sin \widehat{xn};$$
$$\overline{\mathrm{PR}} = \overline{\mathrm{OD}} \sin \widehat{xn} - \overline{\mathrm{OE}} \cdot \cos \widehat{xn};$$

and the *intensities* of these components by

$$\left.\begin{aligned} p_n &= \frac{\overline{\mathrm{OP}}}{\overline{\mathrm{AB}}} = p_{xx} \cdot \cos^2 \widehat{xn} + p_{yy} \cdot \sin^2 \widehat{xn} + 2 p_{xy} \cdot \cos \widehat{xn} \cdot \sin \widehat{xn}; \\ p_t &= \frac{\overline{\mathrm{PR}}}{\overline{\mathrm{AB}}} = (p_{xx} - p_{yy}) \cos \widehat{xn} \cdot \sin \widehat{xn} + p_{xy} (\sin^2 \widehat{xn} - \cos^2 \widehat{xn}). \end{aligned}\right\} (2.)$$

The *obliquity*, $\angle$ N O R $= \widehat{nr}$, of the stress on A B is given by the equation

$$\tan \widehat{nr} = \frac{p_t}{p_n}. \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.)$$

109. **Principal Axes of Stress Parallel to One Plane.**—THEOREM. *For every condition of stress parallel to one plane, there are two planes perpendicular to each other, on which there is no tangential stress.*

As in Article 108, let the three rectangular components, p_{xx}, p_{yy}, p_{xy}, of the stress on two rectangular planes, O Y, O X, be given. The condition, that there shall be no tangential stress on a plane normal to O N, is expressed by making $p_t = 0$ in the second of the equations 2 of that Article; and in order that this may be fulfilled, we must have

$$\frac{\cos \widehat{xn} \cdot \sin \widehat{xn}}{\cos^2 \widehat{xn} - \sin^2 \widehat{xn}} = \frac{p_{xy}}{p_{xx} - p_{yy}};$$

or, what is the same thing,

$$\tan 2 \widehat{xn} = \frac{2 p_{xy}}{p_{xx} - p_{yy}}; \ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

Now for two values of $\widehat{x\,n}$, differing by a right angle, the values of $tan\ 2\,\widehat{x\,n}$ are equal; hence there are two directions of the normal ON perpendicular to each other, which fulfil the condition of having no tangential stress.

Those two directions are called *principal axes of stress*, and the stresses along them (which are conjugate to each other) *principal stresses.*

There may be a third principal stress, conjugate and at right angles to the first two; but as, with one exception, the ensuing investigations of this section relate to stresses upon planes parallel to the direction of this third principal stress, which does not affect such planes, it may be left out of consideration.

The most simple mode of expressing the relations amongst internal stresses parallel to a plane is obtained by taking the two principal axes of stress in that plane for axes of co-ordinates; and this is done in the ensuing Articles.

110. **Equal Principal Stresses—Fluid Pressure.**—THEOREM I. *If a pair of principal stresses be of the same kind and of equal intensity, every stress parallel to the same plane is of the same kind, of equal intensity, and normal to its plane of action.*

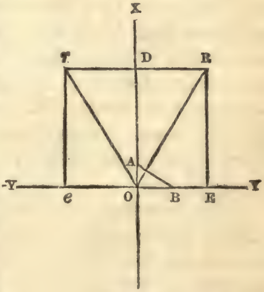


Fig. 53.

In fig. 53, let OX, OY, be the directions of the given principal stresses, and p_x, p_y, their intensities. By the conditions of the question, those intensities are equal, or

$$p_x = p_y.$$

Let it be required to find the direction and intensity of the stress on any plane A B. As in Article 108, consider the condition of the triangular prism O A B; and let the length of that prism, in a direction perpendicular to the plane X O Y be unity. Then the total stresses on the faces $\overline{\mathrm{OB}}$ and $\overline{\mathrm{OA}}$ will be respectively—

$$p_x \cdot \overline{\mathrm{O\,B}} \text{ and } p_y \cdot \overline{\mathrm{O\,A}}.$$

On O X and O Y respectively, take $\overline{\mathrm{O\,D}}$ to represent $p_x \cdot \overline{\mathrm{O\,B}}$, and $\overline{\mathrm{O\,E}}$ to represent $p_y \cdot \overline{\mathrm{O\,A}}$; complete the rectangle O D R E; then its diagonal $\overline{\mathrm{O\,R}}$ will represent the amount and direction of the stress on the face $\overline{\mathrm{A\,B}}$ of the prism, and the intensity of that stress will be

$$\frac{\overline{\mathrm{O\,R}}}{\overline{\mathrm{A\,B}}} = p_r.$$

Now, because $p_x = p_y$, we have

$$\frac{\overline{\mathrm{OD}}}{\overline{\mathrm{OB}}} = \frac{\overline{\mathrm{OE}}}{\overline{\mathrm{OA}}} = \frac{\overline{\mathrm{OR}}}{\overline{\mathrm{AB}}};$$

Sim. triangles

and consequently

$$p_r = p_x = p_y;$$

and because of the similarity of the triangles A O B, O E R, $\overline{\mathrm{OR}}$ is perpendicular to $\overline{\mathrm{AB}}$. Therefore, the stress on each plane perpendicular to X O Y is normal, and of equal intensity in all directions.—Q. E. D.

In this case it is obvious, that every direction in the plane X O Y has the properties of an *axis of stress.*

Corollary. If the stress in all directions parallel to a given plane be normal, it must be of equal intensity in all those directions.

Theorem II. *In a perfect fluid, the pressure at a given point is normal and of equal intensity in all directions.*

Fluid is a term opposed to *solid*, and comprehending the liquid and gaseous conditions of bodies, which have been defined in Article 4. The property common to the liquid and the gaseous conditions is that of *not tending to preserve a definite shape*, and the possession of this property by a body in perfection throughout all its parts, constitutes that body a *perfect fluid.* The parts of a body resisting alteration of shape must exert tangential stress; a perfect fluid does not resist alteration of shape; therefore the parts of a perfect fluid cannot exert tangential stress; therefore the stress exerted amongst and by them at every point and in every direction is normal; therefore at a given point, it is of equal intensity in every direction. —Q. E. D.

This theorem, and its consequences, form the branch of statics called *Hydrostatics*, which is sometimes treated of separately, but which, in this treatise, it has been considered more convenient to include in the subject of the statics of distributed forces in general.

Gaseous fluids always tend to expand, so that the stress in them is always a *pressure.* Liquid fluids are capable of exerting to a slight extent *tension*, or resistance to dilatation, as well as pressure; but in all cases of practical importance in applied mechanics, the only kind of stress in liquids which is of sufficient magnitude to be considered, is *pressure.*

The term *fluid pressure* is used to denote a thrust which is normal and equally intense in all directions round a point.

The idea of perfect fluidity is not absolutely realized by actual liquids, they having all more or less a tendency in their parts to resist distortion, which is called *viscosity*, and which constitutes an approach to the solid condition; nevertheless, in problems of applied hydrostatics, the assumption of perfect fluidity gives results near enough to the truth for practical purposes.

111. **Opposite Principal Stresses.**—THEOREM. *If a pair of principal stresses be of equal intensities, but of opposite kinds, the stress on any plane perpendicular to the plane of the directions of the principal stresses is of the same intensity, and the angles which its direction makes with the normal to its plane are bisected by the axes of principal stress.*

In fig. 53, let the stresses acting along the rectangular axes OX, OY, be as before, of equal intensity; but let them now be, not as before, of the same kind, but of opposite kinds, one being a thrust and the other a pull:—a condition expressed by the equation

$$p_y = -p_x;$$

and let it be required to find the direction and intensity of the stress on the plane A B, to which O R is normal.

In this case $\overline{OD}$ is to be taken as before, to represent $p_x \cdot \overline{OB}$, the total stress on the face $\overline{OB}$ of the triangular prism O A B; but instead of taking $\overline{OE}$ in the direction from O towards B, to represent the total stress on $\overline{OA}$, viz., $p_y \cdot OA$, we are now to take $\overline{Oe}$ of equal length, but in the contrary direction. Complete the rectangle $ODre$; then the diagonal $\overline{Or}$ will represent the total stress on $\overline{AB}$. The intensity of this stress is the same as before, viz.,

$$p_r = p_x;$$

but its *direction* O*r*, instead of being perpendicular to A B, makes an angle XO*r* on one side of the axis OX, equal to the angle XOR which the normal O R makes on the other side of that axis; and O X bisects the angle of obliquity R O*r*.—Q. E. D.

The stress p_r agrees in kind with that one of the principal stresses to which its direction is nearest; and when it makes angles of 45° with each of the axes, it is *shearing* or *tangential;* so that a pull and a thrust of equal intensity, on a pair of planes at right angles to each other, are equivalent to a pair of shearing stresses of the same intensity on a pair of planes at right angles to each other, and making angles of 45° with the first pair.

112. **Ellipse of Stress.**—PROBLEM I. *A pair of principal stresses of any intensities, and of the same or opposite kinds, being given, it is required to find the direction and intensity of the stress on a plane in any position at right angles to the plane parallel to which the two principal stresses act.*

Let O X and O Y (figs. 54 and 55), be the directions of the two principal stresses; O X being the direction of the greater stress.

Let p_x be the intensity of the greater stress;
and p_y that of the less.

$RQ = a$, $RP = b$ $Pm = \sqrt{b^2 - y^2}$

$a : b :: y : \sqrt{b^2-y^2}$

$a\sqrt{b^2-y^2} = by$

$a^2b^2 - a^2y^2 = b^2y^2$

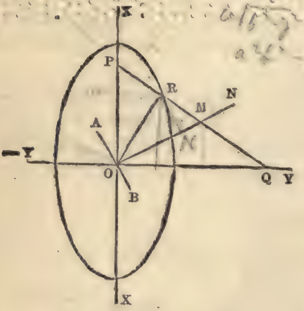


Fig. 54.

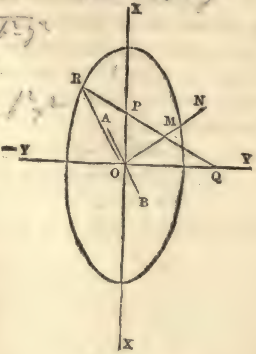


Fig. 55.

The kind of stress to which each of these belongs, pull or thrust, is to be distinguished by means of the algebraical signs. If a pull is considered as positive, a thrust is to be considered as negative, and *vice versâ*. It is in general convenient to consider that kind of stress as positive to which the greater principal stress belongs. Fig. 54 represents the case in which p_x and p_y are of the same kind; fig. 55 the case in which they are of opposite kinds. In all the following equations, the sign of p_y is held to be *implied* in that symbol.

Consider the two equations

$$p_x = \frac{p_x + p_y}{2} + \frac{p_x - p_y}{2};$$

$$p_y = \frac{p_x + p_y}{2} - \frac{p_x - p_y}{2};$$

From these it appears, that the pair of stresses, p_x and p_y, may be considered as made up of two pairs of stresses, viz.:—a pair of stresses of equal intensity and of the same kind, whose common value is $\frac{p_x + p_y}{2}$, and a pair of stresses of equal intensity, but opposite kinds, whose values are $\pm \frac{p_x - p_y}{2}$.

Now let AB be the plane on which it is required to ascertain the direction and intensity of the stress, and ON a normal to that plane, making with the axis of greatest stress the angle

$$\angle \mathrm{XON} = \hat{xn}.$$

On O N take $\overline{\mathrm{OM}} = \frac{p_x + p_y}{2}$; this will represent a normal stress on AB of the same kind with the greater principal stress, and of an intensity which is a mean between the intensities of the two principal stresses; and this, according to Article 110, Theorem I., will be the effect upon the plane AB, of the pair of stresses $\frac{p_x + p_y}{2}$.

Through M draw PMQ, making with the axis of stress the same angles which ON makes, but in the opposite direction; that is to say, take $\overline{\mathrm{MP}} = \overline{\mathrm{MQ}} = \overline{\mathrm{MO}}$. On the line thus found set off from M towards the axis of greatest stress, $\overline{\mathrm{MR}} = \frac{p_x - p_y}{2}$. This, according to Article 111, will represent the direction and the intensity of the oblique stress on AB, which is the effect of the pair of stresses

$$\frac{p_x - p_y}{2}.$$

Join $\overline{\mathrm{OR}}$. Then will that line represent the resultant of the forces represented by OM and MR; that is to say, the direction and intensity of the entire stress on AB.—Q. E. I.

The algebraical expression of this solution is easily obtained by means of the formulæ of plane trigonometry, and consists of the two following equations:—

$$\text{Intensity, } \overline{\mathrm{OR}} \text{ or } p_r = \sqrt{\{p_x^2 \cdot \cos^2 \widehat{x\,n} + p_y^2 \cdot \sin^2 \widehat{x\,n}\}} \ldots\ldots (1.)$$

an equation which might have been obtained by making $p_{xy} = 0$ in equation 1 of Article 108, Problem II.

$$\text{Obliquity, } \angle \mathrm{NOR} \text{ or } \widehat{n\,r}.$$

$$= \text{arc sin} \cdot \left(\sin 2\,\widehat{x\,n} \cdot \frac{p_x - p_y}{2\,p_r}\right) \ldots\ldots\ldots\ldots (2.)$$

This obliquity is always towards the axis of greatest stress.

In fig. 54, p_x and p_y are represented as being of the same kind; and $\overline{\mathrm{MR}}$ is consequently less than $\overline{\mathrm{OM}}$, so that $\overline{\mathrm{OR}}$ falls on the same side of OX with ON, that is to say, $\widehat{n\,r} < \widehat{x\,n}$. In fig. 55, p_x and p_y are of opposite kinds, $\overline{\mathrm{MR}}$ is greater than $\overline{\mathrm{OM}}$, and OR falls on the opposite side of OX to OM; that is to say, $\widehat{n\,r} > \widehat{x\,n}$.

The locus of the point M is obviously a circle of the radius $\frac{p_x + p_y}{2}$, and that of the point R, an ellipse whose semi-axes are p_x and p_y, and which may be called the ELLIPSE OF STRESS, because its semidiameter in any direction represents the intensity of the stress in that direction.

The *principal stresses*, being represented by the semi-axes of this ellipse, are respectively the *greatest* and *least* of the stresses parallel to the plane X O Y.

The *direct* and *shearing*, or *normal* and *tangential* components of $\overline{\mathrm{O\,R}} = p_r$ are found by letting fall a perpendicular from R upon O N, and are as follows:—

$$\textit{Direct, } p_n = p_x \cdot \cos^2 \widehat{n\,x} + p_y \cdot \sin^2 \widehat{x\,n}; \dots\dots\dots\dots (3.)$$

$$\textit{Shearing, } p_t = (p_x - p_y) \cos \widehat{x\,n} \cdot \sin \widehat{x\,n}; \dots\dots\dots (4.)$$

equations which might have been deduced from the equations 2 of Article 108, Problem II.

From equation 3 it is obvious, that *the sum of the normal stresses on a pair of planes at right angles to each other is equal to the sum of the principal stresses*; and from equation 4 follows the principle, already demonstrated otherwise in Article 104, of the equality of the shearing stress on a pair of planes perpendicular to each other.

PROBLEM II. *A pair of principal stresses being given, it is required to find the positions of the planes on which the shear, or tangential component of the stress, is most intense, and the intensity of that shear.* It is evident that the shear is greatest when M R is perpendicular to O M; and then $\overline{\mathrm{M\,R}}$ itself represents the intensity of the shear; that is to say,

$$\text{maximum } p_t = \frac{p_x - p_y}{2} \dots\dots\dots\dots\dots\dots (5.)$$

In this case, A B is either of the two planes which make angles of 45° with the axes of stress.

PROBLEM III. *To find the planes on which the obliquity of the stress is greatest, the intensity of that stress, and the angle of its obliquity.*

CASE 1. *When the principal stresses are of the same kind.* (Fig. 54.) In this case $\overline{\mathrm{M\,R}} \angle \overline{\mathrm{M\,O}}$, and it is evident that the angle of obliquity, $\angle \mathrm{M\,O\,R} = \widehat{n\,r}$ is greatest, when M R is perpendicular to O R, and that its value is given by the equation

$$\text{maximum } \widehat{n\,r} = \text{arc} \cdot \sin \cdot \frac{\overline{\mathrm{M\,R}}}{\overline{\mathrm{O\,M}}}$$

$$= \text{arc} \cdot \sin \frac{p_x - p_y}{p_x + p_y} \dots\dots\dots\dots\dots\dots\dots (6.)$$

To find the *position* of the normal O N to the plane A B, we have to consider that,

$$\widehat{x\,n} = \frac{1}{2} \angle \mathrm{P\,M\,N};$$

$$\text{but } \angle \mathrm{PMN} = \angle \mathrm{MRO} + \angle \mathrm{MOR}$$
$$= 90^\circ + \text{max. } \widehat{nr};$$

consequently in this case,

$$\widehat{xn} = \frac{90^\circ + \text{max. } \widehat{nr}}{2} \dots\dots\dots\dots\dots\dots (7.)$$

(an obtuse angle).

And for the position of the plane AB itself, we have

$$\angle \mathrm{XOA} = 90^\circ - \widehat{xn} = \frac{90^\circ - \text{max. } \widehat{nr}}{2} \dots\dots\dots (8.)$$

(an acute angle).

These equations apply to a pair of planes, making equal angles at opposite sides of O X.

The *intensity* of the most oblique stress is obviously

$$p_r = \sqrt{\left(\overline{\mathrm{OM}}^2 - \overline{\mathrm{MR}}^2\right)}$$
$$= \sqrt{\left\{ \frac{(p_x + p_y)^2}{4} - \frac{(p_x - p_y)^2}{4} \right\}} = \sqrt{(p_x\, p_y)} \dots\dots (9.)$$

or a *mean proportional* between the principal stresses. This is otherwise evident from the consideration, that when $\mathrm{OR} \perp \mathrm{PRQ}$, then $\overline{\mathrm{OR}} = \sqrt{(\overline{\mathrm{PR}} \cdot \overline{\mathrm{RQ}})}$, and that $\overline{\mathrm{RQ}} = p_x$, $\overline{\mathrm{PR}} = p_y$.

Case 2. *When the principal stresses are of opposite kinds* (fig. 55), it is evident, that the most oblique stress possible is a tangential stress, and that the problem amounts to finding the circumstances under which O R lies in the plane AB. In this case it is evident, that the triangle O M R becomes right-angled at O, and consequently, that the intensity of the stress is given by the equation

$$p_r = \sqrt{\left(\overline{\mathrm{MR}}^2 - \overline{\mathrm{OM}}^2\right)} = \sqrt{\left\{ \frac{(p_x - p_y)^2}{4} - \frac{(p_x + p_y)^2}{4} \right\}}$$
$$= \sqrt{(- p_x\, p_y)}, \dots\dots\dots\dots\dots\dots\dots (10.)$$

being, as before, a mean proportional between the principal stresses. The product $- p_x\, p_y$ is a positive quantity, notwithstanding its negative sign, because p_y in this case is implicitly negative.

The position of the normal O N is found by considering, that

$$\widehat{xn} = \frac{1}{2} \angle \mathrm{PMN},$$

and that

$$\angle \mathrm{PMN} = \angle \mathrm{MOR} + \angle \mathrm{MRO}$$
$$= 90^\circ + \text{arc} \cdot \sin \frac{p_x + p_y}{p_x - p_y};$$

consequently,

$$\left.\begin{array}{l} \overset{\wedge}{x n} = \dfrac{1}{2}\left\{90^\circ + \text{arc} \cdot \sin \cdot \dfrac{p_x + p_y}{p_x - p_y}\right\} \\ \text{(an obtuse angle);} \\ \angle \text{XOA} = 90^\circ - \overset{\wedge}{x n} = \dfrac{1}{2}\left\{90^\circ - \text{arc} \cdot \sin \cdot \dfrac{p_x + p_y}{p_x - p_y}\right\} \\ \text{(an acute angle).} \end{array}\right\} \quad (11.)$$

In these, as in the other formulæ applicable to the case in which p_x and p_y are of opposite kinds, it is to be borne in mind that p_y is *implicitly negative,* and that consequently $p_x + p_y$ means the *difference,* and $p_x - p_y$ the *sum,* of the *arithmetical values* of the principal stresses.

PROBLEM IV. *The intensities, kinds, and obliquities, of any two stresses whose planes of action are perpendicular to the plane of their directions, being given, it is required to find the principal stresses and axes of stress.* CASE 1. *When the given stresses are of the same kind, and unequal.*

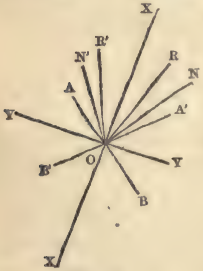


Fig. 56.

In fig. 56, let A B, A′ B′, represent the given planes, O N, O N′, their normals, O R, O R′, the stresses upon them.

Let the intensities be denoted algebraically by

$$p = \text{O R}; \; p' = \text{O R}',$$

and the obliquities by

$$\angle \text{N O R} = \overset{\wedge}{n r}; \; \angle \text{N}' \text{O R}' = \overset{\wedge}{n' r'}.$$

In fig. 57, take O N to represent at once the normals to both planes.

Make $$\angle \text{N O R} = \overset{\wedge}{n r}; \; \angle \text{N O R}' = \overset{\wedge}{n' r'};$$

$$\text{O R} = p; \; \text{O R}' = p'.$$

Join RR′, bisect it in S, from which draw SM ⊥ RR′, cutting O N in M. Join MR, MR′, which lines are evidently equal. Then from a comparison of the construction of this figure with the generation of the ellipse of stress, as described under Problem I., is evident, that

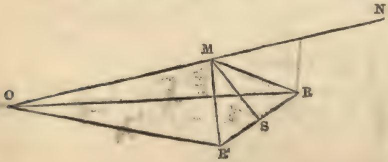


Fig. 57.

$$\overline{\mathrm{OM}} = \frac{p_x + p_y}{2};\ \overline{\mathrm{MR}} = \overline{\mathrm{MR'}} = \frac{p_x - p_y}{2};$$

and consequently that the principal stresses are

$$p_x = \overline{\mathrm{OM}} + \overline{\mathrm{MR}};\ p_y = \overline{\mathrm{OM}} - \overline{\mathrm{MR}}; \ldots\ldots\ldots\ldots (12.)$$

and it is also evident, that the angles made by the axis of greatest stress, with the two normals respectively, are

$$\widehat{xn} = \frac{1}{2} \angle \mathrm{NMR};\ \widehat{xn'} = \frac{1}{2} \angle \mathrm{NMR'}; \ldots\ldots\ldots (13.)$$

which data are sufficient to determine the position of the axes.—Q. E. I.

CASE 2. *When the given stresses are of opposite kinds,* the construction is the same in every respect, except that the lesser of the given stresses must be represented in fig. 57 by a line in the *prolongation* of its direction beyond O, making an obtuse angle with O N, equal to the supplement of its obliquity.

In either of the two cases that have been stated, the angle between the normals to the two given planes must have one or other of the two following values :—

$$\widehat{nn'} = \begin{Bmatrix} \text{either } \widehat{xn'} + \widehat{xn} = \angle \mathrm{NMS} \\ \text{or } \quad \widehat{xn'} - \widehat{xn} = \angle \mathrm{RMS} \end{Bmatrix} \ldots\ldots\ldots (14.)$$

according as the two normals are at opposite sides, or at the same side of the axis of greatest stress.

The solution of cases 1 and 2 is expressed algebraically by the following equations, which are deduced from the geometrical solution by means of well known formulæ of trigonometry :—

$$\frac{p_x + p_y}{2} = \overline{\mathrm{OM}} = \frac{p^2 - p'^2}{2\left(p \cos \widehat{nr} - p' \cos \widehat{n'r'}\right)}; \ldots\ldots. (15.)$$

$$\frac{p_x - p_y}{2} = \overline{\mathrm{MR}} = \overline{\mathrm{MR'}}$$

$$= \sqrt{\left\{ \frac{(p_x + p_y)^2}{4} + p^2 - (p_x + p_y)\, p \cos \widehat{nr} \right\}}$$

Trig. p. 47

$$= \sqrt{\left\{ \frac{(p_x + p_y)^2}{4} + p'^2 - (p_x + p_y)\, p' \cos \widehat{n'r'} \right\}}; \ldots.. (16.)$$

$$\left.\begin{aligned} \cos 2\,\widehat{xn} &= \frac{2p \cos \widehat{nr} - p_x - p_y}{p_x - p_y}; \\ \cos 2\,\widehat{xn'} &= \frac{2p' \cos \widehat{n'r'} - p_x - p_y}{p_x - p_y}. \end{aligned}\right\} \ldots\ldots\ldots. (17.)$$

In using these equations, it is to be observed that the cosine of an obtuse angle is negative.

Simplified Forms of Cases 1 *and* 2.

3

CASE 3. *When the two given stresses are conjugate,* they are of equal obliquity; and the points O, R′, S, R, in fig. 57, are in one straight line, to which M S is perpendicular; the angle between the two normals being

$$\angle \mathrm{NMS} = \widehat{n\,n'} = 90^\circ + \widehat{n\,r} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (18.)$$

In this case, equation 15 becomes Divide full terms of frac. by [illegible]

$$\frac{p_x + p_y}{2} = \overline{\mathrm{OM}} = \frac{p + p'}{2 \cos \widehat{n\,r}}; \ldots\ldots\ldots\ldots\ldots\ldots (19)$$

$m_y = m'_x$

equation 16 becomes Sub. from Eq 19 root of (p_x+p_y)

$$\frac{p_x - p_y}{2} = \overline{\mathrm{MR}} = \overline{\mathrm{MR'}} = \sqrt{\left\{\frac{(p_x + p_y)^2}{4} - pp'\right\}} =$$

Sub. root of (p_x-p_y)

$$\sqrt{\left\{\frac{(p + p')^2}{4 \cos^2 \widehat{n\,r}} - pp'\right\}} \ldots\ldots\ldots\ldots\ldots\ldots (20.)$$

equations 17 are modified only by the equality of $\widehat{n'\,r'}$ to $\widehat{n\,r}$.

CASE 4. *When the planes of action of the two given stresses are perpendicular to each other,* M S is perpendicular and R R′ parallel to O N, in fig. 57, so that we have, for the tangential component of each stress,

$$\overline{\mathrm{MS}} = p \sin \widehat{n\,r} = p' \sin \widehat{n'\,r'} = p_t.$$

Let the normal components of the given stresses be denoted by

$$p_n = p \cos \widehat{n\,r}; \; p'_n = p' \cos \widehat{n'\,r'}.$$

Then equation 15 becomes

$$\frac{p_x + p_y}{2} = \frac{p_n + p'_n}{2}; \ldots\ldots\ldots\ldots\ldots\ldots\ldots (21.)$$

equation 16 becomes

$$\frac{p_x - p_y}{2} = \sqrt{\left\{\frac{(p_n - p'_n)^2}{4} + p_t^2\right\}} \ldots\ldots\ldots\ldots\ldots (22.)$$

The equations 17 become

$$\left.\begin{array}{l} \cos 2\,\widehat{x\,n} = -\cos 2\,\widehat{x'\,n'} = \dfrac{p_n - p'_n}{p_x - p_y}; \\ \text{or, what is equivalent,} \\ \tan 2\,\widehat{x\,n} = -\tan 2\,x'\,n' = \dfrac{2\,p_t}{p_n - p'_n}. \end{array}\right\} \ldots\ldots (23.)$$

being the same with equation 1 of Article 109.

Problem V. *The stress in every direction being a thrust, and the greatest obliquity being given, it is required to find the ratio of two conjugate thrusts whose common obliquity is given.*

Let φ denote the given greatest obliquity. Then according to Problem III.,

$$\frac{p_x - p_y}{p_x + p_y} = \sin\varphi.$$

Let $\widehat{nr}$, which must not exceed φ, denote the common obliquity of a pair of conjugate thrusts, so that, as in Problem IV., case 3,

$$90° + \widehat{nr}$$

shall be the angle between the normals to their planes of action, and

$$90° - \widehat{nr}$$

the angle between those planes themselves. Let p be the intensity of the greater, and p' that of the less, of those conjugate thrusts whose ratio is sought; then dividing equation 20 of this Article by equation 19, and squaring the result, we find

$$\sin^2\varphi = \left(\frac{p_x - p_y}{p_x + p_y}\right)^2 = 1 - \frac{4\,p\,p'\cos^2\widehat{nr}}{(p+p')^2} \text{ ....(24.)}$$

or transposing

$$\frac{(p+p')^2}{4\,p\,p'} = \frac{\cos^2\widehat{nr}}{\cos^2\varphi} \text{ ....(25.)}$$

(Reducing — $p+p' = 2\sqrt{pp'}\left(\frac{\cos nr}{\cos\varphi}\right)$)

Hence it follows that the ratio of the conjugate stresses, p, p', is that of the two roots of a quadratic equation.

$$u^2 - 2\cos\widehat{nr}\cdot u + \cos^2\varphi = 0 \text{ ....(26.)}$$

that is to say, let p be the greater thrust, and p' the less, then

The values of p & p' are readily found from eq [illegible]

$$\frac{p'}{p} = \frac{\cos\widehat{nr} - \sqrt{(\cos^2\widehat{nr} - \cos^2\varphi)}}{\cos\widehat{nr} + \sqrt{(\cos^2\widehat{nr} - \cos^2\varphi)}} \text{ ....(27.)}$$

When $\widehat{nr} = 0$, this becomes the ratio of the principal thrusts, viz. :—

$$\frac{p_y}{p_x} = \frac{1 - \sin\varphi}{1 + \sin\varphi}; \text{ ....(28.)}$$

when $\widehat{nr} = \varphi$, the ratio becomes that of equality.

113. **Combined Stresses in One Plane.**—PROBLEM. *Given the normal intensities and directions of any number of simple stresses whose directions are in the same plane; required, the directions and intensities of the pair of principal stresses resulting from their combination.*

Distinguish the pulls from the thrusts by considering the kind whose sum is greatest as positive, and the opposite kind as negative. Assume two planes at right angles to each other (which may be called planes of reduction), to each of which, by the process of Article 98, reduce all the given stresses; and then resolve, as in Article 99, each of the reduced stresses thus obtained into a direct or normal, and a shearing or tangential component. Compute (attending to the positive and negative signs) the two sums of the direct component stresses on the two planes of reduction respectively; compute also the sum of the shearing components, which will be the same for each plane of reduction: lastly, from the pair of total direct stresses, and the total shearing stress, thus computed, relatively to the assumed rectangular planes of reduction, determine, as in Article 112, Problem IV., case 4, the directions and intensities of the resultant principal stresses.—Q. E. I.

The algebraical expression of this solution is as follows:—Let n be taken to denote the normal to one of the rectangular planes of reduction.

Let p denote the *normal intensity* of any one of the given direct stresses, and $\widehat{np}$ the angle which its direction makes with the normal n. The symbol Σ, as in previous examples, denotes the operation of taking the sum of a set of quantities, with due regard to their algebraical signs, that is to say, adding the positive and subtracting the negative quantities.

The direct and shearing components of a single stress p, as reduced to the rectangular planes of reduction respectively, according to the principles of Article 99, are as follows:—

$$\text{Normal,}\left\{\begin{array}{ll}\text{on the plane normal to } n, & p\cos^2\widehat{np}\,; \\ \text{on the other plane,} & p\sin^2\widehat{np}\,;\end{array}\right.$$

$$\text{Tangential on each plane, } p\cos\widehat{np}\sin\widehat{np}.$$

Consequently, the total direct and shearing stresses on the planes of reduction, are as follows:—

$$\text{Normal,}\left\{\begin{array}{l}p_n = \Sigma\left(p\cos^2\widehat{np}\right); \\ p'_n = \Sigma\left(p\sin^2\widehat{np}\right);\end{array}\right.$$

$$\text{Tangential, } p_t = \Sigma\left(p\cos\widehat{np}\sin\widehat{np}\right).$$

Introducing these values into the equations 21, 22, and 23, of Article 112, and observing that

$$\cos^2 \widehat{np} + \sin^2 \widehat{np} = 1\,;\ \cos^2 \widehat{np} - \sin^2 \widehat{np} = \cos 2\,\widehat{np},$$

$$\cos \widehat{np} \cdot \sin \widehat{np} = \frac{1}{2} \sin 2\,\widehat{np},$$

we obtain the following results :—

$$\frac{p_x + p_y}{2} = \frac{1}{2}\, \Sigma \cdot p \ldots\ldots\ldots\ldots (1.)$$

$$\frac{p_x - p_y}{2} = \frac{1}{2} \surd \left\{ \left(\Sigma \cdot p \cos 2\,\widehat{np}\right)^2 + \left(\Sigma \cdot p \sin 2\,\widehat{np}\right)^2 \right\} \ldots (2.)$$

$$\widehat{nx} = \frac{1}{2} \operatorname{arc} \cdot \tan \frac{\Sigma \cdot p \sin 2\,\widehat{np}}{\Sigma \cdot p \cos 2\,\widehat{np}} \ldots\ldots\ldots (3.)$$

The equation 2 is capable of being expressed in another form, as follows. Let a, a' be *any two* angles. Then

$$\cos a \ \cos a' + \sin a \sin a' = \cos (a - a').$$

Now the quantity under the sign $\surd$, in equation 2, consists of the following classes of terms :—

1. All the squares $p^2 \cos^2 2\,\widehat{np}$;

2. All the products $2\,p\,p' \cos 2\,\widehat{np} \cos 2\,\widehat{np'}$;

where p, p', are *any pair* of the given stresses ;

3. All the squares $p^2 \sin^2 2\,\widehat{np}$;

4. All the products $2\,p\,p' \sin 2\,\widehat{np} \sin 2\,\widehat{np'}$.

The first and third of these classes being added together, make $\Sigma\,(p^2)$; the second and fourth make $2\,\Sigma\,(p\,p' \cdot \cos 2\,\widehat{pp'})$; $\widehat{pp'}$ being the angle between p and p'. Equation 2 thus becomes

$$\frac{p_x - p_y}{2} = \frac{1}{2} \surd \left\{ \Sigma\,(p^2) + 2\,\Sigma \left(pp' \cos 2\,\widehat{pp'}\right) \right\} \ldots\ldots (4.)$$

From the equations (1) and (4) it appears that the *intensities* of the principal stresses p_x and p_y can be computed without assuming planes of reduction ; for the only angles involved in this pair of equations are the several angles $\widehat{pp'}$, which the given stresses make with each other when compared by pairs in every possible combination. To find the *directions*, however, of those principal stresses, planes of reduction must be assumed.

In using the equation (4), it is to be remembered that when $2\,\widehat{p\,p'}$ exceeds 90°, we have

$$\cos 2\,\widehat{p\,p'} = -\cos\left(180° - 2\,\widehat{p\,p'}\right).$$

SECTION 4.—*Of the Internal Equilibrium of Stress and Weight, and the Principles of Hydrostatics.*

114. **Varying Internal Stress.**—The investigations of the preceding section have been conducted as if the internal stress, whether simple or compound, were uniform at all points in the body under consideration; but their results are nevertheless correctly applicable to internal stress which varies from point to point of the body; for those results are arrived at by considering the conditions of equilibrium of a pyramidal or prismatic portion of the body containing the point at which the relations amongst the components of the stress are to be determined; and when the stress varies from point to point, then by supposing the pyramid or prism to be small enough, its condition of stress may be made to deviate from uniformity to an extent less than any assigned limit of deviation; but the truth of the propositions of the preceding section for an uniform stress is independent of the size of the prism or pyramid; therefore they can be proved to deviate from the truth for a varying stress by less than any assignable error; therefore they must be true for a varying as well as for an uniform stress.

115. **Causes of Varying Stress.**—The internal stress exerted amongst the parts of a body, may vary from point to point, from three classes of causes, viz.:—

I. Mutual attractions and repulsions between the parts of the body;

II. Attractions and repulsions exerted between the parts of the body in question and external bodies;

III. Stress exerted between the body in question and external bodies at their surfaces of contact.

I. The first of these classes of causes may be left out of consideration in the present treatise; because the mutual attractions and repulsions of the parts of an artificial structure are too small to be of practical importance in the art of construction.

II. Of the second class of causes, the only force which is of sufficient magnitude to be considered in the art of construction, is *weight*.

III. The consideration of the third class of causes belongs to the subject of the strength of materials, which will be treated of in the sequel.

The subject of the present section, therefore, is the relation between the weight of the parts of a body, and the variation of its condition of stress from point to point.

116. **General Problem of Internal Equilibrium.**—Let w denote the weight per unit of volume of a body, or part of a body, and let it be required to determine what modes of variation of internal stress are consistent with that specific gravity.

Consider the condition of a rectangular molecule A (fig. 58), bounded by ideal planes, whose edges are parallel to three rectangular axes, O X, O Y, O Z. The position of this set of axes is immaterial to the result; but the algebraic formulæ are simplified by assuming one axis to be vertical; let O Z, then, be vertical, and let distances along it be positive upwards. Then weight must be treated as a negative force; and the weight of a portion of the body of the volume V will be denoted by

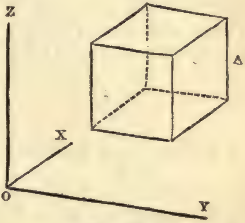


Fig. 58.

$$-wV.$$

Let the dimensions of the molecule A be

Δx parallel to O X,
Δy ” ” O Y,
Δz ” ” O Z.

Then its weight is represented by

$$-w \cdot \Delta x \; \Delta y \; \Delta z.$$

The six faces will be designated as follows:—

	Farthest from O.	Nearest to O.
The pair parallel to Y O Z	$+\Delta y\,\Delta z$	$-\Delta y\,\Delta z$
” ” ” Z O X	$+\Delta z\,\Delta x$	$-\Delta z\,\Delta x$
” ” ” X O Y (That is, the horizontal pair.)	$+\Delta x\,\Delta y$ (the upper.)	$-\Delta x\,\Delta y$ (the lower.)

Let the six intensities of the components of the stress be denoted as in Article 104, viz.:—

Normal, p_{xx}, p_{yy}, p_{zz};
Tangential, p_{yz}, p_{zx}, p_{xy}.

As for the signs of normal stress, let pull be positive and thrust negative. As for the signs of tangential stress, let those stresses be considered as $\left\{\begin{matrix}\text{positive}\\ \text{negative}\end{matrix}\right\}$ which tend to make the pair of corners of the molecule which are nearest and farthest from O $\left\{\begin{matrix}\text{sharper}\\ \text{flatter}\end{matrix}\right\}$.

In the first place, let the *rate of variation* of the stress, of what kind soever, from point to point, be uniform; that is to say, for example, if the mean intensity of any one of the components of the stress at the face $- \Delta x \Delta y$ be p, then at the face $+ \Delta x \Delta y$, whose distance from $- \Delta x \Delta y$ is Δz, let the mean intensity of the same component be

$$p + \frac{d p}{d z} \cdot \Delta z,$$

in which $\frac{d p}{d z}$ is a constant co-efficient or factor, meaning "*the rate of variation of* p *along* z," which is positive or negative, according as the variation of p is of the same or of the contrary kind to that of z. *Rates of variation* are also known by the name of *differential co-efficients.* As there are six components in the stress, and three axes of co-ordinates, there are *eighteen* possible differential co-efficients of the stress with respect to the co-ordinates; but it will presently appear that *nine* only of those co-efficients are concerned in the solution of the present problem.

The relations amongst the weight of the molecule A, and the variations of the intensities of the component stresses on its different faces, depend on this principle, that the *force arising from the variations of stress must balance the weight of the molecule;* that is to say, the resultant force parallel to each of the horizontal axes, which arises from the variation of stress, must be *nothing,* and the resultant force parallel to the vertical axis, which arises from the variation of stress, must be *upward,* and *equal to the weight of the molecule*—a principle expressed by the three following equations:—

$$\left.\begin{aligned}
&\frac{dp_{xx}}{dx} \Delta x \cdot \Delta y \Delta z + \frac{dp_{xy}}{dy} \Delta y \cdot \Delta z \Delta x + \frac{dp_{zx}}{dz} \Delta z \cdot \Delta x \Delta y = 0; \\
&\frac{dp_{xy}}{dx} \Delta x \cdot \Delta y \Delta z + \frac{dp_{yy}}{dy} \Delta y \cdot \Delta z \Delta x + \frac{dp_{yz}}{dz} \Delta z \cdot \Delta x \Delta y = 0; \\
&\frac{dp_{zx}}{dx} \Delta x \cdot \Delta y \Delta z + \frac{dp_{yz}}{dy} \Delta y \cdot \Delta z \Delta x + \frac{dp_{zz}}{dz} \Delta z \cdot \Delta x \Delta y \\
&\qquad = w \cdot \Delta x \Delta y \Delta z.
\end{aligned}\right\} \quad (1.)$$

Each of the nine terms which compose the left sides of the above equations is the product of four factors; the first being the rate of variation of a stress, the second the distance between two faces on which that stress acts, and the third and fourth the dimensions of those faces, whose product is their common area.

Each term of those three equations contains as a common factor the volume of the molecule, $\Delta x \, \Delta y \, \Delta z$; dividing by this, they are reduced to the following:—

$$\left.\begin{aligned} \frac{d p_{xx}}{d x} + \frac{d p_{xy}}{d y} + \frac{d p_{zx}}{d z} &= 0; \\ \frac{d p_{xy}}{d x} + \frac{d p_{yy}}{d y} + \frac{d p_{yz}}{d z} &= 0; \\ \frac{d p_{zx}}{d x} + \frac{d p_{yz}}{d y} + \frac{d p_{zz}}{d z} &= w. \end{aligned}\right\} \text{..........(2.)}$$

In this second form, the equations are applicable to rates of variation which are not uniform as well as to those which are uniform. For as the rectangular molecule, from the conditions of whose equilibrium these equations are deduced, is of arbitrary size, it may be supposed as small as we please; and when the rates of variation of the stress are not uniform, we can always, by supposing the molecule small enough, make the rates of variation of the stresses throughout its bulk deviate from uniform rates to an extent less than any given limit of error.

The equations 2 can easily be modified so as to adapt them to any different arrangement of the axes of co-ordinates. Thus, if z be made positive downwards instead of upwards, $-w$ is to be put for w in the third equation. If x or y, instead of z, be made the vertical axis, w is to be substituted for 0 in the first or the second equation, as the case may be, and 0 for w in the third equation. If the axes of x, y, and z make respectively the angles α, β, and γ, with a line pointing vertically upwards, the force of gravity is to be resolved into three rectangular components, each of which must be separately balanced by variations of stress; so that for

$$0, \qquad 0, \qquad w,$$

in the first, second, and third equations respectively, are to be substituted

$$w \cos \alpha, \qquad w \cos \beta, \qquad w \cos \gamma.$$

The equations of this Article are not in general sufficient of themselves to determine the mode of variation of the intensity of the stress in a solid body, because of their number not being so great as that of the number of unknown quantities to be determined. They have therefore to be combined with other equations, deduced from the relations which are found by experiment to exist between the alterations of figure, which the parts of a solid body undergo when a load acts on it, and the stresses which at the same time act amongst the disfigured parts. These relations belong to the subject of elasticity and of the strength of materials, and not to that of the principles of statics. The remainder of the present section will relate to those more simple problems which can be solved by means of the equations 2 alone.

6

117. **Equilibrium of Fluids.**—It has already been explained in Article 110, that in a fluid the only stress to be considered in practice is a thrust or pressure, normal and of equal intensity in all directions. This is expressed symbolically in the following manner:—

$$\left.\begin{array}{l} p_{yz} = 0;\ p_{zx} = 0;\ p_{xy} = 0; \\ p_{xx} = p_{yy} = p_{zz} = p; \end{array}\right\} \dots\dots\dots\dots\dots (1.)$$

the single symbol p being used, for the sake of convenience and brevity, to denote the *intensity of the fluid pressure* at any given point in the fluid.

In adapting the equations 2 of Article 116 to this case, it is convenient to take x to denote vertical co-ordinates, and to make it *positive downwards.* Then, bearing in mind that p is now a thrust, being positive (and not a pull when positive and a thrust when negative, as in the general problem), we obtain the following equations:—

Rate of var of pr.

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$$\left.\begin{array}{c} \dfrac{dp}{dx} = w; \\ \dfrac{dp}{dy} = 0;\ \dfrac{dp}{dz} = 0; \end{array}\right\} \dots\dots\dots\dots\dots\dots (2.)$$

The first of these equations expresses the fact, that *in a balanced fluid, the pressure increases with the vertical depth, at a rate expressed by the weight of the fluid per unit of volume;* and the second and third express the fact, that *in a balanced fluid, the pressure has no variation in any horizontal direction;* in other words, that *the pressure is equal at all points in the same level surface.*

[The exact figure of a level surface is spheroidal; but for purposes of applied mechanics it may be treated as a plane, without sensible error.]

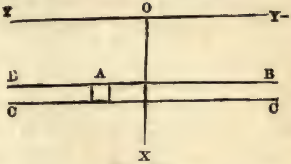

Fig. 59.

Those principles may also be proved directly. Let fig. 59 represent a vertical section of a fluid; Y O Y any horizontal plane, O X a vertical axis. Let B B be a horizontal plane at the depth x below O; C C another horizontal plane at the depth $x + \Delta x$. Let A be a small rectangular molecule contained between those two horizontal planes; and let Δy and Δz be its horizontal dimensions, so that its weight is

$$w \Delta x \Delta y \Delta z.$$

The pressure exerted by the other portions of the fluid against the vertical faces of this molecule are horizontal, and must balance each other; therefore there can be no variation of pressure horizontally. Let p_0, then, be the uniform pressure at the horizontal plane Y O Y, p, that at the plane B B, and $p + \frac{dp}{dx} \Delta x$ that at the plane C C, $\frac{dp}{dx}$ being the rate of increase of pressure with depth. The molecule is pressed downwards by the pressure whose amount is

$$p \Delta y \Delta z,$$

and upwards by the pressure whose amount is

$$\left(p + \frac{dp}{dx} \cdot \Delta x\right) \Delta y \Delta z.$$

The difference between those forces, viz. :—

$$\frac{dp}{dx} \cdot \Delta x \cdot \Delta y \Delta z,$$

has to be balanced by the weight of the molecule; equating it to which, and dividing by the common factor $\Delta x \Delta y \Delta z$, we obtain the first of the equations 2 of this Article.

The pressure p_0 at the surface Y Y being given, the pressure p at any given depth x below Y Y is found by means of the integral,

$$\left.\begin{aligned} p &= p_0 + \int_0^x \frac{dp}{dx} dx; \\ &= p_0 + \int_0^x w \, dx; \end{aligned}\right\} \text{.................. (3.)}$$

that is to say, it is equal to the pressure at the plane Y Y, added to the weight of a vertical column of the fluid whose area of base is *unity*, and which extends from the plane Y Y down to the given depth x below that plane.

It is obviously necessary to the equilibrium of a fluid, that the specific gravity, as well as the pressure, should be the same at all points in the same level surface.

The preceding principles are the base of the science of Hydrostatics.

66 118. **Equilibrium of a Liquid.**—A liquid is a fluid whose parts tend to preserve a definite size; that is to say, a portion of a liquid of a given weight tends to occupy a certain definite volume; and to make it occupy a greater or a less volume, tension or pressure, as the case may be, must be applied to it. The volume occupied by an unit of weight is the reciprocal of the weight of an unit of volume; so that the preceding principle might otherwise be stated by saying, that a liquid tends to preserve a definite specific gravity, which may be increased by pressure, or diminished by tension.

The volume which a given weight of a liquid tends to occupy depends on its temperature according to laws which belong to the science of Heat.

The alterations of the specific gravity of liquids produced by any pressures which occur in practice, are so small, that in most problems respecting the equilibrium of liquids, the specific gravity w may be treated without sensible error as a constant quantity, independent of the pressure p. In the case of water, for example, the compression of volume, and increase of specific gravity, produced by a pressure of *one atmosphere*, or 14·7 pounds per square inch, is about $\frac{1}{20000}$, or $\frac{1}{294000}$ for each pound on the square inch.

If, then, the specific gravity w be treated as a constant in equation 3 of Article 117, it becomes as follows:—

$$p = p_0 + w x; \qquad \text{(1.)}$$

that is to say:—let p_0 be the pressure at the upper surface, Y O Y, (fig. 59) of a mass of liquid; then the pressure p at any given depth x below that surface is greater than the superficial pressure p_0 by an amount found by multiplying that depth by the weight of an unit of volume of the liquid.

When the mass of liquid is in the open air, the superficial pressure p_0 is that arising from the weight of the earth's atmosphere of air, and at places near the level of the sea, is estimated on an average at 14·7 pounds on the square inch. In a close vessel, the superficial pressure may be greater or less than that of the atmosphere.

119. **Equilibrium of different Fluids in contact with each other.**—If two different fluids exist in the same space, they may unite so that each of them shall be distributed throughout the whole space, either by chemical combination or by diffusion; but in such cases they form, in fact, but one fluid, which is a compound or mixture, as the case may be. The present Article has reference to the case

when fluids of different kinds remain in contact, uncombined and unmixed. In this case, the condition of equilibrium is, that the pressures of two fluids at each point of their surface of contact shall be equal to each other,—a condition which, when the two fluids are of different specific gravities, can only be fulfilled when the surface of contact is horizontal.

If, then, two or more fluids of different specific gravities, which do not combine nor mix with each other, be contained in one vessel uninterrupted by partitions, they will arrange themselves in horizontal strata, the heavier fluids being below the lighter.

If two fluids of different specific gravities be contained in the two legs of a tube shaped like the letter U (and called an "inverted siphon"), or if one of the two fluids be contained in a vertical tube open below, and the other in the space surrounding that tube; or, generally, if the two fluids be partially separated from each other by a vertical or nearly vertical partition, below which there is a communication between the spaces on either side of it; the horizontal surface of contact of the fluids will be at that side of the partition at which the lighter fluid is found, so that it may be above, and the heavier fluid below, that surface of contact.

Let p_0 denote the common pressure of the two fluids at their surface of contact, and let any ordinate measured from that surface *upwards*, be denoted by x. Let w' denote the specific gravity, and p' the pressure, of the lighter fluid; w'' the specific gravity, and p'' the pressure, of the heavier fluid. Then at any given elevation x above the surface of contact

$$\left.\begin{aligned} p' &= p_0 - \int_0^x w'\,dx; \\ p'' &= p_0 - \int_0^x w''\,dx; \end{aligned}\right\} \text{.................} (1.)$$

which equations, when the fluids are *liquids*, and w', w'', constants, become

$$p' = p_0 - w x;\ p'' = p_0 - w'' x \text{.................} (2.)$$

As in the case of the barometer, and the mercurial pressure gauge, the height at which a liquid stands in a tube, closed and empty at the upper end, above its surface of contact with another fluid, may be used to determine the pressure exerted by that other fluid at the surface of contact. In this case, $p'' = 0$, or nearly so; consequently

$$p_0 = w'' x, \text{..............................} (3.)$$

Let x', x'', be two heights above the surface of contact at which the respective pressures of the lighter and the heavier fluid are either equal to each other, or both equal to nothing; then $p'' = p'$, and consequently, for fluids in general,

$$\int_0^{x'} w' \, dx = \int_0^{x''} w'' \, dx, \ldots\ldots\ldots\ldots\ldots\ldots \text{(4.)}$$

If the fluids be both liquids, this becomes,

$$w' x' = w'' x'', \ldots\ldots\ldots\ldots\ldots\ldots \text{(5.)}$$

or, the heights are inversely as the specific gravities.

If the heavier fluid be a liquid (such as the mercury in the barometer) and the lighter a gas (such as the atmosphere) the equation becomes

$$\int_0^{x'} w' \, dx = w'' x''; \ldots\ldots\ldots\ldots\ldots\ldots \text{(6.)}$$

and on this last formula is founded the method of determining differences of level by barometric observations of the atmospheric pressure.

120. **Equilibrium of a Floating Body.**—THEOREM. *A solid body floating on the surface of a liquid is balanced, when it displaces a volume of liquid whose weight is equal to the weight of the floating body, and when the centre of gravity of the floating body, and that of the volume from which the liquid is displaced, are in the same vertical line.*

Let fig. 60 represent a solid body (such as a ship), floating in a liquid, whose horizontal upper surface is Y Y. Suppose, in the first place, that there is no pressure on the surface Y Y. Consider a small portion S of the surface of the immersed part of the solid body. The liquid will exert against S a normal pressure, whose amount will be expressed by

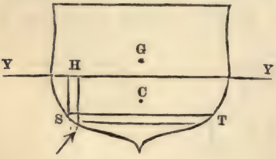

Fig. 60.

$$Sp = S w x,$$

where S is the area of the small portion of the immersed surface, x the depth of immersion of its centre below the level surface Y Y, and w the weight of unity of volume of the liquid.

Let α denote the angle of inclination of the area S to a horizontal plane, or, what is the same thing, the angle of inclination of the pressure on S to the vertical. Conceive a vertical prism H S to stand on the area S; the area of the horizontal transverse section of this prism is what is called the *horizontal projection* of the area S, and its value is

$$S \cos \alpha.$$

Conceive a horizontal prism S T to have its axis in the vertical plane which is perpendicular to S, and to have the area S for an oblique section; the vertical transverse section of this prism is what is called the *vertical projection* of the area S, and its value is

$$S \sin \alpha.$$

This horizontal prism cuts the immersed surface in another small area T, whose projection on a vertical plane perpendicular to the axis of the prism S T is equal to that of S, and which is immersed to the same depth, and sustains pressure of the same intensity.

Resolve the total pressure on S into a horizontal component and a vertical component. The horizontal component is

$$S\,p \cdot \sin \alpha = S\,w\,x \cdot \sin \alpha,$$

being equal to the product of the intensity p by the *vertical projection* of S; but this component is balanced by an equal and opposite component of the total pressure on T; and the same is the case for every portion such as S into which the immersed surface can be divided; therefore the resultant of all the horizontal components of the pressure exerted by the liquid against the solid is *nothing*.

The vertical component of the pressure on S is

$$S\,p \cos \alpha = S\,w\,x \cos \alpha,$$

being equal to the product of the intensity p by the *horizontal projection* of S. But $S\ x \cos \alpha$ is the *volume of the vertical prism* H S, standing upon the small area S, and bounded above by the horizontal surface Y Y, and w is the weight of unity of volume of the liquid; therefore $S\,w\,x \cos \alpha$ is the weight of liquid which the prism H S would contain; so that the vertical component of the pressure on S is an upward force, *equal and opposite to the weight of the liquid displaced by the prismatic portion of the solid body which stands vertically above* S. Then if the whole of the immersed surface be divided into small areas such as S, the resultant of the pressure of the liquid against that entire surface is the sum of all the vertical components of the pressures on the small areas; that is, a force equal and opposite to the sum of the weights of liquid displaced by all the prisms such as H S; that is, a sum equal and opposite to the weight of the whole volume of liquid displaced by the floating body; and the line of action of that resultant traverses the centre of gravity of the volume of liquid so displaced.

Let C denote that centre of gravity, which is also called the *Centre of Buoyancy.* Let G denote the centre of gravity of the floating body. Let W denote the weight of the floating body, and V the volume of liquid displaced by it.

Then the conditions of equilibrium of the floating body are obviously the following:—

First:—$W = w\,V$; or its weight must be equal to the weight of the volume of liquid displaced by it;—

Secondly:—its centre of gravity G, and the centre of buoyancy C, must be in the same vertical line.—Q. E. D.

The preceding demonstration has reference to the case in which the pressure on the horizontal surface Y Y is nothing. In the case of bodies floating on water, that surface, as well as the non-immersed part of the surface of the floating body, have to sustain the pressure of the air. To what extent this fact modifies the conclusions arrived at will appear in the next Article.

121. **Pressure on an Immersed Body.**—THEOREM. *If a solid body be wholly immersed in a fluid, the resultant of the pressure of the fluid on the solid body is a vertical force, equal and directly opposed to the weight of the portion of the fluid which the solid body displaces.*

Let fig. 61 represent a solid body totally immersed in a fluid, whether liquid or gaseous. Conceive a small vertical prism S U to extend from a portion S of the lower surface of the body, to the portion U of the upper surface which is vertically above S. Also let S T be a horizontal prism of which S is an oblique section, and U V a horizontal prism of which U is an oblique section, as in Article 120.

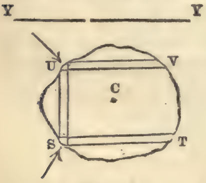

Fig. 61.

Then, as in Article 120, it may be proved that the horizontal component of the pressure on S is balanced by an equal and opposite component of the pressure on T, and the horizontal component of the pressure on U by an equal and opposite component of the pressure on V; so that the horizontal component of the resultant of the pressure of the fluid on the entire body is nothing, and that resultant is vertical.

The vertical component of the pressure on S is upward, and equal to the weight of the prismatic portion of the fluid which would stand vertically above S if a part of it were not displaced by the solid body. The vertical component of the pressure on U is downward, and equal to the weight of the prismatic portion of the fluid which stands vertically above U. The vertical force arising from the pressures on S and on U together is upward, and equal to the difference between those two weights; that is, it is equal and directly opposed to the weight of the portion of the fluid displaced by the prismatic portion S U of the immersed body.

Hence the resultant of the pressure of the fluid over the entire surface of the immersed body is equal and directly opposed to the weight of the portion of fluid displaced by that body.—Q. E. D.

The centre of gravity C, of the portion of fluid which would occupy the position of the body if it were not immersed, is called, as before, the *centre of buoyancy*, and is traversed by the vertical line of action of the resultant of the pressure of the fluid, which is itself called the *buoyancy* of the immersed body, and sometimes the *apparent loss of weight.*

To maintain an immersed body in equilibrio, there must be applied to it a force or couple, as the case may be, equal and directly opposed to the resultant, if any, of its downward weight and upward buoyancy; that resultant being determined according to the principles of Articles 39 and 40.

When a body floats in a heavier fluid (as water) having its upper portion surrounded by a lighter fluid (as air), its total buoyancy is equal and opposite to the resultant of the weights of the two portions of the respective fluids which it displaces.

In practical questions relative to the equilibrium of ships, the buoyancy arising from the displacement of air is too small as compared with that arising from the displacement of water, to require to be taken into account in calculation.

122. **Apparent Weights.**—The only method of testing the equality of the weights of two bodies which is sufficiently delicate for exact scientific purposes, is that of hanging them from the opposite ends of a lever with equal arms.

If this process were performed in a vacuum, the balancing of the bodies would prove their weights to be equal; but as it must be performed in air, the balancing only proves the equality of the *apparent weights* of the bodies *in air*, that is, of the respective excesses of their weights above the weights of the volumes of air which they displace. The real weights of the bodies, therefore, are not equal unless their volumes are equal also. If their volumes are unequal, the real weight of the larger body must be the greater by an amount equal to the weight of the difference between the volumes of air which they displace.

The weight of a cubic foot of pure dry air, under the pressure of one atmosphere (14·7 lbs. on the square inch), and at the temperature of melting ice (32° Fahrenheit) is

0·080728 pound avoirdupois.

Let this be denoted by w_0. Then the weight of a cubic foot of air under any other pressure of p atmospheres, and at the temperature t of Fahrenheit's scale, is given with a degree of accuracy sufficient for most purposes, by the formula,

$$w = w_0 p \frac{493^\circ{\cdot}2}{t + 461^\circ{\cdot}2}; \dotfill (1.)$$

and if w, w', be the weights of a given volume of air, under the respective pressures p, p', and at the temperatures t, t', of Fahrenheit's scale, then

$$\frac{w'}{w} = \frac{p'}{p} \cdot \frac{t + 461^\circ{\cdot}2}{t' + 461^\circ{\cdot}2} \dotfill (2.)$$

Let W_1 denote the true weight of a body, V_1 its volume, w_1 its weight per unit of volume, w the weight of unity of volume of air. Then

$$W_1 = w_1 V_1,$$

and the apparent weight of the same body in air is

$$W' = (w_1 - w) V_1 = \frac{w_1 - w}{w_1} W_1 \text{.............} (3.)$$

Let this body now be balanced against another body in an accurate pair of scales, and let their apparent weights be equal. Then, if W_2 denote the true weight, and w_2 the weight per unit of volume, of the second body, we have

$$\frac{w_1 - w}{w_1} W_1 = \frac{w_2 - w}{w_2} W_2 \text{..................} (4.)$$

so that the proportion between the real weights of the bodies is

$$\frac{W_2}{W_1} = \frac{w_1 w_2 - w_2 w}{w_1 w_2 - w_1 w} \text{.......................} (5.)$$

123. **Relative Specific Gravities.**—If the true weight of a solid body be known, and that body be next weighed while immersed in a liquid, the proportion of the specific gravities of the solid body and of the liquid can be deduced from the apparent loss of weight, which is the weight of the volume of liquid displaced by the body.

Let W_1, as in equation 3 of Article 122, denote the true weight of the solid body, w_1 its weight per unit of volume, w_2 the weight of an unit of volume of the liquid in which its apparent weight is found, and W'' the apparent weight; then by the equation already referred to

$$W'' = \frac{w_1 - w_2}{w_1} W_1 = \left(1 - \frac{w_2}{w_1}\right) W_1;$$

and consequently

$$\frac{w_2}{w_1} = \frac{W_1 - W''}{W_1} \text{.......................} (1.)$$

Let the first weighing take place in air and the second in the liquid, and let W' be the apparent weight in air; then

$$W' = \frac{w_1 - w}{w_1} W_1$$

and consequently

$$\frac{W''}{W'} = \frac{w_1 - w_2}{w_1 - w}; \text{.......................} (2.)$$

so that if $\frac{w}{w_2}$ is known, $\frac{w_1}{w_2}$ may be found by the equation

$$\frac{w_1}{w_2} = \frac{W' - W'' \frac{w}{w_2}}{W' - W''} \text{.....................} (3.)$$

When the object of weighing of this kind is to determine the specific gravities of solids, the liquid usually employed is pure water; and the results obtained are the *ratios* of the specific gravities of solid bodies to that of pure water. If these ratios, or relative specific gravities, be multiplied by the weight of a cubic foot of pure water, the weight of a cubic foot of the solid is obtained.

The weight of a cubic foot of pure water at the temperature of its maximum density (being, according to Playfair and Joule, 39°·1 Fahrenheit) is, according to the best existing data,

62·425 pounds avoirdupois.

For any other temperature t on Fahrenheit's scale, the weight of a cubic foot of pure water is

$$\frac{62{\cdot}425}{v} \text{..............................} (4.)$$

where v denotes the volume to which a mass of water measuring one cubic foot at 39°·1 expands at t^0; a volume which may be computed for temperatures from 32° to 77° Fahrenheit, by means of the following empirical formula, extracted from Prof. W. H. Miller's paper on the Standard Pound in the *Philosophical Transactions* for 1856:—

$$\log. v = 10{\cdot}1\,(t - 39{\cdot}1)^2 - 0{\cdot}0369\,(t - 39{\cdot}1)^3 \div 10{,}000{,}000. \quad (5.)$$

The relative specific gravities of two liquids are determined by weighing the same solid body immersed in them successively and comparing its apparent losses of weight.

124. **Pressure on an Immersed Plane.**—If a horizontal plane surface of any figure be immersed in a fluid, the pressure on that surface is vertical, and uniformly distributed; its amount is the product of the intensity of the pressure at the depth to which the plane is immersed by the area of the plane; and the *centre of pressure* (as already shown in Art. 90) is the centre of gravity of a flat plate of the figure of the plane surface, or, as it is usually termed, the centre of gravity of the plane surface.

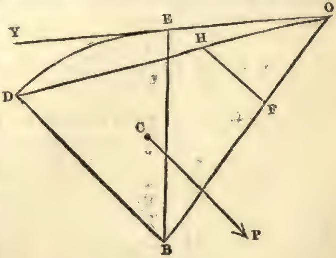

Fig. 62.

If an inclined or vertical plane surface be immersed in a liquid, let O Y (fig. 62), represent a section of the horizontal plane at which the pressure is nothing, and B F a vertical section of the immersed plane. Let $x_1 = \overline{\mathrm{BE}}$ be the depth to which the lower edge of this plane is immersed below O Y. From B draw $\overline{\mathrm{BD}} = \overline{\mathrm{BE}}$, and ⊥ BF; produce the plane BF till it cuts the horizontal plane of no pressure, O Y, in the line represented in section by O; through O and D draw a plane O H D, and conceive the prism B D H F to stand normally upon the base B F and to be bounded above by the plane D H. The pressure on the plane BF will be normal; its amount will be equal to the weight of fluid contained in the volume B D H F; that is to say, let x_0 denote the depth of the centre of gravity of the plane BF below O Y, and w the weight of unity of the volume of liquid; then the *mean intensity* of the pressure on B F is

$$p_0 = w x_0, \ldots\ldots\ldots\ldots\ldots (1.)$$

and the *amount* of the pressure

$$\mathrm{P} = w x_0 \cdot \text{area BF} \ldots\ldots\ldots\ldots\ldots (2.)$$

Let C be the centre of gravity of the volume B D H F; then the *centre of pressure* of the surface B F is the point where it is cut by the perpendicular C P let fall on it from C.

As the intensity of the pressure on any point of BF is proportional to its depth below O Y, and consequently to its distance from O, this is a case of *uniformly varying stress*, and the formulæ of Article 94 are applicable to it. In the application of those formulæ it is to be observed, that the ordinates y are to be measured horizontally in the plane BF, whose centre of gravity is to be taken as the origin; that the co-ordinates x are to be measured in the same plane, along the direction of *steepest declivity*, and reckoned positive downwards; and that the value of the constant a in the equations of Article 94 is given by the formula

$$a = w \sin \alpha \ldots\ldots\ldots\ldots\ldots (3.)$$

where α is the angle of inclination of the plane B F to a horizontal plane.

125. **Pressure in an Indefinite Uniformly Sloping Solid.**—Conceive a mass of homogeneous solid material to be indefinitely extended laterally and downwards, and to be bounded above by a plane surface, making a given angle of declivity θ with a horizontal plane. In fig. 63, let Y O Y represent a vertical section of that upper sloping surface along its direction of greatest declivity, and O X a vertical plane perpendicular to the plane of vertical

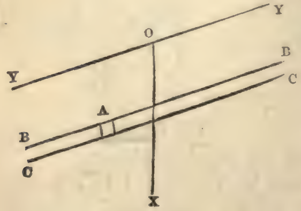

Fig. 63.

section which is represented by the paper. Let w be the uniform weight of unity of volume of the substance. Let B B be any plane parallel to, and at a vertical depth x below the plane Y Y. If the substance is exposed to no external force except its own weight, the only pressure which any portion of the plane B B can have to sustain is the weight of the material directly above it. Hence follows—

THEOREM I. *In an indefinite homogeneous solid bounded above by a sloping plane, the pressure on any plane parallel to that sloping surface is vertical, and of an uniform intensity equal to the weight of the vertical prism which stands on unity of area of the given plane.*

The area of the horizontal section of that prism is $\cos\theta$, consequently, the intensity of the vertical pressure on the plane B B at the depth x is

$$p_x = w\,x\cos\theta \text{ .... (1.)}$$

From the above theorem, combined with the principle of conjugate stresses of Article 101, there follows—

THEOREM II. *The stress, if any, on any vertical plane is parallel to the sloping surface, and conjugate to the stress on a plane parallel to that surface.*

Consider now the condition of a prismatic molecule A, bounded above and below by planes B B, C C, parallel to the sloping surface Y Y, and laterally by two pairs of parallel vertical planes. Let the common area of the upper and lower surfaces of this prism be unity, and its height Δx; then its volume is $\Delta x \cdot \cos\theta$, and its weight $w\,\Delta x \cdot \cos\theta$, which is equal and opposite to, and balanced by the excess of the vertical pressure on its lower face above the vertical pressure on its upper face. Therefore, the pressures parallel to the sloping surface, on the vertical faces of the prism, must balance each other independently; therefore they must be of equal mean intensity throughout the whole extent of the layer between the planes B B, C C; whence follows—

THEOREM III. *The state of stress, at a given uniform depth below the sloping surface, is uniform.*

126. **On the Parallel Projection of Stress and Weight.**—In applying the principles of parallel projection to distributed forces, it is to be borne in mind that those principles, as stated in Chapter IV., are applicable to lines representing the *amounts* or *resultants* of distributed forces, and *not their intensities.* The relations amongst the intensities of a system of distributed forces, whose resultants have been obtained by the method of projection, are to be arrived at by a subsequent process of dividing each projected resultant by the projected space over which it is distributed.

Examples of the application of processes of this kind to practical questions will appear in the Second Part.

CHAPTER VI.

ON STABLE AND UNSTABLE EQUILIBRIUM.

127. **Stable and Unstable Equilibrium of a Free Body.**—Suppose a body, which is in equilibrio under a balanced system of forces, to be free to move, and to be caused to deviate to a small extent from its position of equilibrium. Then if the body tends to deviate further from its original position, its equilibrium is said to be *unstable;* and if it tends to return to its original position, its equilibrium is said to be *stable.*

Cases occur in which the equilibrium of the same body is stable for one kind or direction of deviation, and unstable for another.

When the body neither tends to deviate further, nor to recover its original position, its equilibrium is said to be *indifferent.*

The solution of the question, whether the equilibrium of a given body under given forces is stable, unstable, or indifferent, for a given kind of deviation of position, is effected by supposing the deviation made, and finding the resultant of the forces which act on the body, altered as they may be by the deviation, in amount, in position, or in both. If this resultant acts towards the same direction with the deviation, the equilibrium is unstable—if towards the opposition direction, stable—and if the resultant is still nothing, the equilibrium is indifferent.

The disturbance of a free body from a position of stable equilibrium causes it to oscillate about that position.

128. **Stability of a Fixed Body.**—The term "stability," as applied to the condition of a body forming part of a structure, has, in most cases, a meaning different from that explained in the last Article, viz., the property of *remaining in equilibrio,* without sensible deviation of position, notwithstanding certain deviations of the load, or externally applied force, from its mean amount or position. Stability, in this sense, forms one of the principal subjects of the second part of this treatise.

PART II.

THEORY OF STRUCTURES.

CHAPTER I.

DEFINITIONS AND GENERAL PRINCIPLES.

129. **Structures—Pieces—Joints.**—Structures have already, in Article 15, been distinguished from machines. A structure consists of two or more solid bodies, called its *pieces*, which touch each other, and are connected at portions of their surfaces called *joints.*

130. **Supports—Foundations.**—Although the pieces of a structure are fixed relatively to each other, the structure as a whole may be either fixed or moveable relatively to the earth.

A fixed structure is supported on a part of the solid material of the earth, called the *foundation* of the structure; the pressures by which the structure is supported, being the resistances of the various parts of the foundation, may be more or less oblique.

A moveable structure may be supported, as a ship, by floating in water, or as a carriage, by resting on the solid ground through wheels. When such a structure is actually in motion, it partakes to a certain extent of the properties of a machine; and the determination of the forces by which it is supported requires the consideration of dynamical as well as of statical principles; but when it is not in actual motion, though capable of being moved, the pressures which support it are determined by the principles of statics; and it is obvious that they must be wholly vertical, and have their resultant equal and directly opposed to the weight of the structure.

131. **The Conditions of Equilibrium of a Structure** are the three following:—

1. *That the forces exerted on the whole structure by external bodies shall balance each other.* The forces to be considered under this head are—(1.) the Attraction of the Earth, that is, the *weight* of the structure; (2.) the *External Load,* arising from the pressures exerted against the structure by bodies not forming part of it nor of its foundation; (these two kinds of forces constitute the *gross* or *total load;* (3.) the *Supporting Pressures,* or resistance of the foundation. Those three classes of forces will be spoken of together as the *External Forces.*

II. *That the forces exerted on each piece of the structure shall balance each other.* These consist of—(1.) the *Weight* of the piece, and (2.) the *External Load* on it, making together the *Gross Load;* and (3.) the *Resistances,* or stresses exerted at the joints, between the piece under consideration and the pieces in contact with it.

III. *That the forces exerted on each of the parts into which the pieces of the structure can be conceived to be divided shall balance each other.* Suppose an ideal surface to divide any part of any one of the pieces of the structure from the remainder of the piece; the forces which act on the part so considered are—(1.) its weight, and (2.) (if it is at the external surface of the piece) the external stress applied to it, if any, making together its *gross load;* (3.) the *stress* exerted at the ideal surface of division, between the part in question and the other parts of the piece.

132. **Stability, Strength, and Stiffness.**—It is necessary to the permanence of a structure, that the three foregoing conditions of equilibrium should be fulfilled, not only under one amount and one mode of distribution of load, but under all the variations of the load as to amount and mode of distribution which can occur in the use of the structure.

Stability consists in the fulfilment of the *first* and *second* conditions of equilibrium of a structure under all variations of load within given limits. A structure which is deficient in stability gives way by the displacement of its pieces from their proper positions.

Strength consists in the fulfilment of the *third* condition of equilibrium of a structure for all loads not exceeding prescribed limits; that is to say, the greatest internal stress produced in any part of any piece of the structure, by the prescribed greatest load, must be such as the material can bear, not merely without immediate breaking, but without such injury to its texture as might endanger its breaking in the course of time.

A piece of a structure may be rendered unfit for its purpose not merely by being broken, but by being stretched, compressed, bent, twisted, or otherwise strained out of its proper shape. It is necessary, therefore, that each piece of a structure should be of such dimensions that its alteration of figure under the greatest load applied to it shall not exceed given limits. This property is called *stiffness,* and is so connected with strength that it is necessary to consider them together.

From the foregoing considerations, it is evident that the theory of structures may be divided into two divisions, relating, the first to STABILITY, or the property of resisting displacement of the pieces, and the second to STRENGTH and STIFFNESS, or the power of each piece to resist fracture and disfigurement.

CHAPTER II.

STABILITY.

133. **Resultant Gross Load.**—The mode of distribution of the intensity of the load upon a given piece of a structure affects the strength and stiffness only. So far as *stability* alone is concerned, it is sufficient to know the magnitude and position of the *resultant* of that load, which is to be found by means of the principles explained in the First Part of this work, and may then be treated as a single force.

134. **Centre of Resistance of a Joint.**—In like manner, when stability only is in question, it is sufficient to consider the position and magnitude of the *resultant* of the resistance or stress exerted between two pieces of a structure at the joint where they meet, and to treat that resultant as a single force. The point where its line of action traverses the joint is called the *centre of resistance* of that joint.

135. **A Line of Resistance** is a line, straight, angular, or curved, traversing the centres of resistance of the joints of a structure. It is to be borne in mind, that the direction of this line at any given joint does not *necessarily* coincide with the direction of the resistance at that joint, although it may so coincide in certain cases.

136. **Joints Classed.**—Joints, and the structures in which they occur, may be divided into three classes, according to the limits of the variation of position of which their centres of resistance are capable.

I. *Framework joints* are such as occur in carpentry, in frames of metal bars, and in structures of ropes and chains, fixing the ends of two or more pieces together, but offering little or no resistance to change in the relative angular positions of those pieces. In a joint of this class, the centre of resistance is at the middle of the joint, and does not admit of any variation of position consistently with security,

II. *Blockwork joints* are such as occur in masonry and brickwork, being plane or curved surfaces of contact, of considerable extent as compared with the dimensions of the pieces which they connect, capable of resisting a thrust more or less oblique, according to laws to be afterwards explained, but not of resisting a pull of sufficient intensity to be taken into account in practice. In such joints the position of the centre of resistance may be varied within certain limits.

III. *Fastened joints*, at which, by means of some strong cement, or of bolts, rivets, or other fastenings, two pieces are so connected that the joint fixes their relative angular position, and is capable of resisting a pull as well as a thrust. In this case, the centre of resistance may be at any distance from the centre of the joint; and there may even be no centre of resistance, when the resultant of the stress at the joint is a couple, as explained in Articles 91, 92, and 93. It is obvious that the effect of a joint thus cemented or fastened is to make the two pieces which it connects act as one piece, and that the resistance which it is capable of exerting is a question not of stability but of strength.

SECTION 1.—*Equilibrium and Stability of Frames.*

137. **Frame** is here used to denote a structure composed of bars, rods, links, or cords, attached together or supported by joints of the first class described in the last Article, the centre of resistance being at the middle of each joint, and the line of resistance, consequently, a polygon whose angles are at the centres of the joints. The condition of a single bar will be considered first, then that of a combination of two bars, then of three bars, and then of any number.

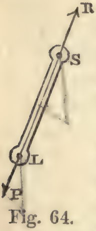

Fig. 64.

138. **Tie.**—Let fig. 64 represent a single bar of a frame, L the centre of resistance where the load is applied, and S the centre of resistance where the supporting force is applied; so that the straight line L S is the "line of resistance."

The bar is represented as being straight itself, that being the figure which connects the points L and S, and gives adequate stiffness and strength, with the least expenditure of material. But the bar may, consistently with the principles of this Article, be of any other figure connecting those two points, provided it is sufficiently strong and stiff to prevent their distance from altering to an extent inconsistent with the purposes of the structure.

The condition of the bar is the same with that of the solid in Article 23; and it is obvious that the load P, and the supporting resistance R, must be equal and directly opposed, and must both act along the line of resistance L S.

In the present case those forces are supposed to be directed outward, or *from* each other. The bar between L and S is in a state of *tension*, and the stress exerted between any two divisions of it is a *pull*, equal and opposite to the loading and supporting forces. A bar in this condition is called a *tie*. It is obvious that a *rope* or *chain* will answer the purpose of a tie.

The equilibrium of a tie is stable; for if its angular position be deviated, the equal forces P and R, which originally were directly opposed, now constitute a *couple* tending to restore the tie to its original position.

139. **Strut.**—If the equal and opposite forces applied to the two ends, L and S, of the line of resistance of a bar be directed (as in fig. 65) *inwards*, or *towards* each other, the bar, between L and S, is in a state of compression, and the stress exerted between any two divisions of it is a *thrust* equal and opposite to the loading and supporting forces. It is obvious that a flexible body will not answer the purpose of a strut.

The equilibrium of a moveable strut is unstable; for if its angular position be deviated, the equal forces P and R, which originally were directly opposed, now constitute a couple tending to make it deviate still farther from its original position.

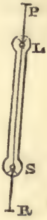

Fig. 65.

In order that a strut may have stability, its ends must be prevented from deviating laterally. Pieces connected with the ends of a strut for this purpose are called *stays*.

140. **Treatment of the Weight of a Bar.**—In the two preceding Articles, the weight of the bar itself has not been taken into account. But the principles of those Articles, *so far as they relate to the equilibrium of the bar as a whole,* continue to be applicable when the weight of the bar is treated in the following manner. Resolve that weight, by the principles of Articles 39 and 40, into two parallel components, acting through L and S respectively. Let P now represent not merely the external load, but the resultant of that load, and of the component of the weight which acts through L. Let R represent not merely the supporting force, but the resultant of that force and of the component of the weight which acts through S. Then P and R, as before, must be equal and directly opposed.

In many cases, the weight of a strut or tie is too small as compared with the load applied to it to require to be specially considered in practice.

141. **Beam under Parallel Forces.**—A bar supported at two points, and loaded in a direction perpendicular or oblique to its length is called a *beam*. In the first place, let the supporting pressures be parallel to each other and to the direction of the load; and let the load act *between* the points of support, as in fig. 66; where P represents the resultant of the gross load, including the weight of the beam itself, L, the point where the line of action of that

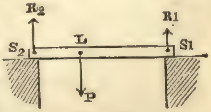

Fig. 66.

resultant intersects the axis of the beam, R_1, R_2, the two supporting pressures or resistances of the props parallel to, and in the same plane with P, and acting through the points S_1, S_2, in the axis of the beam.

Then, according to the Theorem of Article 39, each of those three forces is proportional to the distance between the lines of action of the other two; and the load is equal to the sum of the two supporting pressures; that is to say,

$$P : R_1 : R_2 :: \overline{S_1 S_2} : \overline{L S_2} : \overline{L S_1}; \text{..........} (1.)$$

$$\text{and } P = R_1 + R_2 \text{........................} (2.)$$

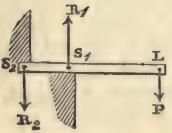

Fig. 67.

Next, let the load act *beyond* the points of support, as in fig. 67, which represents a cantilever or projecting beam, held up by a wall or other prop at S_1, held down by a notch in a mass of masonry or otherwise at S_2, and loaded so that P is the resultant of the load, including the weight of the beam. Then the proportional equation (1) remains exactly as before; but the load is equal to the difference of the supporting pressures; that is to say,

$$P = R_1 - R_2 \text{........................} (3.)$$

In these examples the beam is represented as horizontal; but the same principles would hold if it were inclined; for the proportions amongst the distances between parallel lines in the same plane are the same, whether they be measured in a direction perpendicular or oblique to those lines.

142. **Beam under Inclined Forces.**—Let the directions of the supporting forces R_1, R_2, be now inclined to that of the resultant of the load, P, as in fig. 68. This case is that of the equilibrium of three forces treated of in Articles 51 and 52; and consequently the following principles apply to it.

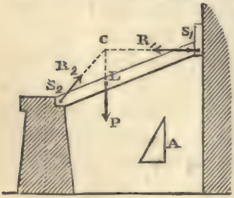

Fig. 68.

I. The lines of action of the supporting forces and of the resultant of the load must be in one plane.

II. They must intersect in one point (C, fig. 68).

III. Those three forces must be proportional to the three sides of triangle A, respectively parallel to their directions; or in other words, to the sides and diagonal of a parallelogram.

Problem. *Given the resultant of the load in magnitude and position,* P, *the line of action of one of the supporting forces,* R_1, *and the centre of resistance of the other,* S_2; *required the line of action of the second supporting force, and the magnitudes of both.*

Produce the line of action of R till it cuts the line of action of P at the point C; join C S_2; this will be the line of action of R_2; construct a triangle A with its sides respectively parallel to those three lines of action; the ratios of the sides of that triangle will give the ratios of the forces.—Q. E. I.

To express this algebraically, let i_1, i_2, be the angles made by the lines of action of the supporting forces with that of the resultant of the load; then because each side of a triangle is proportional to the sine of the angle between the other two,

$$P : R_1 : R_2 : : \sin (i_1 + i_2) : \sin i_2 : \sin i_1.$$

143. **Load supported by Three Parallel Forces.**—THEOREM. *If four parallel forces balance each other, let their lines of action be intersected by a plane, and let the four points of intersection be joined by six straight lines so as to form four triangles; each force will be proportional to the area of the triangle whose angles are in the lines of action of the other three.*

In fig. 69, let the plane of the paper represent the plane which is cut by the lines of action of the four forces in the points L, S_1, S_2, S_3; let P, R_1, R_2, R_3, denote the four parallel forces. Join the four points by six lines as in the figure, and produce each of the three lines S L till it cuts the opposite line S S in one of the points B.

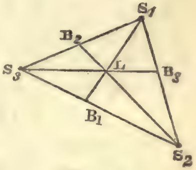

Fig. 69.

Because the forces balance each other, the resultant of R_2 and R_3, whose magnitude is $R_2 + R_3$, must traverse B_1; and because the resultant of that resultant and R_1 is equal and opposite to P, we must have the following proportion:—

$$P : R_1 : : \overline{S_1 B_1} : \overline{L B_1} : : \triangle S_1 S_2 S_3 : \triangle S_2 L S_3;$$

and applying the same reasoning to the forces R_2, R_3, we find the proportions,

$$P : R_1 : R_2 : R_3 : : \triangle S_1 S_2 S_3 : \triangle S_2 L S_3 : \triangle S_3 L S_1 : \triangle S_1 L S_2.$$

—Q. E. D.

By the aid of this Theorem may be determined the proportion in which the load of a given body is distributed amongst three props, exerting parallel supporting forces.

144. **Load supported by Three Inclined Forces.**—The case of a load supported by three inclined forces is that considered in Articles 54 and 56. The lines of action of the three supporting forces must intersect that of the load in one point; and the magnitudes of the three supporting forces are represented by the three edges of a parallelopiped, whose diagonal represents the load.

145. **Frame of Two Bars—Equilibrium.**—PROBLEM. Figures 70, 71, and 72 represent three cases in which a frame consisting of two

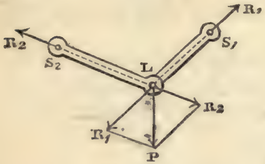

Fig. 70.

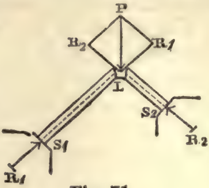

Fig. 71.

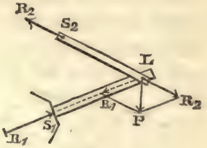

Fig. 72.

bars, jointed to each other at the point L, is loaded at that point with a given force, P, and is supported by the connection of the bars at their farther extremities, S_1, S_2, with fixed bodies. It is required to find the stress on each bar, and the supporting forces at S_1 and S_2.

Resolve the load P (as in Article 55) into two components, R_1, R_2, acting along the respective lines of resistance of the two bars. Those components are the loads borne by the two bars respectively; to which loads the supporting forces at S_1, S_2, are equal and directly opposed.—Q. E. I.

The symbolical expression of this solution is as follows :—let i_1, i_2, be the respective angles made by the lines of resistance of the bars with the line of action of the load ; then

$$P : R_1 : R_2 : : \sin (i_1 + i_2) : \sin i_2 : \sin i_1.$$

The inward or outward direction of the forces acting along each bar indicates that the stress is a thrust or a pull, and the bar a strut or a tie, as the case may be. Fig. 70 represents the case of two ties ; fig. 71 that of two struts (such as a pair of rafters abutting against two walls); fig. 72 that of a strut, L S_1, and a tie, L S_2 (such as the gib and the tie-rod of a crane).

146. **Frame of Two Bars—Stability.**—A frame of two bars is stable as regards deviations in the plane of its lines of resistance.

With respect to *lateral* deviations of angular position, in a direction perpendicular to that plane, a frame of two ties is stable; so also is a frame consisting of a strut and a tie, when the direction of the load inclines *from* the line S_1 S_2, joining the points of support.

A frame consisting of a strut and a tie, when the direction of the load inclines *towards* the line S_1 S_2, and a frame of two struts in all cases, are unstable laterally, unless provided with lateral stays.

These principles are true of *any pair of adjacent bars whose farther centres of resistance are fixed ;* whether forming a frame by themselves, or a part of a more complex frame.

147. **Treatment of Distributed Loads.**—Before applying the principles of Article 145, or those of the following Articles, to frames in which the load, whether external or arising from the weight of

the bars, is distributed over their length, it is necessary to reduce that distributed load to an equivalent load, or series of loads, applied at the centres of resistance. The steps in this process are as follows:—

I. Find the resultant load on each single bar.

II. Resolve that load, as in Article 141, into two parallel components acting through the centres of resistance at the two ends of the bar.

III. At each centre of resistance where two bars meet, combine the component loads due to the loads on the two bars into one resultant, which is to be considered as the total load acting through that centre of resistance.

IV. When a centre of resistance is also a point of support, the component load acting through it, as found by step II. of the process, is to be left out of consideration until the supporting force required by the system of loads at the other joints has been determined; with this supporting force is to be compounded a force equal and opposite to the component load acting directly through the point of support, and the resultant will be the total supporting force.

In the following Articles of this section, all the frames will be supposed to be loaded only at those centres of resistance which are *not* points of support; and therefore, in those cases in which components of the load act directly through the points of support also, forces equal and opposite to such components must be combined with the supporting forces as determined in the following Articles, in order to complete the solution.

148. **Triangular Frame.**—Let fig. 73 represent a triangular frame, consisting of the three bars A, B, C, connected at the three joints 1, 2, 3, viz.: C and A at 1, A and B at 2, B and C at 3. Let a load P_1 be applied at the joint 1 in any given direction; let supporting forces, P_2, P_3, be applied at the joints 2, 3; the lines of action of those two forces must be in the same plane with that of P_1, and must either be parallel to it or intersect it in one point. The latter case is taken first, because its solution comprehends that of the former.

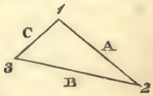

Fig. 73.

The three external forces, in virtue of Article 131, condition I., balance each other, and are therefore proportional to the three sides of a triangle respectively parallel to their directions. In fig. 73* let A B C be such a triangle, in which

$\overline{CA}$ represents P_1,
$\overline{AB}$ ... P_2,
$\overline{BC}$ ... P_3,

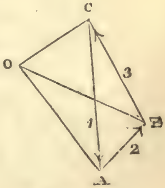

Fig. 73*.

Then by the conditions of equilibrium of a frame of two bars (Article 145), the external force P_1 applied at the joint 1, and the resistances or stresses along the bars C and A which meet at that joint, are represented in magnitude by the sides of a triangle respectively parallel to their directions. Therefore, in fig. 73*, draw $\overline{CO}$ parallel to the bar C, and $\overline{AO}$ parallel to the bar A, meeting in the point O, and those two lines will represent the stresses on the bars C and A respectively. In the same manner it is proved, that $\overline{BO}$ represents the stress on the bar B. The three lines C O, A O, B O, meet in one point O, because the components along the line of direction of a given bar, of the external forces applied at its two extremities, are equal and directly opposed.

Hence follows the following

THEOREM. *If three forces be represented by the three sides of a triangle, and if three straight lines radiating from one point be drawn to the three angles of that triangle, then a triangular frame whose lines of resistance are parallel to the three radiating lines will be in equilibrio under the three given forces, each force being applied to the joint where the two lines of resistance meet, which are parallel to the radiating lines contiguous to that side of the original triangle which represents the force in question.*

Also, the lengths of the three radiating lines will represent the stresses on the bars to which they are respectively parallel.

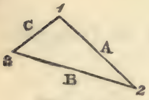

Fig. 74.

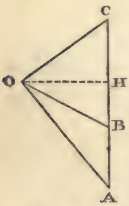

Fig. 74*.

149. **Triangular Frame under Parallel Forces.**—When the three external forces are parallel to each other, the triangle of forces A B C of fig. 73* becomes a straight line C A, as in fig. 74*, divided into two segments by the point B. Let straight lines radiate from O to A, B, C; and let fig. 74 represent a triangular frame whose sides 1 2 or A, 2 3 or B, 3 1 or C, are respectively parallel to O A, O B, O C; then if the load $\overline{CA}$ be applied at 1 (fig. 74), $\overline{AB}$ applied at 2, and $\overline{BC}$ applied at 3, are the supporting forces required to balance it; and the radiating lines $\overline{OA}$, $\overline{OB}$, $\overline{OC}$, represent the stresses on the bars A, B, C, respectively.

From O let fall $\overline{OH}$ perpendicular to C A, the common direction of the external forces. Then that line will represent a component of the stress, which is of equal amount in each bar. When $\overline{CA}$, as is usually the case, is vertical, $\overline{OH}$ is horizontal; and the force represented by it is called the "*horizontal thrust*" of the frame. *Horizontal Stress* or *Resistance* would be a more precise term; because the force in question is a pull in some parts of the frame, and a thrust in others.

In fig. 74, A and C are *struts*, and B a *tie*. If the frame were exactly inverted, all the forces would bear the same proportions to each other; but A and C would be *ties*, and B a *strut*.

The trigonometrical expression of the relations amongst the forces acting in a triangular frame, under parallel forces, is as follows:—

Let a, b, c, denote the respective angles of inclination of the bars A, B, C, to the line $\overline{OH}$ (that is, in general, to a horizontal line).

Then,

$$\left.\begin{array}{rl} \text{Load} & \overline{CA} = \overline{OH}\cdot(\tan c \pm \tan a); \\ \text{Supporting} \left\{ \begin{array}{c} \\ \end{array}\right. & \overline{AB} = \overline{OH}\cdot(\tan a \mp \tan b); \\ \text{Forces} \quad & \overline{BC} = \overline{OH}\cdot(\tan b \pm \tan c); \end{array}\right\}\ldots\ldots\ldots(1.)$$

The sign $\left\{\begin{array}{c} + \\ - \end{array}\right\}$ is to be used when the two inclinations are in $\left.\begin{array}{l} \text{opposite directions} \\ \text{the same direction.} \end{array}\right.$

$$\text{Stresses}\left\{\begin{array}{l} \overline{OA} = \overline{OH}\cdot\sec a \\ \overline{OB} = \overline{OH}\cdot\sec b \\ \overline{OC} = \overline{OH}\cdot\sec c \end{array}\right\}\ldots\ldots\ldots\ldots(2.)$$

$$\overline{OH} = \frac{\overline{CA}}{\tan c \pm \tan a}\ldots\ldots\ldots\ldots\ldots\ldots(3.)$$

150. **Polygonal Frame—Equilibrium.**—The Theorem of Article 148 is the simplest case of a general theorem respecting polygonal frames consisting of any number of bars, which is arrived at in the following manner. In fig. 75, let A, B, C, D, E, be the lines of resistance of the bars of a polygonal frame, connected together at the joints, whose centres of resistance are, 1 between A and B, 2 between B and C, 3 between C and D, 4 between D and E, and 5 between E and A. In the figure, the frame consists of five bars; but the demonstration is applicable to any number. From a point O, in fig. 75* (which may be called the *Diagram of Forces*), draw radiating lines $\overline{OA}$, $\overline{OB}$, $\overline{OC}$, $\overline{OD}$, $\overline{OE}$, parallel respectively to the lines of resistance of the bars; and on those radiating lines take any lengths whatsoever, to represent the stresses on the several bars, which may have any magnitudes within the limits of strength of the material. Join the points thus found by straight lines, so as to form a closed polygon ABCDEA; then it is evident that $\overline{AB}$ is the ex-

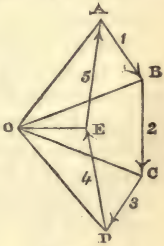

Fig. 75*.

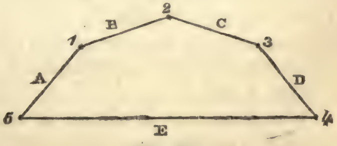

Fig. 75.

ternal force, which being applied at the joint 1 of A and B, will produce the stress $\overline{OA}$ on A and $\overline{OB}$ on B; that $\overline{BC}$ is the external force which being applied at the joint 2 of B and C, will produce the stress $\overline{OB}$ on B (already mentioned) and $\overline{OC}$ on C; and so on for all the sides of the polygon of forces A B C D E A. Hence follows this

THEOREM. *If lines radiating from a point be drawn parallel to the lines of resistance of the bars of a polygonal frame, then the sides of any polygon whose angles lie in those radiating lines will represent a system of forces, which, being applied to the joints of the frame, will balance each other; each such force being applied to the joint between the bars whose lines of resistance are parallel to the pair of radiating lines that enclose the side of the polygon of forces, representing the force in question. Also, the lengths of the radiating lines will represent the stresses along the bars to whose lines of resistance they are respectively parallel.*

151. **Open Polygonal Frame.**—When the polygonal frame, instead of being closed, as in fig. 75, is converted into an OPEN frame, by the omission of one bar, such as E, the corresponding modification is made in the diagram of forces by omitting the lines O E, D E, E A. Then the polygon of external forces becomes A B C D O A; and $\overline{DO}$ and $\overline{OA}$ represent the *supporting forces* respectively, equal and directly opposed to the stresses along the extreme bars of the frame, D and A, which must be exerted by the foundations (called in this case *abutments*), at the points 4 and 5, against the ends of those bars, in order to maintain the equilibrium.

152. **Polygonal Frame—Stability.**—The stability or instability of a polygonal frame depends on the principles already stated in Articles 138 and 139, viz., that if a bar be free to change its angular position, then if it is a tie it is stable, and if a strut, unstable; and that a strut may be rendered stable by fixing its ends.

For example, in the frame of fig. 75, E is a tie, and stable; A, B, C, and D, are struts, free to change their angular position, and therefore unstable.

But these struts may be rendered stable in the plane of the frame by means of stays; for example, let two stay-bars connect the joints 1 with 4, and 3 with 5; then the points 1, 2, and 3, are all fixed, so that none of the struts can change their angular positions. The same effect might be produced by two stay-bars connecting the joint 2 with 5 and 4.

The frame, as a whole, is unstable, as being liable to overturn laterally, unless provided with lateral stays, connecting its joints with fixed points.

Now, suppose the frame to be exactly inverted, the loads at 1, 2, and 3, and the supporting forces at 4 and 5, being the same as before. Then E becomes a strut; but it is stable, because its ends are fixed in position; and A, B, C, and D become ties, and are stable without being stayed.

An open polygon consisting of ties, such as is formed by A, B, C, and D when inverted, is called by mathematicians a *funicular polygon*, because it may be made of ropes.

It is to be observed, that the stability of an *unstayed* polygon of ties is of the kind described in Article 127, and admits of *oscillation* to and fro about the position of equilibrium. This oscillation may be injurious in practice, and stays may be required to prevent it.

153. **Polygonal Frame under Parallel Forces.**—When the external forces are parallel to each other, the polygon of forces of fig. 75* becomes a straight line A D, as in fig. 75**, divided into segments by the radiating lines; and each segment represents the external force which acts at the joint of the bars whose lines of resistance are parallel to the radiating lines that bound the segment. Moreover, the segment of the straight A D which is intercepted between the radiating lines parallel to the lines of resistance of *any two bars whether contiguous or not*, represents the resultant of the external forces which act at points *between the bars*.

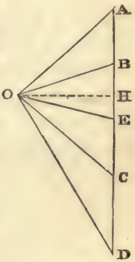

Fig. 75**.

Thus, $\overline{AD}$ represents the total load, consisting of the three portions $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, applied at 1, 2, 3 respectively. $\overline{DA}$ represents the total supporting force, equal and opposite to the load, consisting of the two portions $\overline{DE}$, $\overline{EA}$, applied at 4 and 5 respectively. $\overline{AC}$ represents the resultant of the load applied between the bars A and C; and similarly for any other pair of bars.

From O draw $\overline{OH}$ perpendicular to A D; then that line represents a component of the stress, whose amount is the same in each bar of the frame. When the load, as is usually the case, is vertical, that component is called the "*horizontal thrust*" of the frame, and, as in Article 149, might more correctly be called *horizontal stress* or *resistance*, seeing that it is a pull in some of the bars and a thrust in others.

The trigonometrical expression of these principles is as follows:—

Let the force $\overline{OH}$ be denoted simply by H.

Let i, i', denote the inclinations to O H of the lines of resistance of *any two bars*, contiguous or not.

Let R, R′, be the respective stresses which act along those bars.

Let P be the resultant of the external forces acting through the joint or joints between those two bars.

Then $$\mathrm{R} = \mathrm{H} \cdot \sec i; \ \mathrm{R}' = \mathrm{H} \cdot \sec i';$$
$$\mathrm{P} = \mathrm{H}(\tan i \pm \tan i').$$

The $\left\{\begin{matrix}\text{sum}\\ \text{difference}\end{matrix}\right\}$ of the tangents of the inclinations is to be used according as they are $\left\{\begin{matrix}\text{opposite}\\ \text{similar}\end{matrix}\right\}$.

154. **Open Polygonal Frame under Parallel Forces.**—When the frame becomes an *open* polygon by the omission of the bar E, the diagram of forces 75** is modified by omitting the line O E.

Then the supporting forces exerted by the abutments at 4 and 5, are no longer represented by the segments $\overline{\mathrm{DE}}$ and $\overline{\mathrm{EA}}$ of the line $\overline{\mathrm{AD}}$, but by the inclined lines $\overline{\mathrm{DO}}$ and $\overline{\mathrm{OA}}$, equal and directly opposed respectively to the stresses along the extreme bars of the frame, D and A.

Let i_d and i_a denote the angles of inclination of those bars.

Let $\mathrm{R}_d = \overline{\mathrm{OD}}$ and $\mathrm{R}_a = \overline{\mathrm{OA}}$ be the stresses along them.

Let $\Sigma \cdot \mathrm{P} = \overline{\mathrm{AD}}$ denote the total load on the frame. Then by the equations of Article 153,

$$\mathrm{H} = \frac{\Sigma \cdot \mathrm{P}}{\tan i_d + \tan i_a};$$

$$\mathrm{R}_d = \mathrm{H} \cdot \sec i_d; \ \mathrm{R}_a = \mathrm{H} \cdot \sec i_a.$$

155. **Bracing of Frames.**—A *brace* is a stay-bar on which there is a permanent stress. When the external forces applied to a polygonal frame, although balancing each other as an entire system, are distributed in a manner not consistent with the equilibrium of each bar separately, then by connecting two or more joints together by means of *braces*, which may be either struts or ties, the resistances of those braces may be made to supply, at the joints which they connect, the forces wanting to produce equilibrium of each bar.

The resistance of a brace introduces a pair of equal and opposite forces, acting along the line of resistance of the brace, upon the pair of joints which it connects. It therefore does not alter the *resultant* of the forces applied to that pair of joints in amount nor in position; but only the *distribution* of the components of that resultant on the pair of joints.

The same remark applies to any number of joints connected by a system of braces.

To exemplify the use of braces and the mode of determining the stresses on them, let fig. 76 represent a frame such as frequently occurs in iron roofs, consisting of two struts or rafters, A and E, and three tie-bars, B, C, and D, forming a polygon of five sides, jointed at 1, 2, 3, 4, 5, loaded vertically at 1, and supported by the vertical resistance of a pair of walls at 2 and 5. The joints 3 and

4, having no loads applied to them, are connected with 1 by the

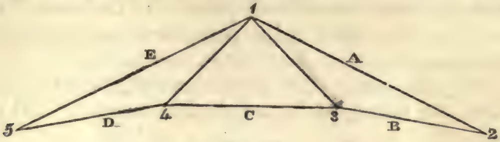

Fig. 76.

braces 1 4 and 1 3. It is required to find the stresses on those braces, and on the other pieces of the frame.

To make the diagram of forces (fig. 76*), draw the vertical line E A, as in Article 153, to represent the direction of the load and of the supporting forces.

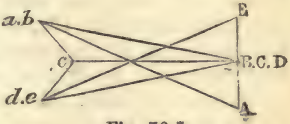

Fig. 76.*

The two segments of that line, $\overline{AB}$ and $\overline{DE}$, are to be taken to represent the supporting forces at 2 and 5; and the whole line E A will represent the load at 1. From the ends, and from the point of division of the *scale of external forces* E A, draw straight lines parallel respectively to the lines of resistance of the frame, each line being drawn from the point in E A that is marked with the corresponding letter. Then A *a* and B *b*, meeting at *a*, *b*, will represent the stresses along A and B respectively; and E *e* and D *d*, meeting in *d*, *e*, will represent the stresses along D and E respectively; but those four lines, instead of meeting each other and C *c* parallel to C in one point, leave *gaps*, which are to be filled up by drawing straight lines parallel *to the braces:* that is say, from *a*, *b*, to *c*, parallel to 13; and from *d*, *e*, to *c*, parallel to 4 1. Then those straight lines will represent the stresses along the braces to which they are respectively parallel; and C *c* will represent the tension along C. Upon analyzing the diagram of forces so constructed, it will be found that to each joint in the frame, fig. 76, there corresponds in fig. 76*, a triangle, or other closed polygon, having its sides respectively parallel, and therefore proportional, to the forces that act at that joint. For example,

Joints,	1,	2,	3,	4,	5,
Polygons,	E A *a c e* E;	A B *b* A;	B *c b* B;	D *d c* D;	D E *e* D.

The order of the letters indicates the directions in which the forces act relatively to the joints.*

The method of arranging the positions of braces, and determining the stresses along them, of which an example has been given, may be thus described in general terms.

If the distribution of the loads on the joints of a polygonal frame, though consistent with its equilibrium as a whole, be not consistent with the equilibrium of each bar, then, in the diagram of forces, when converging lines respectively parallel to the lines of resistance are drawn from the angles of the polygon of external forces, those converging lines, instead of meeting in one point, will be found to have gaps between them. The lines necessary to fill up those gaps will indicate the forces to be supplied by means of the resistance of braces.

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* This method of treating *braced* frames contains an improvement suggested by Mr. Clerk Maxwell in 1867.

156. **Rigidity of a Truss.**—The word *truss* is applied in carpentry and iron framing to a triangular frame, and to a polygonal frame to which rigidity is given by staying and bracing, so that its figure shall be incapable of alteration by turning of the bars about their joints. If each joint were *absolutely* of the kind described as the first class in Article 136, that is, like a hinge, incapable of offering any resistance to alteration of the relative angular position of the bars connected by it, it would be necessary, in order to fulfil the condition of rigidity, that every polygonal frame should be divided by the lines of resistance of stays and braces into triangles and other polygons so arranged, that every polygon of four or more sides should be surrounded by triangles on all but two sides and the included angle at farthest. For every unstayed polygon of four sides or more, with flexible joints, is flexible, unless all the angles except one be fixed by being connected with triangles.

Sometimes, however, a certain amount of stiffness in the joints of a frame, and sometimes the resistance of its bars to bending, is relied upon to give rigidity to the frame, when the load upon it is subject to small variations only in its mode of distribution. For example, in the truss of fig. 81 (for which see Article 161, farther on), the tie-beam A A is made in one piece, or in two or more pieces, so connected together as to act like one piece; and part of its weight is suspended from the joints C, C, by the rods C B, C B. These rods also serve to make the resistance of the tie-beam C C to being bent, act so as to prevent the struts A C, C C, C A, from deviating from their proper angular positions, by turning on the joints A, C, C, A. If A B, B B, and B A, were three distinct pieces, with flexible joints at B, B, it is evident that the frame might be disfigured by distortion of the quadrangle B C C B.

157. **Variations of Load on Truss.**—The object of stiffening a truss by braces is to enable it to sustain loads variously distributed; for were the load always distributed in one way, a frame might be designed of a figure exactly suited to that load, so that there should be no need of bracing.

The variations of load produce variations of stress on all the pieces of the frame, but especially on the braces; and each piece must be suited to withstand the greatest stress to which it is liable.

Some pieces, and especially braces, may have to act sometimes as struts and sometimes as ties, according to the mode of distribution of the load.

158. **Bar common to several Frames.**—When the same bar forms at the same time part of two or more different frames, the stress along it is determined by the aid of the following

THEOREM. *The stress on a bar common to two or more frames, is the resultant of the different stresses to which it is subject, in virtue of its position in the different frames.*

Illustrations of this will be found in the following Articles.

159. **Secondary Trussing.**—A *secondary truss* is a truss which is supported by another truss.

When a load is distributed over a great number of centres of resistance, it may be advantageous, instead of connecting all those centres by one polygonal frame, to sustain them by means of several small trusses, which are supported by larger trusses, and so on, the whole structure of secondary trusses resting finally on one large truss, which may be called the *primary truss.* In such a combination, the same piece may often form part of different trusses; and then the stress upon it is to be determined according to the Theorem of Article 158.

Example I. Fig. 77 represents a kind of secondary trussing common in the framework of iron roofs.

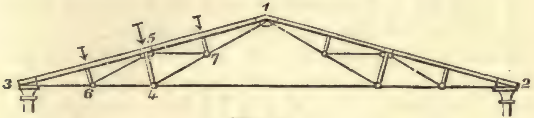

Fig. 77.

The entire frame is supported by pillars at 2 and 3, each of which sustains in all, half the weight.

1 2 3 is the *primary truss,* consisting of two rafters 1 3, 1 2, and a tie-rod 2 3.

The weight of a division of the roof is distributed over the rafters.

The middle point of each rafter is supported by a *secondary truss;* one of those is marked 1 4 3; it consists of a strut, 1 3 (the rafter itself), two ties 4 1, 4 3, and a strut-brace, 5 4, for transmitting the load, applied at 5, to the point where the ties meet.

Each of the two larger secondary trusses just described supports two *smaller secondary trusses* of similar form and construction to itself; two of those are marked 1 7 5, 5 6 3; and the subdivision of the load might be carried still farther.

In determining the stresses on the pieces of this structure, it is indifferent, so far as mathematical accuracy is concerned, whether we commence with the primary truss or with the secondary trusses; but by commencing with the primary truss, the process is rendered more simple.

(1.) *Primary Truss* 1 2 3. Let W denote the weight of the roof; then $\frac{1}{2}$ W is distributed over each rafter, the resultants acting through the middle points of the rafters. Divide each of those resultants into two equal and parallel components, each equal to $\frac{1}{4}$ W, acting through the ends of the rafter; then $\frac{1}{4}$ W is to be considered as directly supported at 3, $\frac{1}{4}$ W at 2, and $\frac{1}{4}\mathrm{W} + \frac{1}{4}\mathrm{W} = \frac{1}{2}$ W at 1; therefore the load at the joint 1 is

$$\mathrm{P} = \tfrac{1}{2}\mathrm{W}.$$

Let i be the inclination of the rafters to the horizon; then by the equations of Article 149

$$\mathrm{H} = \frac{\frac{1}{2}\mathrm{W}}{2\tan i} = \frac{\mathrm{W}}{4\tan i};\ \dots\dots\dots\dots(1.)$$

This is the pull upon the horizontal tie-rod of the primary truss, 2 3; and the thrust on each of the rafters 1 3, 1 2, is given by the equation

$$\mathrm{R} = \mathrm{H}\sec i = \frac{\mathrm{W}\operatorname{cosec} i}{4}\ \dots\dots\dots\dots(2.)$$

(2.) *Secondary Truss* 1 4 3 5. The rafter 1 3 has the load $\frac{1}{2}$ W distributed over it; and reasoning as before, we are to leave two quarters of this out of the calculation, as being directly supported at 1 and 3, and to consider one-half, or $\frac{1}{4}$ W, as being the vertical load at the point 5. The truss is to be considered as consisting of a polygon of four pieces, 5 1, 1 4, 4 3, 3 5, two of which happen to be in the same straight line, and of the strut-brace, 5 4, which exerts obliquely upwards against 5, and obliquely downwards against 4, a thrust equal to the component perpendicular to the rafter of the load $\frac{1}{4}$ W; which thrust is given by the equation

$$\mathrm{R}_{54} = \tfrac{1}{4}\mathrm{W}\cos i\ \dots\dots\dots\dots(3.)$$

Then we easily obtain the following values of the stresses on the rafter and ties, in which each stress is distinguished by having affixed to the letter R the numbers denoting the two joints between which it acts.

$$\left.\begin{array}{ll}
\begin{array}{c}\text{Pulls}\\ \text{on ties}\end{array} & \left\{\mathrm{R}_{43} = \mathrm{R}_{41} = \dfrac{\mathrm{R}_{54}}{2\sin i} = \dfrac{1}{8}\mathrm{W}\cot\!\text{an}\, i;\right. \\[2ex]
\begin{array}{c}\text{Thrusts}\\ \text{on}\\ \text{rafter}\end{array} & \left\{\begin{array}{l}
\mathrm{R}_{35} = \dfrac{\mathrm{R}_{54}}{2\tan i} + \dfrac{1}{8}\mathrm{W}\sin i = \dfrac{1}{8}\mathrm{W}\operatorname{cosec} i \\[2ex]
\mathrm{R}_{51} = \dfrac{\mathrm{R}_{54}}{2\tan i} - \dfrac{1}{8}\mathrm{W}\sin i = \dfrac{1}{8}\mathrm{W}(\operatorname{cosec} i - 2\sin i)
\end{array}\right.
\end{array}\right\}(4.)$$

The difference between the thrusts on the two divisions of the rafter,

$$R_{35} - R_{51} = \tfrac{1}{4} W \sin i,$$

is the component *along the rafter* of the load at the point 5.

(3.) *Smaller Secondary Trusses,* 1 7 5, 5 6 3.—These trusses are similar in every respect to the larger secondary trusses, except that the load on each point is one-half, and consequently each of the stresses is reduced to one-half of the corresponding stress in the equations 3 and 4.

(4.) *Resultant Stresses.* The pull on the middle division of the great tie-rod 2 3 is simply that due to the primary truss, 1 2 3. The pull on the tie 4 7 is simply that due to the secondary truss 1 4 3. The pulls on the ties 5 7, 5 6, are simply those due to the smaller secondary trusses, 1 5 7, 5 6 3. But agreeably to the Theorem of Art. 158, the pull on the tie 1 7 is the sum of those due to the larger secondary truss 1 4 3, and the smaller secondary truss 1 7 5. The pull on 6 4 is the sum of those due to the primary truss 1 2 3 and to the larger secondary truss 1 4 3. The pull on 6 3 is the sum of those due to the primary truss 1 2 3, to the larger secondary truss 1 4 3, and to the smaller secondary truss 5 6 3. The thrust on each of the four divisions of the rafter 1 3, is the sum of three thrusts, due respectively to the primary truss, the larger secondary truss, and one or other of the smaller secondary trusses.

Example II. Fig. 78 represents another form of truss common in roofs. Let W be the weight of the roof, as before, distributed over the rafters 1 2, 1 3. 2 3 is the great tie-rod; 1 7, 6 5, 8 9, suspension-rods; 7 6, 7 8, 5 4, 9 10, struts.

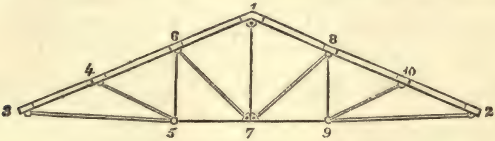

Fig. 78.

(1.) *Primary Truss* 1 2 3. The load at 1, as before, is to be taken as $= \frac{1}{2}$ W.

(2.) *Secondary Trusses* 7 6 3, 7 8 2. The load at 6 is to be held to consist of one-half of the load between 6 and 1, and one-half of the load between 6 and 3; that is, one-half of the load between 1 and 3, or $\frac{1}{4}$ W. The trusses are triangular, each consisting of two struts and a tie, and the stresses are to be found as in Article 149.

The suspension-rod 1 7 supports two-thirds of the load on 7 6 3, and two-thirds of the load on 7 8 2; that is, $\frac{2}{3} \cdot \frac{2}{4} \cdot W = \frac{1}{3} W$; and this, together with $\frac{1}{6}$ W which rests *directly* on 1, makes up the load of $\frac{1}{2}$ W, already mentioned.

(3.) *Smaller Secondary Trusses* 3 4 5, 9 10 2. Each of the points 4 and 10 sustains a load of $\frac{1}{6}$ W, from which the stresses on the bars of those smaller trusses can be determined.

One-half of the load on 4, that is $\frac{1}{12}$ W, hangs by the suspension-rod 6 5; and this, together with $\frac{1}{6}$ W, which rests directly on 6, makes up the load of $\frac{1}{4}$ W on that point, formerly mentioned. The same remarks apply to the suspension-rod 8 9.

(4.) *Resultant Stresses.* The pull between 5 and 9 is the sum of those due to the primary and larger secondary trusses; that between 5 and 3, and between 9 and 2, is the sum of the pulls due to the primary, larger secondary, and smaller secondary trusses.

The thrust on 1 6 is due to the primary truss alone; that on 6 4 to the primary and larger secondary truss; that on 4 3 to the primary, larger secondary, and smaller secondary trusses; and similarly for the divisions of the other rafter.

Example III. Suppose that instead of only three divisions, there are n divisions in each of the rafters 1 3, 1 2, of fig. 78; so that besides the middle suspension-rod 1 7, there are $n - 2$ suspension-rods under each rafter, or $2n - 4$ in all; and $n - 1$ sloping struts under each rafter, or $2n - 2$ in all. There will thus be $2n - 1$ centres of resistance; that is, the ridge-joint 1, and $n - 1$ on each rafter; and the load *directly supported* on each of these points will be $\frac{W}{2n}$.

The total load on the ridge-joint, 1, will be as before, $\frac{W}{2}$; that is to say, $\frac{W}{2n}$ directly supported, and $\frac{W}{2}\left(1 - \frac{1}{n}\right)$ hung by the middle suspension-rod.

The total load on the upper joint of any secondary truss, distant from the ridge-joint by m divisions of the rafter, will be, $\frac{n-m+1}{4n}$ W; that is to say, $\frac{W}{2n}$ directly supported, and $\frac{n - m - 1}{4n}$ W hung by a suspension-rod.

The stresses on the struts and tie of each truss, primary and secondary, being determined as in Article 149, are to be combined as in the preceding examples.

160. **Compound Trusses.**—Several frames, without being distinguishable into primary and secondary, may be combined and con-

nected in such a manner, that certain pieces are common to two or more of them, and require to have their stresses determined by the Theorem of Article 158.

Example I. In fig. 79, 8 9 represents part of the horizontal platform of a suspension bridge, supported and balanced by being hung from the top of a central pier, 1, by pairs of equally inclined rods or ropes, viz.:—1 8 and 1 9; 1 6 and 1 7; 1 4 and 1 5; 1 2 and 1 3.

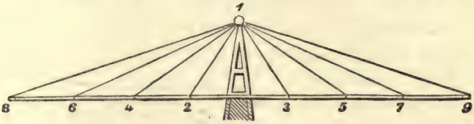

Fig. 79.

Here 8 1 9 is to be considered as a distinct triangular frame, consisting of a strut 8 9, and two ties 1 8 and 1 9, loaded with equal weights at 8 and 9, and supported at 1. Let x denote the height of the point of suspension 1 above the level of the loaded points, $y_8 = y_9$, the distance of those points on either side of the middle of the pier, P the load at each point, $R_8 = R_9$ the pull on each of the ties, 1 8, 1 9, T_{89} the thrust between 8 and 9 along the platform. Then we have

$$T_{89} = \frac{P y_8}{x};\ R_8 = \frac{P\sqrt{x^2 + y_8^2}}{x};$$

and similar equations for each of the other distinct frames 6 1 7, 4 1 5, 2 1 3.

Then using a similar notation in each case, the thrust along the platform

$$\left.\begin{array}{r}\text{between } 8 \text{ and } 6\\ ,,\quad 7 \text{ and } 9\end{array}\right\}\text{ is } T_{89},$$
$$\left.\begin{array}{r},,\quad 6 \text{ and } 4\\ ,,\quad 5 \text{ and } 7\end{array}\right\}\text{ is } T_{89} + T_{67},$$

and so on for as many pairs of divisions as the platform consists of.

Example II. Fig. 80 represents the framework for supporting

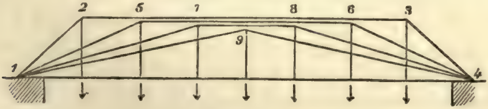

Fig. 80.

one side of a timber bridge, resting on two piers at 1 and 4. It consists of four distinct trusses, viz.,

1 2 3 4 loaded at 2 and 3,
1 5 6 4 „ 5 „ 6,
1 7 8 4 „ 7 „ 8,
1 9 4 „ 9;

but all those trusses have the same tie-beam, 1 4; and the pull along that tie-beam is the sum of the pulls due to the four trusses.

161. **Resistance of Frame at a Section.**—THEOREM. *If a frame be acted upon by any system of external forces, and if that frame be conceived to be completely divided into two parts by an ideal surface, the stresses along the bars which are intersected by that surface, balance the external forces which act on each of the two parts of the frame.*

This theorem, which requires no demonstration, furnishes in some cases the most convenient method of determining the stresses along the pieces of a frame. The following consideration shows to what extent its use is limited.

CASE 1. When the lines of resistance of the bars, and the lines of action of the external forces, are all in one plane, let the frame be supposed to be intersected anywhere by a plane at right angles to its own plane. Take the line of intersection of these two planes for an axis of co-ordinates; say for the axis of y, and any convenient point in it for the origin O; let the axis of x be perpendicular to this, and in the plane of the frame, and the axis of z perpendicular to both, and in the plane of section.

The external forces applied to the part of the frame at one side of the plane of section (either may be chosen) being treated as in Article 59, give three data, viz., the total force along $x = \mathrm{F}_x$; the total force along $y = \mathrm{F}_y$, and the moment of the couple acting round $z = \mathrm{M}$; and the bars which are cut by the plane of section must exert resistances capable of balancing those two forces and that couple. If not more than three bars are cut by the plane of section, there are not more than three unknown quantities, and three relations between them and given quantities, so that the problem is determinate; if more than three bars are cut by the plane of section, the problem is or may be indeterminate.

The formulæ to which this reasoning leads are as follows:—Let x be positive in a direction from the plane of section towards the part of the structure which is considered in determining F_x, F_y, and M; let $+ y$ lie to the right of $+ x$ when looking from z; let angles measured from O x towards $+ y$, that is, towards the right, be positive; and let the lines of resistance of the three bars cut by the plane of section make the angles i_1, i_2, i_3, with x. Let n_1, n_2, n_3, be the perpendicular distances of those three lines of resistance from O, distances towards the

$$\left.\begin{matrix}\text{right}\\ \text{left}\end{matrix}\right\}\text{ of O } x \text{ being considered as }\left\{\begin{matrix}\text{positive}\\ \text{negative}\end{matrix}\right\}.$$

Let R_1, R_2, R_3, be the resistances, or total stresses, along the three bars, pulls being positive, and thrusts negative. Then we have the following three equations :—

$$\left.\begin{aligned} F_x &= R_1 \cos i_1 + R_2 \cos i_2 + R_3 \cos i_3; \\ F_y &= R_1 \sin i_1 + R_2 \sin i_2 + R_3 \sin i_3; \\ -M &= R_1 n_1 + R_2 n_2 + R_3 n_3; \end{aligned}\right\} \ldots\ldots\ldots (1.)$$

from which the three quantities sought, R_1, R_2, R_3, can be found.

Speaking with reference to the given plane of section, F_x may be called the *normal stress*, F_y the *shearing stress*, and M the *moment of flexure* or *bending stress;* for it tends to bend the frame at the section under consideration.

Case 2. When the bars of the frame, and the forces applied to them, act in any direction, the forces applied to one of the two divisions of the frame are to be reduced to rectangular components; and the three resultant forces along these rectangular axes, F_x, F_y, F_z, and the three resultant couples round these three axes, M_x, M_y, M_z, are to be found as in Article 60. Those forces and couples must be equal and opposite to the corresponding forces and couples arising from the stresses along the bars cut by the section; and thus are obtained six equations between those stresses and known quantities; so that if the section cuts not more than six bars, the problem is determinate; if more, it is or may be indeterminate.

The equations are obtained as follows :—Let R denote the stress along any one of the bars, pull being positive and thrust negative. Let α, β, γ, be the inclinations of the line of resistance of that bar to the axes of x, y, z. Let n be its perpendicular distance from O. Conceive a plane to pass through O and through the line of resistance of the bar, and a normal to be drawn to that plane in such a direction, that looking from the end of that normal towards O, the bar is seen to lie to the right of O, and let λ, μ, ν, be the angles of inclination of that normal to the three axes. Let Σ denote the summation of six corresponding quantities for the six bars. Then the six equations are,

$$\left.\begin{aligned} F_x = \Sigma \cdot R \cos \alpha;\ F_y = \Sigma \cdot R \cos \beta;\ F_z = \Sigma \cdot R \cos \gamma; \\ -M_x = \Sigma \cdot R\, n \cos \lambda;\ -M_y = \Sigma \cdot R\, n \cos \mu; \\ -M_z = \Sigma \cdot R\, n \cos \nu; \end{aligned}\right\} (2).$$

from which the six stresses sought can be computed by elimination.

The plane of $y\,z$ being as before, that of the section, F_x is the *total direct stress* on it; F_y and F_z are the total shearing stresses; M_y and M_z are *bending couples*, and M_x a *twisting couple.*

Remarks.—Every problem respecting the equilibrium of frames which can be solved by the *method of sections* explained in this

Article, can also be solved by the *method of polygons* explained in the previous Articles; and the choice between the two methods is a question of convenience and simplicity in each particular case.

The following is one of the simplest examples of the solution of a problem in both ways. Fig. 81 represents a truss of a form very common in carpentry (already referred to in Article 156), and consisting of three struts, A C, C C, C A, a tie-beam A A, and two suspension-rods, C B, C B, which serve to suspend part of the weight of the tie-beam from the joints C C, and also to stiffen the truss in the manner mentioned in Article 156.

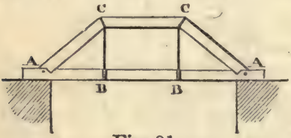

Fig. 81.

Let i denote the equal and opposite inclinations of the rafters A C, C A, to the horizontal tie-beam A A; and leaving out of consideration the portions of the load directly supported at A A, let P, P, denote equal vertical loads applied at C C, and — P, — P, equal upward vertical supporting forces applied at A A, by the resistance of the props. Let H denote the pull on the tie-beam, R the thrust on each of the sloping rafters, and T the thrust on the horizontal strut C C.

Proceeding by the *method of polygons*, as in Article 153, we find at once,

$$\left.\begin{array}{l} \mathrm{H} = -\mathrm{T} = \mathrm{P} \operatorname{cotan} i; \\ \mathrm{R} = -\mathrm{P} \operatorname{cosec} i. \end{array}\right\} \dots\dots\dots\dots\dots\dots (3.)$$

(Thrusts being considered as negative.)

To solve the same question by the *method of sections*, suppose a vertical section to be made by a plane traversing the centre of the right hand joint C; take that centre for the origin of co-ordinates; let x be positive towards the right, and y positive downwards; let x_1, y_1, be the co-ordinates of the centre of resistance at the right hand point of support A. When the plane of section traverses the centre of resistance of a joint, we are at liberty to suppose either of the two bars which meet at that joint on opposite sides of the plane of section to be cut by it at an insensible distance from the joint.

First, consider the plane of section as cutting C A. The forces and couple acting on the part of the frame to the right of the section are

$$\mathrm{F}_x = 0; \ \mathrm{F}_y = -\mathrm{P}$$
$$\mathrm{M} = -\mathrm{P}\, x_1.$$

Then, observing that for the strut A C, $n = 0$, and that for the tie A A, $n = y_1$, we have, by the equations 1 of this Article

$$R \cos i + H = F_x = 0;$$
$$R \sin i = -P;$$
$$H y_1 = -M = +P x_1;$$

whence we obtain, from the last equation,

$$\left.\begin{array}{l} H = \dfrac{P x_1}{y_1} = P \operatorname{cotan} i \\ \text{from the first, or from the second} \\ R = -\dfrac{H}{\cos i} = -P \operatorname{cosec} i. \end{array}\right\} \text{..........(4.)}$$

Next, conceive the section to cut C C at an insensible distance to the left of C. Then the equal and opposite applied forces + P at C, and — P at A, have to be taken into account; so that

$$F_x = 0;\ F_y = 0;\ M = -P x_1;$$

from the first of which equations we obtain

$$H + T = F_x = 0, \text{ and}$$
$$T = -H = -P \operatorname{cotan} i \text{..................(5.)}$$

In the example just given, the method of sections is tedious and complex as compared with the method of polygons, and is introduced for the sake of illustration only; but in the problems which are to follow, the reverse is the case, the solution by the method of sections being by far the more simple.

162. **A Half-Lattice Girder**, sometimes called a "Warren Girder," is represented in fig. 82. It consists essentially of a horizontal upper bar, a horizontal lower bar, and a series of diagonal bars sloping alternately in opposite directions, and dividing the space between the upper and lower bars into a series of triangles. In the example to be considered, the girder is supposed to be supported by the vertical resistance of piers at its ends A and B, and loaded with weights acting at or through the joints at the angles of the several triangles.

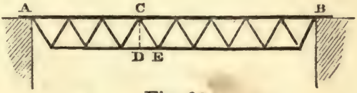

Fig. 82.

This girder might be treated as a case of secondary trussing, by considering the upper and lower and endmost diagonal bars as forming a polygonal truss like fig. 81, but inverted, supporting a smaller erect truss of the same kind, which supports a still smaller inverted truss, which supports a still smaller erect truss, and so on to the smallest truss, which is the middle triangle. But it is more simple to proceed by the method of sections, which must be applied successively to each division of the girder.

The load at each joint being known, the two supporting forces at A and B, are to be determined by the principles of the equilibrium of parallel forces in one plane (Articles 43, 44). Let P_A, P_B, denote those supporting forces, upward forces being treated as positive, and downward as negative; and let $-P$ denote the load at any joint, which may be a constant or a varying quantity for different joints.

Suppose now that it is required to find the stress along any one of the diagonals, such as C E, along the top bar immediately to the right of C, and along the bottom bar immediately to the left of E. Conceive the girder to be divided by a vertical plane of section C D, at an insensibly small distance to the right of C; take the intersection of this plane with the line of resistance of the top bar for the origin of co-ordinates, which sensibly coincides with C.

Let x denote the distance of any one of the joints to the left of the plane of section, from that plane. Let x_1 be the distance of the point of support A to the left of the same plane. Let y be positive upwards; so that for the joints of the upper bar, $y = 0$, and for those of the lower bar, $y = -h$, h denoting the vertical depth between the lines of resistance of the upper and lower bars.

Let i be the inclination of the diagonal C E to the horizontal axis of x. In the present instance this is positive; but had C E sloped the other way, it would have been negative.

Let the symbol $-\Sigma_C^A \cdot P$ denote the sum of the loads acting at the joints between the plane of section and the point of support A, *the load at the joint* C *being included.* Then for the total forces and couple acting on the division of the girder to the left of the plane of section, we have,—*direct force,* $F_x = 0$, because the applied forces are all vertical;—*shearing force,* $F_y = P_A - \Sigma_C^A \cdot P$; a force which is $\left\{\begin{array}{c}\text{positive or upward}\\ \text{negative or downward}\end{array}\right\}$ according as the plane of section lies $\left\{\begin{array}{c}\text{nearer to}\\ \text{farther from}\end{array}\right\}$ the point of support A, than a plane which divides the load into two portions equal respectively to the supporting pressures;—*bending couple* $M = P_A x_1 - \Sigma_C^A \cdot P x$; which is *upward,* and right-handed with respect to the axis of z.

Now let R_1 denote the stress along the upper bar at C, R_2 that along the lower bar at D, and R_3 that along the diagonal C E; then the equations 1 of Article 161 become the following:—

$$R_1 + R_2 + R_3 \cos i = 0; \text{ or } R_1 + R_3 \cos i = -R_2 \ldots (a.)$$

that is, the stress along the upper bar, and the horizontal component of the stress along the diagonal, are equal and opposite to the stress along the lower bar;

$$\mathrm{R}_3 \sin i = \mathrm{F}_y = \mathrm{P}_{\mathrm{A}} - \Sigma_{\mathrm{C}}^{\mathrm{A}} \cdot \mathrm{P}\,; \dots\dots\dots\dots\dots (b.)$$

that is, the vertical component of the stress along the diagonal, balances the shearing force;

$$-\mathrm{R}_2\, y = \mathrm{R}_2\, h = \mathrm{M} = \mathrm{P}_{\mathrm{A}}\, x_1 - \Sigma_{\mathrm{C}}^{\mathrm{A}} \cdot \mathrm{P}\, x\,; \dots\dots\dots (c.)$$

that is, the couple formed by the equal and opposite horizontal stresses of equation (*a*), acting at the ends of the arm *h*, balances the bending couple.

Finally, from the equations (*a*), (*b*), (*c*), are deduced the following values of the stresses:—

Pull on lower bar,

$$\mathrm{R}_2 = \frac{1}{h}(\mathrm{P}_{\mathrm{A}}\, x_1 - \Sigma_{\mathrm{C}}^{\mathrm{A}} \cdot \mathrm{P}\, x)\,;$$

Stress on diagonal,

$$\mathrm{R}_3 = \operatorname{cosec} i\, (\mathrm{P}_{\mathrm{A}} - \Sigma_{\mathrm{C}}^{\mathrm{A}} \cdot \mathrm{P})\,;$$

Thrust on upper bar,

$$\left.\begin{aligned} \mathrm{R}_1 &= -\mathrm{R}_2 - \mathrm{R}_3 \cos i \\ &= -\frac{1}{h}(\mathrm{P}_{\mathrm{A}}\, x_1 - \Sigma_{\mathrm{C}}^{\mathrm{A}} \cdot \mathrm{P}\, x) - \operatorname{cotan} i\, (\mathrm{P}_{\mathrm{A}} - \Sigma_{\mathrm{C}}^{\mathrm{A}} \cdot \mathrm{P}). \end{aligned}\right\} (1.)$$

Another, and sometimes a more convenient form, can be found for the second and third of those expressions. Let *s* denote the length of the diagonal C E, and x_1' the horizontal distance of its lower end E from the point of support A; then

$$s = \sqrt{(h^2 + (x_1' - x_1)^2)},$$

and also

$$\operatorname{cosec} i = \frac{s}{h}\,;\ \operatorname{cotan} i = \frac{x_1' - x_1}{h}\,; \dots\dots\dots\dots\dots (2.)$$

which substitutions having been made, give

$$\left.\begin{aligned} \mathrm{R}_3 &= \frac{s}{h}(\mathrm{P}_{\mathrm{A}} - \Sigma_{\mathrm{C}}^{\mathrm{A}} \cdot \mathrm{P}) \\ \mathrm{R}_1 &= -\frac{1}{h}\left\{\mathrm{P}_{\mathrm{A}}\, x_1 - \Sigma_{\mathrm{C}}^{\mathrm{A}} \cdot \mathrm{P}\, x + (x_1' - x_1)(\mathrm{P}_{\mathrm{A}} - \Sigma_{\mathrm{C}}^{\mathrm{A}} \cdot \mathrm{P})\right\} \\ &= -\frac{1}{h}(\mathrm{P}_{\mathrm{A}}\, x_1' - \Sigma_{\mathrm{C}}^{\mathrm{A}} \cdot \mathrm{P}\, x') \end{aligned}\right\} (3.)$$

in which x' is taken to denote the *horizontal distance of any joint to the left of a vertical plane traversing* E. The last expression for R_1 is exactly what would have been obtained by supposing the plane of section to traverse E instead of C.

Any given diagonal is $\left\{ \begin{matrix} \text{a tie} \\ \text{a strut} \end{matrix} \right\}$ according as it slopes $\left\{ \begin{matrix} \text{with} \\ \text{against} \end{matrix} \right\}$ the direction of the shearing force F_y acting on a plane of section traversing it.

163. **Half-Lattice Girder—Uniform Load.**—CASE 1. *Every joint loaded.* When the joints of a half-lattice girder are at equal distances apart horizontally, and loaded with equal weights, the equations take the following form:—

Let N denote the even number of divisions into which vertical lines drawn through the joints divide the total length or *span* between the points of support. Let l be the length of one of these divisions, so that $N\,l$ is the total span. The total number of loaded joints is $N - 1$; this must be an odd number, and there must be a middle joint dividing the girder into two halves, symmetrical to each other in every respect, figure, load, support, and stress, so that it is sufficient to consider one half only; let the left hand half be chosen. Let the middle joint be denoted by O, and the other joints by numbers in the order of their distances from the middle joint, so that the joint numbered n shall be at the distance $n\,l$ from O. The even numbers denote joints on the same horizontal bar with O; the odd numbers those on the other.

The total load on the girder is

$$-(N-1)\,P,$$

of which one-half is supported on each pier; that is to say,

$$P_A = P_B = \frac{N-1}{2}\,P. \qquad\qquad (1.)$$

The stress on the upper bar is everywhere a thrust;—that on the lower bar a pull. Diagonals which $\left\{ \begin{matrix} \text{rise} \\ \text{fall} \end{matrix} \right\}$ from the middle towards the ends are $\left\{ \begin{matrix} \text{ties} \\ \text{struts} \end{matrix} \right\}$. By these principles the *kind* of stress on each piece is determined; it remains only to compute the *amount.*

Let n be the number of any joint; it is required to find the stress along the diagonal which runs from that joint towards the middle of the girder, and the stress along that part of either of the horizontal bars which is opposite the joint.

Suppose a vertical section to be made at an insensible distance from the joint, intersecting the diagonal in question and the horizontal bars.

Between O and either pier there are $\frac{N}{2} - 1$ loaded joints; between O and the plane of section in question, there are $n-1$ joints; hence between the plane of section and the pier there are $\frac{N}{2} - n$ joints. Consequently

$$\Sigma_{C}^{A} \cdot P = \left(\frac{N}{2} - n\right) P;$$

and the *shearing force* is

$$F_y = P_A - \Sigma_{C}^{A} \cdot P = \left(n - \frac{1}{2}\right) \cdot P; \ldots\ldots\ldots\ldots(2.)$$

So that it increases at an uniform rate from the middle towards the ends.

The distance of the n^{th} joint from the pier is $x_1 = \left(\frac{N}{2} - n\right) \cdot l$. Hence the upward moment of the supporting force is

$$P_A x_1 = \left(\frac{N}{2} - \frac{1}{2}\right) \left(\frac{N}{2} - n\right) Pl.$$

The downward moment of the load at the joints between the plane of section and the pier is found from the consideration, that the leverage of the nearest portion of that load is nothing, and that of the farthest $\left(\frac{N}{2} - 1 - n\right) l$, so that the mean leverage is $\frac{1}{2}\left(\frac{N}{2} - 1 - n\right) l$; which being multiplied by the load $\Sigma_{C}^{A} \cdot P$ as found above, gives for the moment

$$-\Sigma_{C}^{A} \cdot P x = -\frac{1}{2}\left(\frac{N}{2} - 1 - n\right)\left(\frac{N}{2} - n\right) \cdot P l.$$

hence the bending couple is

$$M = P_A x_1 - \Sigma_{C}^{A} \cdot P x = \frac{1}{2}\left(\frac{N}{2} + n\right)\left(\frac{N}{2} - n\right) \cdot P l$$

$$= \frac{1}{2}\left(\frac{N^2}{4} - n^2\right) \cdot P l; \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.)$$

that is to say, it is proportional to the *product of the segments into which the plane of section divides the length of the girder*, and is greatest at the middle, where it is $\frac{N^2}{8} \cdot P l$.

The uniform inclination of the diagonals, in one direction or the other, being denoted by i, we have

$$\operatorname{cosec} i = \frac{s}{h} = \frac{\sqrt{h^2 + l^2}}{h};$$

and hence the amounts of the stresses are,

$$\left.\begin{array}{l} \textit{Along the diagonal,} \\ \quad \mathrm{R}' = \mathrm{F}_y \cdot \operatorname{cosec} i = \dfrac{s}{h}\left(n - \dfrac{1}{2}\right) \mathrm{P}; \\ \textit{Along the horizontal bar,} \\ \quad \mathrm{R} = \dfrac{\mathrm{M}}{h} = \left(\dfrac{\mathrm{N}^2}{4} - n^2\right) \cdot \dfrac{\mathrm{P}l}{2h}. \end{array}\right\} \text{..........(4.)}$$

These stresses are stated irrespective of their signs, which are to be determined by the rules laid down after equation 1.

The least value of R′ is for the diagonals next the middle point, for which $n = 1$, and $\mathrm{R}' = \frac{s\mathrm{P}}{2h}$. Its greatest value is for the diagonals next the piers, for which $n = \frac{\mathrm{N}}{2}$, and $\mathrm{R}' = \frac{(\mathrm{N}-1)s\mathrm{P}}{2h}$; in fact, these diagonals sustain the entire load.

The least value of the horizontal stress R is at the divisions of one of the horizontal bars next the piers, for which $n = \frac{\mathrm{N}}{2} - 1$, and $\mathrm{R} = \frac{(\mathrm{N}-1)\mathrm{P}l}{2h}$.

The greatest value of R is at the division of one of the horizontal bars opposite the middle joint, for which $n = 0$, and $\mathrm{R} = \frac{\mathrm{N}^2 \mathrm{P} l}{8h}$.

CASE 2. *Every alternate joint loaded.* Suppose those joints only to be loaded which are distant by an even number of divisions from the piers. The total number of loaded joints is $\frac{\mathrm{N}}{2} - 1$, the load on the girder $- \left(\frac{\mathrm{N}}{2} - 1\right)\mathrm{P}$, and the supporting pressures

$$\mathrm{P}_\mathrm{A} = \mathrm{P}_\mathrm{B} = \left(\frac{\mathrm{N}}{4} - \frac{1}{2}\right)\mathrm{P} \text{..................(5.)}$$

Let n be the number of any *loaded* joint, $n - 1$ that of the unloaded joint nearest to it on the side next the middle of the girder, O. If a plane of section traverse the girder at an insensible distance from either of those joints on the side next O, the shearing force is the same, being the excess of the supporting pressure, P_A (equation 5) above the load on n, and the other loaded joints between it and A, whose number is one-half of what it was in case 1, that is $\frac{N}{4} - \frac{n}{2}$. Hence we find

$$F_y = \frac{n-1}{2} . P \text{.........................} (6.)$$

The upward moment of the supporting force is

$$\text{at the joint } n,\ P_A x_1 = \left(\frac{N}{4} - \frac{1}{2}\right)\left(\frac{N}{2} - n\right) \cdot P l;$$

$$\text{at the joint } n-1,\ P_A (x_1 + l) = \left(\frac{N}{4} - \frac{1}{2}\right)\left(\frac{N}{2} - n + 1\right) \cdot P l.$$

The downward moment of the load from the joint n inclusive to the pier, relatively to the plane of section near that joint, is found by considering that the leverage of the nearest portion of that load is nothing, and that of the farthest $\left(\frac{N}{2} - 2 - n\right) l$; so that the mean leverage is $\frac{1}{2}\left(\frac{N}{2} - 2 - n\right) l$, which being multiplied by the load $-\left(\frac{N}{4} - \frac{n}{2}\right) P$, gives for the moment,

$$- \Sigma_C^A \cdot P x = - \frac{1}{4}\left(\frac{N}{2} - 2 - n\right) \cdot \left(\frac{N}{2} - n\right) \cdot P l.$$

The corresponding moment for the joint $n - 1$ is

$$- \Sigma_C^A P (x + l) = - \frac{1}{4}\left(\frac{N}{2} - n\right)^2 \cdot P l.$$

Hence the bending couples are—

$$\left.\begin{array}{l} \text{At the loaded joint } n, \\ M = \frac{1}{4}\left(\frac{N}{2} + n\right)\left(\frac{N}{2} - n\right) P l = \frac{1}{4}\left(\frac{N^2}{4} - n^2\right) P l; \\ \text{At the unloaded joint } n - 1, \\ M_1 = \frac{1}{4}\left\{\frac{N^2}{4} - (n-1)^2 - 1\right\} P l. \end{array}\right\} (7.)$$

Using these data, we obtain for the stress *along the diagonal* connecting the joints n and $n - 1$,

$$R' = F_y \operatorname{cosec} i = \frac{n-1}{2} \cdot \frac{s\,P}{h} \ldots\ldots\ldots\ldots\ldots(8.)$$

(The stress along the diagonal connecting the joints $n - 1$ and $n - 2$ is of equal amount and opposite kind).

$$\left.\begin{array}{l}\textit{Along the bar opposite the loaded joint } n,\\ \qquad R = \dfrac{M}{h} = \dfrac{1}{4}\left(\dfrac{N^2}{4} - n^2\right)\dfrac{P\,l}{h};\\ \textit{Along the bar opposite the unloaded joint } n - 1,\\ \qquad R_1 = \dfrac{M_1}{h} = \dfrac{1}{4}\left\{\dfrac{N^2}{4} - (n-1)^2 - 1\right\}\dfrac{P\,l}{h}.\end{array}\right\}\ldots(9.)$$

The last two stresses are of opposite kinds; and the kind of each stress is to be determined, as before, by the rule given after equation 1 of this Article.

164. **Lattice Girder—Any Load.**—In a lattice girder, as in a half-lattice girder, there are a horizontal upper and lower bar; but whereas a half-lattice girder contains but one zig-zag set of diagonal bars, a lattice girder contains two or more sets, crossing each other, usually at equal inclinations to the horizon. Fig. 83 represents the simplest form of a lattice girder, in which there are two sets of diagonals, crossing each other midway between the upper and lower horizontal bars.

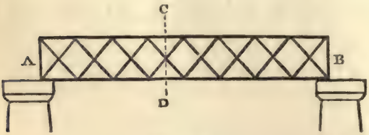

Fig. 83.

The load is supposed to be applied at the joints.

Suppose the girder to be cut by a vertical plane of section C D, traversing one of the joints where the diagonals cross. The shearing force and bending couple at this plane of section are to be determined exactly in the same manner as for a half-lattice girder, in Article 162.

In the present case, because the plane of section C D cuts *four* bars, the problem, in a strict mathematical sense, is indeterminate, according to the principles stated in Article 161; but it is solved by taking for granted what is the fact in well-constructed lattice girders, that each of the two diagonals which cross each other at the section C D bears one-half of the shearing force; and in like manner, when several pairs of diagonals cross each other at the same cross section, it is assumed that the resistance to the shearing force is equally distributed amongst them.

To fulfil this condition where a pair of diagonals, as in fig. 83, cross each other, with equal and opposite inclinations, the stresses along them must be equal, and of opposite kinds. Then let R′ and — R′ be the stresses along the pair of diagonals, and i and — i their inclinations to the horizon, we shall have for the vertical component of the force sustained by them

$$F_y = R' \sin i - R' \sin(-i) = 2 R' \sin i; \dots\dots (1.)$$

and for the horizontal component,

$$R' \cos i - R' \cos(-i) = 0;$$

so that the horizontal components of the stresses along the two diagonals at the plane of section balance each other.

Let $2m$ be the number of diagonal bars which cross each other at a given vertical section, the amount of the stress along each bar is

$$R' = \frac{F_y \operatorname{cosec} i}{2m}; \dots\dots (2.)$$

which is a $\left\{\begin{matrix}\text{pull}\\ \text{thrust}\end{matrix}\right\}$ for bars which slope $\left\{\begin{matrix}\text{with}\\ \text{against}\end{matrix}\right\}$ the shearing force.

The pull along the lower bar, and the thrust along the upper bar, at the given vertical section, must constitute a couple which balances the bending couple M, hence their common amount is

$$R = \frac{M}{h} \dots\dots (3.)$$

165. **Lattice Girder—Uniform Load.**—If N denote the even number of equal divisions into which the length of a lattice girder is divided by vertical lines traversing all the joints, whether of meeting of diagonal and horizontal bars, or of crossing of diagonal bars, and l the length of one of those divisions, so that $N l$, as before, is the span of the girder, then the effect of a load equally distributed amongst all those vertical lines, or amongst the alternate lines, may be found by means of the formulæ for a half-lattice girder, Article 163, as follows:—

I. When the load is distributed over all the vertical lines, the formulæ for case 1, equations 1, 2, 3, 4, are to be applied to vertical sections, such as C D, traversing the joints of crossing of diagonals; observing only, that the resistance to the shearing force is distributed amongst the diagonals as shown by equation 2 of Article 164.

II. When the load is distributed over those vertical lines only which traverse joints of meeting of diagonal and horizontal bars, the formulæ of case 2, equations 5, 6, 7, 8, 9, *so far as they relate to sections made at unloaded joints*, are to be applied to vertical sections, such as C D, traversing the joints of crossing of diagonals; attending as before to the distribution of the stress amongst the diagonals by equation 2 of this Article.

166. **Transformation of Frames.**—The principle explained in Article 66, of the transformation of a set of lines representing one balanced system of forces into another set of lines representing another system of forces which is also balanced, by means of what is called "PARALLEL PROJECTION," being applied to the theory of frames, takes obviously the following form :—

THEOREM. *If a frame whose lines of resistance constitute a given figure, be balanced under a system of external forces represented by a given system of lines, then will a frame whose lines of resistance constitute a figure which is a parallel projection of the original figure, be balanced under a system of forces represented by the corresponding parallel projection of the given system of lines; and the lines representing the stresses along the bars of the new frame, will be the corresponding parallel projections of the lines representing the stresses along the bars of the original frame.*

This Theorem is called the "Principle of the Transformation of Frames." It enables the conditions of equilibrium of any unsymmetrical frame which happens to be a parallel projection of a symmetrical frame (for example, a sloping lattice girder), to be deduced from the conditions of equilibrium of the symmetrical frame,—a process which is often much more easy and simple than that of finding the conditions of equilibrium of the unsymmetrical frame directly.

SECTION 2.—*Equilibrium of Chains, Cords, Ribs, and Linear Arches.*

167. **Equilibrium of a Cord.**—Let D A C in fig. 84 represent a flexible cord supported at the points C and D, and loaded by forces in any direction, constant or varying, distributed over its whole length with constant or varying intensity. Let A and B be any two points in this cord; from those points draw tangents to the cord, A P and B P, meeting in P. The load acting on the cord between the points A and B is balanced by the pulls along the

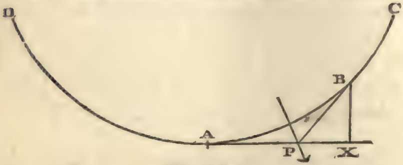

Fig. 84.

cord at those two points respectively; those pulls must respectively act along the tangents A P, B P; hence follows—

THEOREM I. *The resultant of the load between two given points in a balanced cord acts through the point of intersection of the tangents to the cord at those points; and that resultant, and the pulls along the cord at the two given points, are proportional to the sides of a triangle which are respectively parallel to their directions.*

The more the number of loaded points in a *funicular polygon* (as defined in Article 150) is increased,—or, in other words, the more the number of sides in the polygon is multiplied,—the more nearly does it approximate to the condition of a cord continuously loaded; while at the same time, the number of lines radiating from the point O in the diagram of forces (exemplified in fig. 75*) increases with the number of sides of the funicular polygon, and the polygon of external forces of fig. 75* approximates to a continuous line, curved or straight.

A *diagram of forces* for a continuously loaded cord may be constructed in the following manner (fig. 84*). Let radiating lines be drawn from the point O parallel to the tangents of the cord at any points which may be under consideration:—for example, let O C, O D, be parallel to the tangents at the points of support, and O A, O B, parallel to the tangents at the points A and B of fig. 84 respectively. Let the lengths of those radiating lines represent the pulls along the cord at the points to whose tangents they are parallel; and let a line D A B C, curved or straight, as the case may be, be drawn so as to pass through the extremities of all the radiating lines which represent the pulls along the cord at different points. Then from Theorem I. it appears, that a straight line drawn from B to A in fig. 84*, will represent in magnitude and direction the resultant of the load on the cord between A and B (fig. 84). Now, suppose the point marked A in fig. 84 to be taken gradually nearer and nearer to B; then will O A in fig. 84* approach gradually nearer and nearer to O B; and while the direction of the straight line drawn from B to A gradually approaches nearer and nearer to the direction of the tangent at the point B to the line C B A D in fig. 84*, the resultant load between B and A represented by that straight line gradually approaches nearer and nearer in direction to the direction of the load at the point B in fig. 84; therefore, the direction of the load at any point B of the cord (fig. 84), is represented by the direction of a tangent at B (fig. 84*), to the line C B A D. Hence follows—

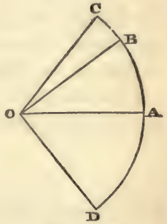

Fig. 84*.

THEOREM II. *If a line (called a* line of loads) *be drawn, such* *that while its radius-vector from a given point is parallel to a tangent to a loaded cord at a given point, its own tangent is parallel to the direction of the load at the point in the cord; then will the length of a radius-vector of the line of loads represent the pull at the corresponding point of the cord; and a straight line drawn between any two points in the line of loads will represent in magnitude and direction the resultant load between the two corresponding points in the cord.*

The supporting forces required at the points C and D (fig. 84), are obviously represented in magnitude and direction by the extreme radiating lines, $\overline{OC}$, $\overline{OD}$.

A loaded cord, hanging freely, is obviously *stable*, but capable of oscillation.

168. **Cord under Parallel Loads.**—If the direction of the load be everywhere parallel and vertical, the line of loads becomes a vertical straight line, as C B A D (fig. 84**).

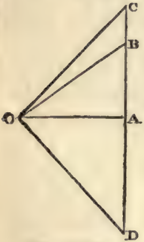

Fig. 84**

To express this case algebraically, let A in fig. 84 be the lowest point of the cord, so that the tangent A P is horizontal. Then in fig. 84**, O A will be horizontal, and perpendicular to C D. Let

$\mathrm{H} = \overline{\mathrm{OA}}$ = *horizontal tension* along the cord at A;

$\mathrm{R} = \overline{\mathrm{OB}}$ = pull along the cord at B;

$\mathrm{P} = \overline{\mathrm{AB}}$ = load on the cord between A and B;

$i = \angle$ X P B (fig. 84) $= \angle$ A O B (fig. 84**) = inclination of cord at B;

then,

$$\mathrm{P} = \mathrm{H} \tan i; \ \mathrm{R} = \sqrt{(\mathrm{P}^2 + \mathrm{H}^2)} = \mathrm{H} \sec i \ldots\ldots\ldots (1.)$$

To deduce from these formulæ an equation by which the form of the curve assumed by the cord can be determined when the distribution of the load is known, let that curve be referred to rectangular horizontal and vertical co-ordinates, measured from the lowest point A, the co-ordinates of B being, $\overline{\mathrm{AX}} = x$, $\overline{\mathrm{XB}} = y$; then

$$\tan i = \frac{dy}{dx};$$

whence we obtain

$$\frac{dy}{dx} = \frac{\mathrm{P}}{\mathrm{H}}; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.)$$

a differential equation which enables the form assumed by the cord to be determined when the distribution of the load is known.

169. **Cord under Uniform Vertical Load.**—By an *uniform vertical load* is here meant a vertical load uniformly distributed along a

horizontal straight line; so that if A (fig. 85), be the lowest point of the rope or cord, the load suspended between A and B shall be proportional to $\overline{AX} = x$, the horizontal distance between those points, and capable of being expressed by the equation

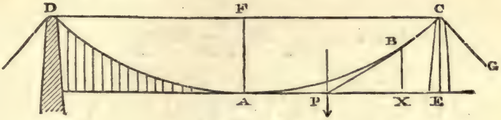

Fig. 85.

$$P = p\,x; \qquad (1.)$$

where p is a constant quantity, denoting the *intensity of the load in units of weight per unit of horizontal length:* in pounds per lineal foot, for example. It is required to find the form of the curve D A B C, and the relations amongst the load P, the horizontal pull at A (H), the pull at B (R), and the co-ordinates $\overline{AX} = x$, $\overline{BX} = y$.

First Solution.—Because the load between A and B is uniformly distributed, its resultant bisects A X; therefore, the tangent B P bisects A X: this is a property characteristic of a PARABOLA whose vertex is at A; therefore, the curve assumed by the cord is such a parabola.

Also, the proportions of the load, and the horizontal and oblique tensions are as follows:—

$$P : H : R :: \overline{BX} : \overline{XP} : \overline{PB} :: y : \frac{x}{2} : \surd\left(y^2 + \frac{x^2}{4}\right)$$

$$:: p\,x : \frac{p\,x^2}{2\,y} : px \cdot \surd\left(1 + \frac{x^2}{4\,y_2}\right) \qquad (2.)$$

Second Solution.—In the present case equation 2 of Article 168 becomes

$$\frac{d\,y}{d\,x} = \frac{p\,x}{H} \qquad (3.)$$

which being integrated with due regard to the condition that when $x = 0$, $y = 0$, gives

$$y = \frac{p\,x^2}{2\,H} \qquad (4.)$$

the equation of a parabola whose focal distance (or *modulus,* to use the term adopted in Dr. Booth's paper on the "Trigonometry of the Parabola," Reports of the British Association, 1856), is,

$$m = \frac{x^2}{4\,y} = \frac{H}{2\,p} \qquad (5.)$$

For a parabola we have also the inclination i to the horizon related to the co-ordinates by the following equations:—

$$\left.\begin{aligned} \tan i &= \frac{dy}{dx} = \frac{x}{2m} = \frac{2y}{x}; \\ \sec i &= \sqrt{\left(1 + \frac{dy^2}{dx^2}\right)} = \sqrt{\left(1 + \frac{x^2}{4m^2}\right)} = \sqrt{\left(1 + \frac{4y^2}{x^2}\right)}; \end{aligned}\right\} \text{(6.)}$$

whence we have the proportions

$$\mathrm{P : H : R} :: \tan i : 1 : \sec i :: \frac{2y}{x} : 1 : \sqrt{\left(1 + \frac{4y^2}{x^2}\right)}$$

$$:: px : \frac{px^2}{2y} : px \cdot \sqrt{\left(1 + \frac{x^2}{4y^2}\right)}, \ldots\ldots\ldots\ldots \text{(7.)}$$

as before.

The following are the solutions of some useful problems respecting uniformly loaded cords.

Problem I. *Given the elevations*, y_1, y_2, *of the two points of support of the cord above its lowest point, and also the horizontal distance, or span* a, *between those points of support; it is required to find the horizontal distances*, x_1, x_2, *of the lowest point from the two points of support; also the modulus* m.

In a parabola,

$$y_1 : y_2 :: x_1^2 : x_2^2;$$

therefore,

$$x_1 = a \cdot \frac{\sqrt{y_1}}{\sqrt{y_1} + \sqrt{y_2}}; \; x_2 = a \; \frac{\sqrt{y_2}}{\sqrt{y_1} + \sqrt{y_2}}; \ldots\ldots \text{(8.)}$$

also

$$m = \frac{x_1^2}{4y_1} = \frac{x_2^2}{4y_2} = \frac{x_1^2 + x_2^2}{4(y_1 + y_2)} = \frac{a^2}{4y_1 + 4y_2 + 8\sqrt{y_1 y_2}} \ldots \text{(9.)}$$

When the points of support are at the same level,

$$y_1 = y_2; \; x_1 = \frac{a}{2}; \; m = \frac{a^2}{16 y_1} \ldots\ldots\ldots\ldots\ldots\ldots \text{(10.)}$$

Problem II. *Given the same data, to find the inclinations* i_1, i_2, *of the cord at the points of support.*

By equations 6, we have,

$$\tan i_1 = \frac{2y_1}{x_1} = \frac{2y_1 + 2\sqrt{y_1 y_2}}{a}; \; \tan i_2 = \frac{2y_2}{x_2} = \frac{2y_2 + 2\sqrt{y_1 y_2}}{a}. \text{(11.)}$$

when

$$y_1 = y_2, \; \tan i_1 = \tan i_2 = \frac{4y_1}{a} \ldots\ldots\ldots\ldots\ldots \text{(12.)}$$

PROBLEM III. *Given the same data, and the load per unit of length; required the horizontal tension* H, *and the tensions* R_1, R_2, *at the points of support.*

By equation 5, we find,

$$H = 2pm = \frac{pa^2}{2y_1 + 2y_2 + 4\sqrt{y_1 y_2}}; \dots\dots\dots (13.)$$

and by the proportional equation 7,

$$R_1 = H \sec i_1 = H \sqrt{\left(1 + \frac{4y_1^2}{x_1^2}\right)};$$

$$R_2 = H \sec i_2 = H \sqrt{\left(1 + \frac{4y_2^2}{x_2^2}\right)} \dots\dots\dots\dots\dots (14.)$$

When $y_1 = y_2$, those equations become

$$H = \frac{pa^2}{8y_1}; \; R_1 = R_2 = H \sec i_1 = H \cdot \sqrt{\left(1 + \frac{4y_1^2}{x_1^2}\right)}$$

$$= H \sqrt{\left(1 + \frac{16y_1^2}{a^2}\right)} \dots\dots\dots\dots\dots\dots\dots (15.)$$

PROBLEM IV. *Given the same data as in Problem* I., *to find the length of the cord.*

The following are two well known formulæ for the length of a parabolic arc, commencing at the vertex, one being in terms of the co-ordinates x and y of the farther extremity of the arc, and the other in terms of the modulus m, and the inclination i of the farther extremity of the arc to a tangent at the vertex.

$$s = \sqrt{\left(y^2 + \frac{x^2}{4}\right)} + \frac{x^2}{4y} \cdot \text{hyp. log.} \frac{y + \sqrt{\left(y^2 + \frac{x^2}{4}\right)}}{\frac{x}{2}}$$

$$= m\{\tan i \cdot \sec i + \text{hyp. log.} (\tan i + \sec i)\} \dots (16.)$$

The *length of the cord* is $s_1 + s_2$, where s_1 is found by putting x_1 and y_1 in the first of the above formula, or i_1 in the second, and s_2 by putting x_2 and y_2 in the first formula, or i_2 in the second.

The following *approximate formula* for the length of a parabolic arc is in many cases sufficiently near the truth for practical purposes;

$$s = x + \frac{2y^2}{3x} \textit{ nearly}; \dots\dots\dots\dots\dots (17.)$$

which gives for the total length of the cord

$$s_1 + s_2 = a + \frac{2}{3}\left(\frac{y_1^2}{x_1} + \frac{y_2^2}{x_2}\right) \textit{ nearly.} \dots\dots\dots (18.)$$

and when $y_1 = y_2$, this becomes

$$2s_1 = a + \frac{8}{3} \cdot \frac{y_1^2}{a} \textit{ nearly;} \dots\dots\dots (19.)$$

Problem V. *Given the same data, to find, approximately, the small elongation of the cord* d (s_1 + s_2) *required to produce a given small depression* d y *of the lowest point* A, *and conversely.*

Differentiating equation 18, we find

$$d\,(s_1 + s_2) = \frac{4}{3}\left(\frac{y_1}{x_1} + \frac{y_2}{x_2}\right) d\,y \dots\dots\dots (20.)$$

which serves to compute the elongation from the depression; and conversely,

$$d\,y = \frac{3}{4} \cdot \frac{d\,(s_1 + s_2)}{\frac{y_1}{x_1} + \frac{y_2}{x_2}}; \dots\dots\dots (21.)$$

which serves to compute the depression of the lowest point from the elongation of the cord. When $y_1 = y_2$, those formulæ become,

$$\left.\begin{aligned} 2\,d\,s_1 &= \frac{16y_1}{3a} \cdot d\,y \\ d\,y &= \frac{3a}{16y_1} \cdot 2\,d\,s_1 \end{aligned}\right\} \dots\dots\dots (22.)$$

The preceding formulæ serve to compute the depression which the middle point of a suspension bridge undergoes in consequence of a given elongation of the cable or chain, whether caused by heat or by tension.

170. **Suspension Bridge with Vertical Rods.**—In a suspension bridge the load is not continuous, the platform being hung by rods from a certain number of points in each cable or chain: neither is it uniformly distributed; for although the weight of the platform per unit of length is uniform or sensibly so, the load arising from the weight of the cables or chains and of the suspending rods is more intense near the piers. Nevertheless, in most cases which occur in practice, the condition of each cable or chain approaches sufficiently near to that of a cord continuously and uniformly loaded to enable the formulæ of Article 169 to be applied without material error.

When the piers of a suspension bridge are slender and vertical (as is usually the case), the resultant pressure of the chain or cable on the top of the pier ought to be vertical also. Thus, let C E, in fig. 85, represent the vertical axis of a pier, and C G the portion of the chain or cable behind the pier, which either supports another division of the platform, or is made fast to a mass of rock, or of masonry, or otherwise. If the chain or cable passes over a curved plate on the top of the pier called a *saddle*, on which it is free to slide, the tensions of the portions of the chain or cable on either side of the saddle will be equal; and in order that those tensions may compose a vertical pressure on the pier, their inclinations must be equal and opposite. Let i be the common value of those inclinations; R the common value of the two tensions; then the vertical pressure on the pier is

$$\mathrm{V} = 2\,\mathrm{R}\sin i = 2\,\mathrm{H}\tan i = 2\,p\,x; \text{..........} \quad (1.)$$

that is, twice the weight of the portion of the bridge between the pier and the lowest point, A, of the curve C B A D.

But if the two divisions of the chain or cable D A C, C G, which meet at C, be *made fast* to a sort of truck, which is supported by rollers on a *horizontal* cast iron platform on the top of the pier, then the pressure on the pier will be vertical, whether the inclinations of the two divisions of the chain or cable be equal or unequal; and it is only necessary that the *horizontal components* of their tension should be equal; that is to say, let i, i', be the inclinations of the two divisions of the chain or cable in opposite directions at C, and R, R′, their tensions, then

$$\mathrm{R} = \mathrm{H}\sec i;\ \ \mathrm{R}' = \mathrm{H}\sec i';$$

$$\mathrm{V} = \mathrm{R}\sin i + \mathrm{R}'\sin i' = \mathrm{H}\,(\tan i + \tan i') \text{......} \quad (2.)$$

171. **Flexible Tie.**—Let a vertical load, P, be applied at A, fig. 86, and sustained by means of a horizontal strut, A B, abutting against a fixed body at B, and a sloping rope or chain, or other flexible tie, A D C, fixed at C. The weight of the strut, A B, is supposed to be divided into two components, one of which is supported at B, while the other is *included* in the load P. The weight, W, of the

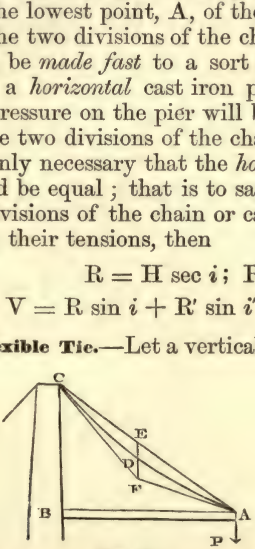

Fig. 86.

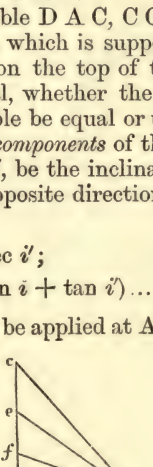

Fig. 86*.

flexible tie, A D C, is supposed to be known, and to be considered separately; and with these data there is proposed the following

PROBLEM. W *being small compared with* P, *to find approximately the vertical depression* E D *of the flexible tie below the straight line* A C, *the pulls along it at* A, D, *and* C, *and the horizontal thrust along* A B.

Because W is small compared with P, the curvature of the tie will be small, and the distribution of its weight along a horizontal line may be taken as *approximately* uniform; therefore its figure will be *nearly* a parabola; the tangent at D will be sensibly parallel to A C, and the tangents at A and C will meet in a point which will be near the vertical line E D F, which line bisects A C, and is bisected in D. Hence we have the following construction:—

Draw the diagram of forces, fig. 86*, in the following manner. On the vertical line of loads $b\,c$, take $\overline{b\,f} = \mathrm{P}$; $\overline{b\,e} = \mathrm{P} + \frac{\mathrm{W}}{2}$; $\overline{b\,c} = \mathrm{P} + \mathrm{W}$. From b draw b O parallel to the strut A B; that is, horizontal; from e draw e O parallel to C A, cutting b O in O; join c O, f O.

In fig. 86, bisect A C in E, through which draw a vertical line; through A and C respectively draw A F || O f, C F || O c, cutting that vertical line in F; bisect E F in D. Then will A F and C F be tangents to the flexible tie at A and C, D will be its most depressed point, and $\overline{\mathrm{D\,E}}$ its greatest depression; and the pulls along the tie at C, D, and A, and the thrust along the strut A B, will, in virtue of the principle of Article 168, be represented by the radiating lines O c, O e, O f, and O b, in fig. 86*.

This solution is in general sufficiently near the truth for practical purposes. To express it algebraically, let R_a, R_d, R_c, be the tensions of the tie at A, D, C, respectively, and H the horizontal thrust; then

$$\left.\begin{aligned}
\mathrm{H} &= \left(\mathrm{P} + \frac{\mathrm{W}}{2}\right)\frac{\mathrm{O}\,b}{b\,e} = \left(\mathrm{P} + \frac{\mathrm{W}}{2}\right)\frac{\overline{\mathrm{A\,B}}}{\overline{\mathrm{B\,C}}};\\
\mathrm{R}_a &= \sqrt{\left(\mathrm{H}^2 + \mathrm{P}^2\right)};\\
\mathrm{R}_d &= \sqrt{\left\{\mathrm{H}^2 + \left(\mathrm{P} + \frac{\mathrm{W}}{2}\right)^2\right\}};\\
\mathrm{R}_c &= \sqrt{\left\{\mathrm{H}^2 + (\mathrm{P} + \mathrm{W})^2\right\}};\\
\overline{\mathrm{D\,E}} &= \tfrac{1}{2}\,\overline{\mathrm{E\,F}} = \tfrac{1}{8}\,\overline{\mathrm{B\,C}}\cdot\frac{\mathrm{W}}{\mathrm{P}+\frac{\mathrm{W}}{2}}.
\end{aligned}\right\}\quad\text{......(1.)}$$

The *difference of length* between the curve A D C and the straight line A E C is found very nearly, by substituting, in the second term of equation 19, Article 169, $\overline{AC}$ for a, and $\frac{\overline{AB}\cdot\overline{DE}}{\overline{AC}}$ for y_1; that is to say,

$$\overline{ADC}-\overline{AEC}=\frac{8}{3}\cdot\frac{\overline{AB}^2\cdot\overline{DE}^2}{\overline{AC}^3}=\frac{1}{24}\cdot\frac{\overline{AB}^2\cdot\overline{BC}^2}{\overline{AC}^3}\cdot\left\{\frac{W}{P+\frac{W}{2}}\right\}^2 \quad ...(2.)$$

172. **Suspension Bridge with Sloping Rods.**—Let the uniformly-loaded platform of a suspension bridge be hung from the chains by parallel sloping rods, making an uniform angle j with the vertical. The condition of a chain thus loaded is the same with that of a chain loaded vertically, except in the direction of the load; and the form assumed by the chain is a parabola, having its axis parallel to the direction of the suspension rods.

In fig. 87, let C A represent a chain, or portion of a chain, supported or fixed at C, and horizontal at A, its lowest point. Let A H be a horizontal tangent at A, representing the platform of the bridge; and let the suspension rods be all parallel to C E, which makes the angle $\angle$ E C H $= j$ with the vertical. Let B X represent any rod, and suppose a vertical load v to be supported at the point X. Then, by the principles of the equilibrium of a *frame of two bars* (Article 145), this load will produce a *pull*, p, on the rod X B, and a *thrust*, q, on the platform between X and H; and the three forces v, p, q, will be proportional to the sides of a triangle parallel to their directions, such as the triangle C E H; that is to say,

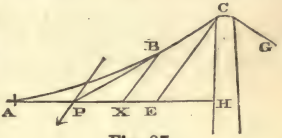

Fig. 87.

$$v:p:q::\overline{CH}:\overline{CE}:\overline{EH}::1:\sec j:\tan j. \quad ......(1.)$$

Next, instead of considering the load on one rod B X, consider the entire vertical load V between A and X. This being the sum of the loads supported by the rods between A and X, it is evident that the proportional equation (1) may be applied to it; and that if P represent the amount of the pull acting on the rods between A and X, and Q the total thrust on the platform at the point X, we shall have

$$V:P:Q::\overline{CH}:\overline{CE}:\overline{EH}::1:\sec j:\tan j. \quad .....(2.)$$

The *oblique load* $P = V \sec j$ is what hangs from the chain between A and B. Being uniformly distributed, its resultant bisects A X in P, which is also the point of intersection of the tangents A P, B P; and the ratio of the oblique load P, the horizontal tension H along the chain at A, and the tension R along the chain at B, is that of the sides of the triangle B X P; that is to say,

$$\mathrm{P} : \mathrm{H} : \mathrm{R} :: \overline{\mathrm{BX}} : \overline{\mathrm{XP}} = \frac{\overline{\mathrm{AX}}}{2} : \overline{\mathrm{BP}} \ldots\ldots\ldots\ldots (3.)$$

Comparing this with the case of Article 169 and fig. 85, it is evident that the form of the chain in fig. 87 must be similar to that of the chain in fig. 85, with the exception that the ordinate $\overline{\mathrm{XB}} = y$ is oblique to the abscissa $\overline{\mathrm{AX}} = x$, instead of perpendicular; that is to say, C B A is a parabola, having its axis parallel to the inclined suspension rods.

The equation of such a parabola, referred to its oblique co-ordinates, with the origin at A, is as follows:—

$$y = \frac{x^2 \cdot \cos^2 j}{4\,m} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4.)$$

where *m*, as in Article 169, denotes the *modulus* of the parabola, given by the equation

$$m = \frac{x^2 \cdot \cos^2 j}{4\,y} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5.)$$

x and y being the co-ordinates of any *known* point in the curve. The length of the tangent $\overline{\mathrm{BP}} = t$ is given by the following equation:—

$$t = \sqrt{\left(\frac{x^2}{4} + y^2 + x\,y \cdot \sin j\right)} \ldots\ldots\ldots\ldots\ldots (6.)$$

Hence are deduced the following formulæ for the relations amongst the forces which act in a suspension bridge with inclined rods. Let v now be taken to denote the *intensity* of the vertical load per unit of length of horizontal platform—per foot, for example; p the intensity of the oblique load; q the rate at which the thrust along the platform increases from A towards H. Then

$$\left.\begin{array}{l} \mathrm{V} = v\,x;\ \mathrm{P} = p\,x = v\,x \cdot \sec j; \\ \mathrm{Q} = q\,x = v\,x \cdot \tan j; \end{array}\right\} \ldots\ldots\ldots\ldots (7.)$$

$$\mathrm{H} = \frac{x\,\mathrm{P}}{2y} = \frac{p\,x^2}{2y} = \frac{2\,p\,m}{\cos^2 j} = 2\,v\,m \cdot \sec^3 j \ldots\ldots\ldots\ldots (8.)$$

$$\mathrm{R} = \frac{t\,\mathrm{P}}{y} = \frac{2\,t\,\mathrm{H}}{x} = \frac{p\,t\,x}{y} \ldots\ldots\ldots\ldots\ldots\ldots (9.)$$

The horizontal pull H at the point A may be sustained in three different ways, viz.:—

I. The chain may be *anchored* or made fast at A to a mass of rock or masonry.

II. It may be attached at A to another equal and similar chain, similarly loaded by means of oblique rods, sloping at an equal angle in the direction opposite to that of the rods B X, &c., so that A may be in the middle of the span of the bridge.

III. The chain may be made fast at A to the horizontal platform A H, so that the pull at A shall be balanced by an equal and opposite thrust along the platform, which must be strong enough and stiff enough to sustain that thrust. In this case, the total thrust at any point, X, of the platform is no longer simply $\mathrm{Q} = q\,x$, but

$$\mathrm{H} + \mathrm{Q} = \left(\frac{\mathrm{P}}{2\,y} + q\right) x$$

$$= v\,(2\,m \cdot \sec^3 j + x \cdot \tan j) \ldots\ldots\ldots\ldots\ldots (10.)$$

The *length of the parabolic arc,* A B, is given exactly by the following formulæ. Let i denote the inclination of the parabola at the point B to a line perpendicular to its axis. Then

$$i = \operatorname{arc} \cdot \cos \left(\frac{x}{2\,t} \cdot \cos j\right) \ldots\ldots\ldots\ldots\ldots\ldots (11.)$$

which, when B coincides with A, becomes simply $i = j$. Then from the known formulæ for the lengths of parabolic arcs, we have

$$\text{parabolic arc A B} = m \left\{ \tan i \sec i - \tan j \sec j \right.$$

$$\left. + \text{hyp. log.} \frac{\tan i + \sec i}{\tan j + \sec j} \right\} \ldots\ldots\ldots\ldots (12.)$$

In most cases which occur in practice, however, it is sufficient to use the following approximate formula:—

$$\text{arc A B} = x + y \cdot \sin j + \frac{2}{3} \cdot \frac{y^2 \cdot \cos^2 j}{x + y \cdot \sin j}, \text{ nearly} \ldots\ldots (13.)$$

The formulæ of this Article are applicable to Mr. Dredge's suspension bridges, in which the suspending rods are inclined, and although not exactly parallel, are nearly so.

173. **Extrados and Intrados.**—When a cord is loaded with parallel vertical forces, and ordinates are drawn downwards from the cord, of lengths proportional to the intensity of the vertical load at the points of the cord from which they are drawn, a line, straight or

curved as the case may be, which traverses the lower ends of all these ordinates, is called the *extrados* of the given load. The curve formed by the cord itself is called the *intrados.* The load suspended between any two points of the cord is proportional to the vertical plane area, bounded laterally by the vertical ordinates at those two points, above by the cord or intrados, and below by the extrados; and may be regarded as equal to the weight of a flexible sheet of some heavy substance, of uniform thickness, bounded above by the intrados, and below by the extrados. The following is the algebraical expression of the relations between the extrados and the intrados.

Assume the horizontal axis of x to be taken at or below the level of the lowest point of the extrados; and let the vertical axis of y, as in Articles 168, 169, and 170, traverse the point where the intrados is lowest. For a given abscissa x, let y' be the ordinate of the extrados, and y that of the intrados, so that $y - y'$ is the length of the vertical ordinate intercepted between those two lines, to which the intensity of the load is proportional. Let w be the weight of unity of area of the vertical sheet by which the load is considered to be represented. Then we have for the load between the axis of y and a given ordinate at the distance x from that axis,

$$\mathrm{P} = w \int_0^x (y - y')\, d\, x; \ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

the integral representing the area between the axis of y, the given ordinate, the extrados and the intrados. Combining this equation with equation 2 of Article 168, we obtain the following equation :—

$$\tan i = \frac{d\, y}{d\, x} = \frac{\mathrm{P}}{\mathrm{H}} = \frac{w}{\mathrm{H}} \int_0^x (y - y')\, d\, x; \ldots\ldots\ldots (2.)$$

an equation which affords the means of determining, by an indirect process, the equation of the intrados, when the horizontal tension H, and the equations of the extrados are given, and also, by a somewhat more indirect process, the equation of the intrados and the horizontal tension, when the equation of the extrados and one of the points of the intrados are given. Both these processes are in general of considerable algebraical intricacy.

$\frac{\mathrm{H}}{w}$ obviously represents the area of a portion of the sheet above mentioned, whose weight is equal to the horizontal tension. Let that area be the square of a certain line, a; that is, let

$$\frac{\mathrm{H}}{w} = a^2; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.)$$

Then that line is called the *parameter* of the intrados, or curve in which the cord hangs.

When the vertical load is of uniform intensity, as in Article 169, so that the intrados is a parabola, it is obvious that the extrados is an equal and similar parabola, situated at an uniform depth below the intrados.

[The reader who has not studied the properties of exponential functions may pass at once to Article 176.]

174. **Cord with Horizontal Extrados.**—If the extrados be a horizontal straight line, that line may itself be taken for the axis of x. Thus, in fig. 87 A, let O X be the straight horizontal extrados, A the lowest point of the intrados, and let the vertical line O A be the axis of y. Denote the length of $\overline{\mathrm{OA}}$, which is the least ordinate of the intrados, by y_0. Let $\overline{\mathrm{BX}} = y$ be any other ordinate, at the end of the abscissa $\overline{\mathrm{OX}} = x$. Let the area O A B X be denoted by u. Then equations 1 and 2 of Article 172 become the following:—

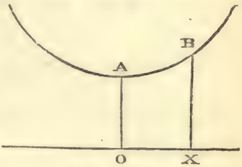

Fig. 87 A.

$$\left.\begin{aligned} \mathrm{P} = w\,u = w\int_o^x y\,d\,x; \\ \frac{d\,y}{d\,x} = \frac{d^2 u}{d\,x^2} = \frac{\mathrm{P}}{\mathrm{H}} = \frac{u}{a^2}. \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots(1.)$$

The general integral of the latter of these equations is

$$u = \mathrm{A}\,e^{\frac{x}{a}} - \mathrm{B}\,e^{-\frac{x}{a}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(a.)$$

in which A and B are constants, which are determined by the special conditions of the problem in the following manner. When $x = 0$, $e^{\frac{x}{a}} = e^{-\frac{x}{a}} = 1$; but at the same time $u = 0$, therefore $\mathrm{A} = \mathrm{B}$, and equation $(a.)$, may be put in the form,

$$u = \mathrm{A}\left(e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots(b.)$$

This gives for the ordinate,

$$y = \frac{\mathrm{A}}{a}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots(c.)$$

which, for $x = 0$, becomes $y_0 = \dfrac{2\,\mathrm{A}}{a}$; and therefore

$$\mathrm{A} = \frac{a\,y_0}{2}, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(d.)$$

which value being introduced into the various preceding equations, gives the following results, as to the geometrical properties of the intrados:—

$$\left.\begin{aligned} \textit{Area, } u &= \frac{a\,y_0}{2}\left(e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right);\\ \textit{Ordinate, } y &= \frac{y_0}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right);\\ \textit{Slope, } \tan i &= \frac{d\,y}{d\,x} = \frac{u}{a^2} = \frac{y_0}{2\,a}\left(e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right);\\ \textit{Deviation, } \frac{d^2\,y}{d\,x^2} &= \frac{y}{a^2} = \frac{y_0}{2\,a^2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right). \end{aligned}\right\} \text{(2.)}$$

The relations amongst the forces which act on the cord are given by the equations

$$\left.\begin{aligned} \mathrm{H} &= w\,a^2;\ \mathrm{P} = \mathrm{H}\cdot\frac{d\,y}{d\,x} = w\,u;\\ \mathrm{R}\ (\text{tension at B}) &= \sqrt{\mathrm{P}^2 + \mathrm{H}^2} = \mathrm{H}\sqrt{1 + \frac{d\,y^2}{d\,x^2}} \end{aligned}\right\} \text{(3.)}$$

In the course of the application of these principles, the following problem may occur:—*given, the extrados* O X, *the vertex* A *of the intrados, and a point of support* B; *it is required to complete the figure of the intrados.* For this purpose it is necessary and sufficient to find the parameter a; so that the problem in fact amounts to this; given the least ordinate y_0, and the ordinate y corresponding to *one given value* of the abscissa x, it is required to find a, so as to fulfil the equation

$$\left.\begin{aligned} \frac{y}{y_0} &= \frac{e^{\frac{x}{a}} + e^{-\frac{x}{a}}}{2}\\ &= \textit{hyperbolic cosine } \text{of } \frac{x}{a}, \end{aligned}\right\} \text{........(4.)}$$

as this function is called. Supposing a table of hyperbolic cosines to be at hand, $\frac{x}{a}$ is found by its being the number whose hyperbolic cosine is $\frac{y}{y_0}$; so that

$$a = \frac{x}{\text{number to hyp. cos. } \frac{y}{y_0}} \text{ ....(5.)}$$

but such a table is rarely to be met with; and in its absence a is found as follows:—

The value of x is given in terms of y by the equation

$$x = a \cdot \text{hyp. log.} \left(\frac{y}{y_0} + \sqrt{\frac{y^2}{y_0^2} - 1} \right); \text{..........(6.)}$$

and hence

$$a = \frac{x}{\text{hyp. log.} \left(\frac{y}{y_0} + \sqrt{\frac{y^2}{y_0^2} - 1} \right)} \text{.............(7.)}$$

175. **Catenary** is the name given to the curve in which a cord or chain of uniform material and sectional area (so that the weight of any part is proportional to its length) hangs when loaded with its own weight alone.

Let fig. 87 A, serve to represent this curve; but let A be taken as the origin of co-ordinates, so that the axis of x is a horizontal tangent at A. Let s denote the length of any given arc A B. Then if p be the weight of an unit of length of the cord or chain, the load suspended between A and B is $\mathrm{P} = p\,s$. The inclination i of the curve at B to a horizontal line is expressed by the equations

$$\left.\begin{aligned} \cos i &= \frac{dx}{ds} \\ \sin i &= \frac{dy}{ds} = \sqrt{1 - \frac{dx^2}{ds^2}}; \\ \tan i &= \frac{dy}{dx} = \frac{\sqrt{1 - \frac{dx^2}{ds^2}}}{\frac{dx}{ds}}. \end{aligned}\right\} \text{..... (1.)}$$

Let the horizontal tension be equal to the weight of a *certain length of chain, m,* so that

$$\mathrm{H} = p\,m \text{.............................. (2.)}$$

From these equations, and from the general equation 2 of Article 168, we deduce the following:—

$$\tan i = \frac{\sqrt{1 - \frac{dx^2}{ds^2}}}{\frac{dx}{ds}} = \frac{\mathrm{P}}{\mathrm{H}} = \frac{s}{m} \text{..........(3.)}$$

which, by a few reductions, is brought to the following form:—

$$\frac{dx}{ds} = \frac{m}{\sqrt{m^2 + s^2}} \text{.......................} (4.)$$

the integral of which (paying due regard to the conditions that when $s = 0$, $x = 0$) is known to be

$$x = m \cdot \text{hyp. log.} \left(\frac{s}{m} + \sqrt{1 + \frac{s^2}{m^2}} \right) \text{.........} (5.)$$

This equation gives the abscissa x of the extremity of an arc A B $= s$, when the *parameter* of the catenary (as m is called) is known. Transforming the equation so as to have s in terms of x, we obtain

$$s = \frac{m}{2} \left(e^{\frac{x}{m}} - e^{-\frac{x}{m}} \right) \text{.....................} (6.)$$

The ordinate y is found in terms of x by integrating the equation

$$\frac{dy}{dx} = \sqrt{\frac{ds^2}{dx^2} - 1} = \frac{s}{m} = \frac{1}{2} \left(e^{\frac{x}{m}} - e^{-\frac{x}{m}} \right), \text{......} (7.)$$

which gives

$$y = \frac{m}{2} \left(e^{\frac{x}{m}} + e^{-\frac{x}{m}} - 2 \right) = \sqrt{s^2 + m^2} - m \cdot ; \text{....} (8.)$$

the term -2 being introduced in order that when $x = 0$, y may be also $= 0$. This is the equation of the catenary, so far as its form is concerned. The mechanical condition is given by the equations

$$\left. \begin{array}{c} \mathrm{H} = p\,m; \ \mathrm{P} = p\,s; \\ \mathrm{R} = p\sqrt{m^2 + s^2} = \frac{p\,m}{2} \left(e^{\frac{x}{m}} + e^{-\frac{x}{m}} \right) = p\,(y + m); \end{array} \right\} (9.)$$

so that *the tension at any point is equal to the weight of a piece of the chain, whose length is the ordinate added to the parameter.*

Suppose the axis of x, instead of being a tangent at the vertex of the curve, to be situated at a depth $\overline{\mathrm{A\,O}} = m$ below the vertex, and let y' denote any ordinate measured from this lowered axis; then

$$y' = y + m = \frac{m}{2} \left(e^{\frac{x}{m}} + e^{-\frac{x}{m}} \right); \text{...............} (10.)$$

which, being compared with the expression for the ordinate amongst equations 2, Article 174, shows, that *the intrados for a horizontal ex-*

trados when the least ordinate is equal to the parameter ($y_0 = a$), *becomes identical with a catenary, having the same parameter* ($m = a = y_0$).

PROBLEM. *Given, two points in a catenary, and the length of chain between them; required the remainder of the curve.*

Let k be the horizontal distance between the two points, v their difference of level, l the length of chain between them. Those three quantities are the data.

The unknown quantities may be expressed in the following manner. Let x_1, y_1, be the co-ordinates of the higher given point, and s_1 the arc terminating at it, all measured from the yet unknown vertex of the catenary, and x_2, y_2, s_2, the corresponding quantities for the lower given point. (The particular case when the points are at the same level will be afterwards considered). Also let

$$x_1 + x_2 = h \text{ (an unknown quantity).}$$

Then we have

$$x_1 = \frac{h+k}{2};\ x_2 = \frac{h-k}{2} \text{..................} (11.)$$

Putting these values of x in the equations 6 and 8, we find

$$\left.\begin{aligned} l = s_1 - s_2 &= m \left(e^{\frac{h}{2m}} + e^{-\frac{h}{2m}}\right) \cdot \left(e^{\frac{k}{2m}} - e^{-\frac{k}{2m}}\right) \\ v = y_1 - y_2 &= m \left(e^{\frac{h}{m}} + e^{-\frac{h}{2m}}\right) \cdot \left(e^{\frac{k}{2m}} - e^{-\frac{k}{2m}}\right) \end{aligned}\right\} \text{...(12.)}$$

Square those two equations and take the difference of the squares; then,

$$\sqrt{l^2 - v^2} = m \left(e^{\frac{k}{2m}} - e^{-\frac{k}{2m}}\right); \text{...............} (13.)$$

In this equation the only unknown quantity is the *parameter* m, which is to be determined by a series of approximations.

Next, divide the sum of the equations (12) by their difference. This gives

$$e^{\frac{h}{m}} = \frac{l+v}{l-v}$$

and consequently

$$h = m \cdot \text{hyp. log. } \frac{l+v}{l-v} \text{.....................} (14.)$$

Either or both of the abscissæ x_1 and x_2, being computed by the equations 11, we find the position of the vertical axis. Then computing by equation 8, either or both of the ordinates, y_1, y_2, we find the vertex of the catenary, which, together with the parameter, being known, completely determines the curve.—Q. E. I.

When the given points are at the same level, that is, when $v = 0$, the vertical axis must be midway between them, so that

$$x_1 = - x_2 = \frac{k}{2};\ h = 0 \dots\dots\dots\dots (15.)$$

In this case equation 13 becomes

$$l = m \left(e^{\frac{k}{2m}} - e^{-\frac{k}{2m}} \right) \dots\dots\dots\dots (16.)$$

from which m is to be found by successive approximations. Then the computation of $y_1 = y_2$ by means of equation 8 determines the vertex of the curve, and completes the solution.

The following are some of the geometrical properties of the catenary:—

I. The radius of curvature at the vertex is equal to the parameter, and at any other point is given by the equation

$$r = m \cdot \sec^2 i \dots\dots\dots\dots (17.)$$

II. The length of a normal to the catenary, at any point, cut off by a horizontal line at the depth m below the vertex, is equal to the radius of curvature at that point.

III. The involute of a catenary commencing at its vertex, is the tractory of the horizontal line before mentioned, with the constant tangent m.

IV. If a parabola be rolled on a straight line, the focus of the parabola traces a catenary whose parameter is equal to the focal distance of the parabola.

176. **Centre of Gravity of a Flexible Structure.**—In every case in which a perfectly flexible structure, such as a cord, a chain, or a funicular polygon, is loaded with weights only, the figure of stable equilibrium in the structure is that which corresponds to the lowest possible position of the centre of gravity of the entire load. This principle enables all problems respecting the equilibrium of vertically loaded flexible structures to be solved by means of the "Calculus of Variations."

177. **Transformation of Cords and Chains.**—The principle of *Transformation by Parallel Projection* is applicable to continuously loaded cords as well as to polygonal frames: it being always borne in mind, that in order that forces may be correctly transformed by parallel projection, their magnitudes must be represented by the *lengths of straight lines parallel to their directions*, so that if in any case

the magnitude of a force is represented by an *area* (as in Articles 173 and 174) or by the length of a curve (as in Article 175), we must, in transforming that force by projection, first consider what length and position a straight line should have in order to represent it.

Some of the cases already given might have been treated as examples of transformation by parallel projection. For instance, the bridge-chain with sloping rods of Article 172 might be treated as a parallel projection of a bridge-chain with vertical rods, made by substituting oblique for rectangular co-ordinates; and the intrados for a horizontal extrados of Article 174, where the least ordinate y_0 and parameter a have any ratio, might be treated as a parallel projection deduced, by altering the proportions of the rectangular co-ordinates, from the corresponding curve in which the least co-ordinate is equal to the parameter; that is, from the catenary.

The algebraical expressions for the alterations made by parallel projection in the co-ordinates of a loaded chain or cord, and in the forces applied to it, are as follows:—

In the original figure, let y be the vertical co-ordinate of any point, and x the horizontal co-ordinate. Let P be the vertical load applied between any point B of the chain and its lowest point A; let $p = \frac{d\,\mathrm{P}}{d\,x}$ be its intensity per horizontal unit of length; let H be the horizontal component of the tension; let R be the tension at the point B.

Suppose that in the transformed figure, the vertical ordinate y', and the vertical load P′, which is represented by a vertical line, are unchanged in length and direction, so that we have

$$y' = y;\ \mathrm{P}' = \mathrm{P};\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.)$$

but for each horizontal co-ordinate x, let there be substituted an *oblique co-ordinate* x', inclined at the angle j to the horizon, and altered in length by the constant ratio $\frac{x'}{x} = a$. Then for the horizontal tension H, there will be substituted an *oblique tension* H′, parallel to x', and altered in the same proportion with that co-ordinate; that is to say,

$$x' = a\,x;\ \mathrm{H}' = a\,\mathrm{H}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.)$$

The original tension at B is the resultant of the vertical load P and the horizontal tension H. Let R be its amount, and i its inclination to H; then

$$\mathrm{R} = \sqrt{\mathrm{P}^2 + \mathrm{H}^2};\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.)$$

and the ratios of those three forces are expressed by the proportion

$$\mathrm{P} : \mathrm{H} : \mathrm{R} :: \tan i : 1 : \sec i :: \sin i : \cos i : 1 \ldots\ldots (4.)$$

Let R′ be the amount of the tension at the point B in the new structure, corresponding to B, and let i' be its inclination to the *oblique co-ordinate* x'; then

$$\mathrm{R}' = \sqrt{(\mathrm{P}'^2 + \mathrm{H}'^2 \pm 2\,\mathrm{P}'\mathrm{H}' \sin j)} \ldots\ldots (5.)$$

$$\mathrm{P}' : \mathrm{H}' : \mathrm{R}' :: \sin i' : \cos(i' \pm j) : \cos j \ldots\ldots (6.)$$

The alternative signs $\pm$ are to be used according as i' and j $\left\{\begin{matrix}\text{agree}\\\text{differ}\end{matrix}\right\}$ in direction.

The *intensity* of the load in the transformed structure *per unit of oblique length* measured along $d\,x'$, is

$$p' = \frac{d\,\mathrm{P}'}{d\,x'} = \frac{p}{a}; \ldots\ldots (7.)$$

but if the intensity of the load be estimated *per unit of horizontal length*, it becomes

$$p' \sec j = \frac{p}{a \cdot \cos j}. \ldots\ldots (8.)$$

178. **Linear Arches or Ribs.**—Conceive a cord or chain to be exactly inverted, so that the load applied to it, unchanged in direction, amount, and distribution, shall act inwards instead of outwards; suppose, further, that the cord or chain is in some manner stayed or stiffened, so as to enable it to preserve its figure and to resist a thrust; it then becomes a *linear arch*, or *equilibrated rib;* and for the pull at each point of the original cord is now substituted an exactly equal *thrust* along the rib at the corresponding point.

Linear arches do not actually exist; but the propositions respecting them are applicable to the lines of resistance of real arches and arched ribs, in those cases in which the direction of the thrust at each joint is that of a tangent to the line of resistance, or curve connecting the centres of pressure at the joints.

All the propositions and equations of the preceding Articles, respecting cords or chains, are applicable to linear arches, substituting only a *thrust* for a *pull*, as the stress along the line of resistance.

The principles of Article 167 are applicable to linear arches in general, with external forces applied in any direction.

The principles of Article 168 are applicable to linear arches under *parallel loads;* and in such arches, the quantity denoted by

H in the formulæ represents a *constant thrust,* in a direction perpendicular to that of the load.

The form of equilibrium for a linear arch under an uniform load is a *parabola,* similar to that described in Article 169.

In the case of a linear arch under a vertical load, *intrados* denotes the figure of the arch itself, and *extrados* a line traversing the *upper* ends of ordinates, drawn *upwards* from the intrados, of lengths proportional to the intensities of the load; and the principles of Article 173 are applicable to relations between the intrados and the extrados.

The curve of Article 174 is the figure of equilibrium for a linear arch with a horizontal extrados; and from Article 175 it appears, that the figures of all such arches may be deduced from that of a catenary, by inverting it and altering its horizontal and vertical co-ordinates in given constant proportions for each case.

The principles of Article 177, relative to the transformation of cords and chains, are applicable also to linear arches or ribs. This subject will be further considered in the sequel.

The preceding Articles of this section contain propositions which, though applicable both to cords and to linear arches, are of importance in practice chiefly in relation to cords or chains. The following Articles contain propositions which, though applicable also to cords as well as linear arches, are of importance in practice chiefly in relation to linear arches.

179. **Circular Arch for Uniform Fluid Pressure.**—It is evident that a linear arch, to resist an uniform normal pressure from without, should be circular; because, as the force to which it is subjected is similar all round, its figure ought to be similar to itself all round—a property possessed by the circle alone.

In fig. 88, let A B A B be a circular linear arch, rib, or ring,

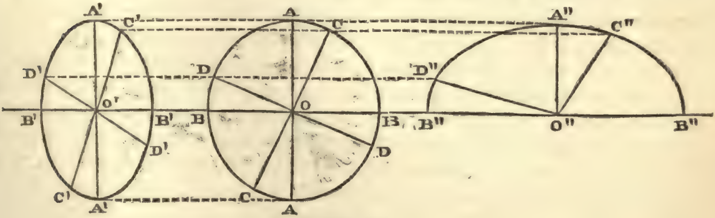

Fig. 88.

whose centre is O, pressed upon from without by a normal pressure of uniform intensity.

In order that the intensity of that pressure may be conveniently expressed in units of force per unit of area, conceive the ring in question to represent a vertical section of a cylindrical shell, whose length, in a direction perpendicular to the plane of the figure, is *unity*. Let p denote the intensity of the external pressure, in units of force per unit of area; r the radius of the ring; T the thrust exerted round it, which, because its length is unity, is a thrust *per unit of length*.

The uniform normal pressure p, if not actually caused by the thrust of a fluid, is similar to fluid pressure; and, according to Article 110, it is equivalent to a pair of conjugate pressures in any two directions at right angles to each other, of equal intensity. For example, let x be vertical, y horizontal, and let p_x, p_y, be the intensities of the vertical and horizontal pressure respectively, then

$$p_x = p_y = p; \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

and the same is true for any pair of rectangular pressures.

To find the thrust of the ring, conceive it to be divided into two parts by any diametral plane, such as C C. The thrust of the ring at the two ends of this diameter, of the amount 2 T, must balance the component, in a direction perpendicular to the diameter, of the pressure on the ring; the normal intensity of that component is p, as already shown; and the area on which it acts, projected on the plane, C C, which is normal to its direction, is $2r$; hence we have the equation

$$2\,\mathrm{T} = 2\,p\,r; \text{ or } \mathrm{T} = p\,r \quad \ldots\ldots\ldots\ldots\ldots\ldots (2.)$$

for the thrust all round the ring; which is expressed in words by this

THEOREM. *The thrust round a circular ring under an uniform normal pressure is the product of the pressure on an unit of circumference by the radius.*

180. **Elliptical Arches for Uniform Pressures.**—If a linear arch has to sustain the pressure of a mass in which the pair of conjugate thrusts at each point are uniform in amount and direction, but not equal to each other, all the forces acting parallel to any given direction will be altered from those which act in a fluid mass, by a given constant ratio; so that they may be represented by *parallel projections* of the lines which represent the forces that act in a fluid mass. Hence the figure of a linear arch which sustains such a system of pressures as that now considered, must be a parallel projection of a circle; that is, an *ellipse*. To investigate the relations which must exist amongst the dimensions of an elliptic linear arch under a pair of conjugate pressures of uniform intensity, let A′ B′ A′ B′, B″ A″ B″, in fig. 88, represent elliptic ribs, transformed from the circular rib A B A B by parallel projection, the vertical dimensions being unchanged, and the horizontal dimensions either expanded (as B″ B″),

or contracted (as B′ B′), in a given uniform ratio denoted by c; so that r shall be the vertical and $c\,r$ the horizontal semi-axis of the ellipse; and if x, y, be respectively the vertical and horizontal coordinates of any point in the circle, and x' y', those of the corresponding point in the ellipse, we shall have

$$x' = x;\ y' = c\,y. \ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

If C C, D D, be any pair of diameters of the circle at right angles to each other, their projections will be a pair of conjugate diameters of the ellipse, as C′ C′, D′ D′.

Let P_x be the total vertical pressure, and P_y the total horizontal pressure, on one quadrant of the circle A B.

Then

$$P_x = P_y = T = p\,r.$$

Let P'_x be the total vertical pressure, and P'_y the total horizontal pressure, on one quadrant of the ellipse, as A′ B′, or A″ B″; and let T'_x be the vertical thrust on the rib at B′ or B″, and T'_y the horizontal thrust at A′ or A″.

Then, by the principle of transformation,

$$\left.\begin{array}{l} T'_x = P'_x = P_x = T = p\,r; \\ T'_y = P'_y = c\ P_y = c\,T = c\,p\,r; \end{array}\right\} \ldots\ldots\ldots\ldots (2.)$$

or, *the total thrusts are as the axes to which they are parallel.*

Further, let $P' = T'$ be the total pressure, parallel to any semidiameter of the ellipse (as O′ D′ or O″ D″) on the quadrant D′ C′ or D″ C″, which force is also the thrust of the rib at C′ or C″, the extremity of the diameter conjugate to O′ D′ or O″ D″; and let O′ D′ or O″ D″ $= r'$; then

$$P' = T' = \frac{r'}{r}\,P = p\,r'; \ldots\ldots\ldots\ldots\ldots\ldots (3.)$$

or, *the total thrusts are as the diameters to which they are parallel.*

Next, let p'_x, p'_y, be the *intensities* of the conjugate horizontal and vertical pressures on the elliptic arch; that is, of the "*principal stresses*" (Articles 109, 112). Each of those intensities being found by dividing the corresponding total pressure by the area of the plane to which it is normal, they are given by the following equation:—

$$\left.\begin{array}{l} p'_x = \dfrac{P'_x}{c\,r} = \dfrac{p}{c} \\[2ex] p'_y = \dfrac{P'_y}{r} = c\,p; \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (4.)$$

so that *the intensities of the principal pressures are as the squares of the axes of the elliptic arch to which they are parallel.*

Hence the "ellipse of stress" of Article 112 is an ellipse whose axes are proportional to the squares of the axes of the elliptic arch; and to adapt an elliptic arch to uniform vertical and horizontal pressures, *the ratio of the axes of the arch must be the square root of the ratio of the intensities of the principal pressures;* that is,

$$c = \sqrt{\frac{p'_y}{p'_x}}. \quad \text{.............................(5.)}$$

The external pressure on any point, D′ or D″, of the elliptic arch, is directed towards the centre, O′ or O″, and its intensity, per unit of area of the plane to which it is conjugate (O′ C′ or O″ C″), is given by the following equation, in which r' denotes the semidiameter (O′ D′ or O″ D″) parallel to the pressure in question, and r'' the conjugate semidiameter (O′ C′ or O″ C″):—

$$p' = \frac{P'}{r''} = p \cdot \frac{r'}{r''}; \quad \text{.........................(6.)}$$

that is, *the intensity of the pressure in the direction of a given diameter is directly as that diameter and inversely as the conjugate diameter.*

Let p'' be the intensity of the external pressure in the direction of the semidiameter r'. Then it is evident that

$: : p\frac{r'}{r''} : p\frac{r''}{r'} = \frac{r'}{r''} : \frac{r''}{r'} = r'^2 : r''^2$

$$p' : p'' : : r'^2 : r''^2; \quad \text{.......................(7.)}$$

that is, *the intensities of a pair of conjugate pressures are to each other as the squares of the conjugate diameters of the elliptic rib to which they are respectively parallel.*

These results might also have been arrived at by means of the principles relative to the ellipse of stress, which have been explained in Article 112.

181. **Distorted Elliptic Arch.**—To adapt an elliptic linear arch to the sustaining of the pressure of a mass in which, while the state of stress is uniform, the pressure conjugate to a vertical pressure is not horizontal, but inclined at a given angle j, the figure of the ellipse must be derived from that of a circle by the substitution of inclined for horizontal co-ordinates.

In fig. 89, let B A C be a semicircular arch on which the external pressures are normal and uniform, and of the intensity p, as before; the radius being r, and the thrust round the arch, and load on a quadrant, being as before, $P = T = p\,r$. Let D be any point in the circle, whose co-ordinates are, vertical, $\overline{OE} = x$, horizontal,

$\overline{ED} = y$. Let B′ A′ C′ be a semi-elliptic arch, in which the vertical ordinates are the same with those of the circle, while for each

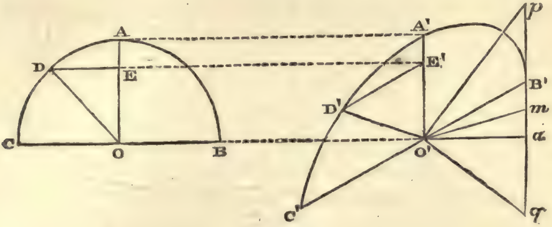

Fig. 89.

horizontal ordinate is substituted an ordinate inclined to the horizon by the constant angle j, and bearing to the corresponding horizontal ordinate of the circle the constant ratio c; that is to say, let

$$\left.\begin{array}{l}\overline{O'E'} = x' = x;\\ \overline{E'D'} = y' = c\,y;\\ \angle\, E\,E'\,D' = j.\end{array}\right\}\dots\dots\dots\dots\dots\dots\dots(1.)$$

Then for the vertical semidiameter of the circle $\overline{OA} = r$, will be substituted the equal vertical semidiameter of the ellipse $\overline{O'A'} = r$; and for the horizontal diameter of the circle $\overline{CB} = 2\,r$, will be substituted the inclined diameter of the ellipse $\overline{C'B'} = 2\,c\,r$, which is *conjugate* to the vertical semidiameter.

The forces applied to the elliptic arch are to be resolved into vertical and *inclined* components, parallel to O′ A′ and C′ B′, instead of vertical and horizontal components. Let P'_x denote the total vertical pressure, and P'_y the total inclined pressure, on either of the elliptic quadrants, C′ A′, A′ B′; T'_y the inclined thrust of the arch at A′, T'_x the vertical thrust at B′ or C′. Then

$$\left.\begin{array}{l}T'_x = P'_x = T = P = p\,r;\\ T'_y = P'_y = c\,T = c\,P = c\,p\,r;\end{array}\right\}\dots\dots\dots\dots\dots(2.)$$

that is to say, those forces are, as before, *proportional to the diameters to which they are parallel.*

Let p'_x be the intensity of the vertical pressure on the elliptic arch per unit of area of the inclined plane to which it is conjugate, $\overline{C'B'}$; let p'_y be the intensity of the inclined pressure per unit of area of the vertical plane to which it is conjugate; then

$$\left.\begin{aligned} p'_x &= \frac{P'_x}{cr} = \frac{p}{c}; \\ p'_y &= \frac{P'_y}{r} = cp; \\ c &= \sqrt{\frac{p'_y}{p'_x}}. \end{aligned}\right\} \text{..........................} (3.)$$

so that, as before, *the intensities of the conjugate pressures are as the squares of the diameters to which they are parallel.*

The thrust of the arch at any point D′ is as before, proportional to the diameter conjugate to O′ D′.

It is sometimes convenient to express the intensity of the vertical pressure per unit of area of the *horizontal projection* of the space over which it is distributed ; this is given by the equation

$$p'_x \cdot \sec j = \frac{p}{c \cdot \cos j} \text{......................} (4.)$$

It is to be borne in mind, that this is not the pressure on unity of area of a horizontal plane (which pressure is inversely as the horizontal diameter of the ellipse and directly as the diameter conjugate to that diameter, to which latter diameter it is parallel), but the pressure on that area of a plane inclined at the angle j, whose horizontal projection is unity.

The following geometrical construction serves to determine the major and minor axes of the ellipse B′ A′ C′.

Draw $O'a \perp$ and $= \overline{O'A'}$; join $B'a$, which bisect in m; in $B'a$ produced both ways take $\overline{mp} = \overline{mq} = O'm$; join $O'p$, $O'q$; these lines, which are perpendicular to each other, are the *directions* of the axes of the ellipse, and the *lengths* of the semiaxes are respectively equal to the segments of the line $p\,q$, viz., $\overline{B'p} = \overline{a\,q}$, $\overline{B'q} = \overline{a\,p}$.

The following is the algebraical expression of this solution. Let A denote the major and B the minor semi-axis of the ellipse.

Then

$$A + B = 2\,\overline{O'm} = r\sqrt{(1 + c^2 + 2c \cdot \cos j)};$$
$$A - B = \overline{B'a} = r\sqrt{(1 + c^2 - 2c \cdot \cos j)};$$

whence we have for the lengths of the semi-axes,

$$\left.\begin{aligned} A &= \frac{r}{2}\left\{ \sqrt{(1 + c^2 + 2c \cdot \cos j)} + \sqrt{(1 + c^2 - 2c \cdot \cos j)} \right\}; \\ B &= \frac{r}{2}\left\{ \sqrt{(1 + c^2 + 2c \cdot \cos j)} - \sqrt{(1 + c^2 - 2c \cdot \cos j)} \right\}; \end{aligned}\right\} (5.)$$

The angle $\angle \mathrm{B}'\mathrm{O}'p = k$, which the nearest axis makes with the diameter C′ B′, is found by the equation

$$\sin k = \frac{\mathrm{B}}{c\,r}\sqrt{\left(\frac{\mathrm{A}^2-c^2r^2}{\mathrm{A}^2-\mathrm{B}^2}\right)} \text{ or } \frac{\mathrm{A}}{c\,r}\sqrt{\left(\frac{\mathrm{B}^2-c^2r^2}{\mathrm{A}^2-\mathrm{B}^2}\right)};\ldots(6.)$$

according as that axis is the longer; —— the shorter.

The axes of the elliptic arch are parallel to, and proportional to the square roots of, the axes of the ellipse of stress in the pressing mass; so that they might be found by the aid of case 3 of Problem IV., Article 112.

182. **Arches for Normal Pressure in General.**—The condition of a linear arch of any figure at any point where the pressure is normal, is similar to that of a circular arch of the same curvature under a pressure of the same intensity; and hence modifying the Theorem of Article 179 to suit this case, we have the following:—

THEOREM I. *The thrust at any normally pressed point of a linear arch is the product of the radius of curvature by the intensity of the pressure;* that is, denoting the radius of curvature by ϱ, the normal pressure per unit of length of curve by p, and the thrust by T,

$$\mathrm{T} = p\,\varrho \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.)$$

Example. This Theorem is verified by the vertically and horizontally pressed elliptic arches of Article 180; for the radii of curvature of an ellipse at the ends of its two axes, r and $c\,r$, are respectively,

$\frac{B^2}{\cancel{A}}$ Ol [illegible]

$$\left.\begin{array}{l} \text{At the ends of } r;\ \varrho_x = \dfrac{c^2r^2}{r} = c^2r; \\[2ex] \text{At the ends of } c\,r;\ \varrho_y = \dfrac{r^2}{c\,r} = \dfrac{r}{c}; \end{array}\right\}\ldots\ldots\ldots\ldots(2.)$$

$\frac{A^2}{\cancel{B}}$

Introducing these values into the equations of Article 180, and into equation 1 of this Article, we find,

$$\left.\begin{array}{l} \mathrm{T}'_x = p'_y\varrho_y = c\,p\cdot\dfrac{r}{c} = p\,r \text{ as before;} \\[2ex] \mathrm{T}'_y = p'_x\varrho_x = \dfrac{p}{c}\cdot c^2r = c\,p\,r \text{ as before:} \end{array}\right\}\ldots\ldots\ldots(3.)$$

It is further evident, that if the pressure be normal at *every* point of the arch (which it is not in the cases cited), the thrust must be constant at every point; for it can vary only by the application of a tangential pressure to the arch; and hence follows

THEOREM II. *In a linear arch sustaining a pressure which is everywhere normal, the thrust is uniform, and the radius of curvature is inversely as the pressure*—a theorem expressed symbolically thus:—

$$\mathrm{T} = p\varrho = \text{constant} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4.)$$

The only arch of this class which has hitherto been considered is the circular arch under uniform normal pressure. Another instance will be given in the following Article.

183. **The Hydrostatic Arch** is a linear arch suited for sustaining normal pressure at each point proportional, like that of a liquid in repose, to the depth below a given horizontal plane; and is sometimes called "the arch of Yvon-Villarceaux," from the name of the mathematician who first thoroughly investigated the properties of its figure by the aid of elliptic functions.

The radius of curvature at a given point in the hydrostatic arch being, in virtue of Theorem II. of the last Article, inversely proportional to the intensity of the pressure, is also inversely proportional to the depth below the horizontal plane at which vertical ordinates representing that intensity commence.

In fig. 90, let Y O Y represent the level surface from which the pressure increases at an uniform rate downwards, so as to be similar to the pressure of a liquid having its upper surface at Y O Y. Let A be the crown of the hydrostatic arch, being the point where it is nearest the level surface, and consequently horizontal. Let co-ordinates be measured from the point O in the level surface, directly above the crown of the arch; so that $\overline{\mathrm{OX}} = \overline{\mathrm{YC}} = x$ shall be the vertical ordinate, and $\overline{\mathrm{OY}} = \overline{\mathrm{XC}} = y$ the horizontal ordinate, of any point, C, in the arch. Let $\overline{\mathrm{OA}}$, the least depth of the arch below the level surface, be denoted by x_0, the radius of curvature at the crown by r_0, and the radius of curvature at any point C by r.

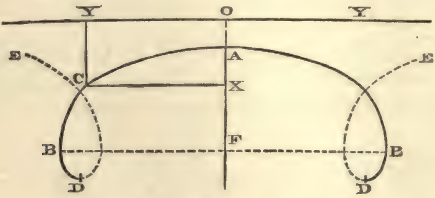

Fig. 90.

Let w be the weight of an unit of volume of the liquid, to whose pressure the load on the arch is equivalent. Then the intensities of the external normal pressure at the crown A, and at any point C, are expressed respectively by

$$p_0 = w x_0;\ p = w x. \text{.....................} (1.)$$

The thrust of the arch, which, in virtue of the principles of Article 182, is a constant quantity, is given by the equation

$$T = p_0 r_0 = w x_0 r_0 = p r = w x r; \text{...........} (2.)$$

from which follows the following geometrical equation, being that which characterizes the figure of the arch:—

$$x r = x_0 r_0. \text{..............................} (3.)$$

When x_0 and r_0 are given, the property of having the radius of curvature inversely proportional to the vertical ordinate from a given horizontal axis enables the curve to be drawn approximately, by the junction of a number of short circular arcs. It is found to present some resemblance to a trochoid (with which, however, it is by no means identical). At a certain point, B, it becomes vertical, beyond which it continues to turn, until at D it becomes horizontal; at this point its depth below the level surface is greatest, and its radius of curvature least. Then ascending, it forms a loop, crosses its former course, and proceeds towards E to form a second arch similar to B A B. Its coils, consisting of alternate arches and loops, all similar, follow each other in an endless series.

It is obvious that only one coil or division of this curve, viz., from one of the lowest points, D, through a vertex, A, to a second point, D, is available for the figure of an arch; and that the portion B A B, above the points where the curve is vertical, is alone available for supporting a load.

Let x_1, y_1, be the co-ordinates of the point B. The vertical load above the semi-arch A B is represented by

$$w \int_0^{y_1} x\, d y;$$

and this being sustained by the thrust T of the arch at B, must obviously be equal to that thrust; whence follows the equation

$$x r = x_0 r_0 = \int_0^{y_1} x \cdot d y \text{..................} (4.)$$

That is to say, *the area of the figure between the shortest vertical ordinate, and the vertical tangent ordinate, is equal to the constant product of the vertical ordinate and radius of curvature.*

The vertical load above any point, C, is

$$w \int_0^{y} x\, d y;$$

and this is sustained by and equal to the vertical component of the thrust of the arch at C, which is T · sin i (i being the inclination of the arch to the horizon).

Hence follows the equation

$$\int_0^y x\,d\,y = x_0\,r_0 \cdot \sin i = \frac{x_0\,r_0}{\sqrt{1+\frac{d\,y^2}{d\,x^2}}}; \dots\dots\dots\dots (5.)$$

That is to say, *the area of the figure between the shortest vertical ordinate and any vertical ordinate, varies as the sine of the angle of inclination to the horizon of the curve at the latter ordinate.*

The horizontal external pressure on the semi-arch from B to A is the same with that on a vertical plane, A F, immersed in a liquid of the specific gravity w with its upper edge at the depth x_0 below the surface (see Article 124), so that its amount is

$$w\int_{x_0}^{x_1} x\,d\,x = w \cdot \frac{x_1^2 - x_0^2}{2};$$

and this is balanced by the thrust of the arch, T, at the crown. Hence follows the equation

$$x\,r = x_0\,r_0 = \frac{x_1^2 - x_0^2}{2} \dots\dots\dots\dots\dots\dots\dots\dots (6.)$$

That is to say, *half the difference of the squares of the least vertical ordinate and of the tangent vertical ordinate is equal to the constant product of the vertical ordinate and radius of curvature.*

Equation 6 gives for the value of the vertical tangent ordinate,

$$x_1 = \sqrt{x_0^2 + 2\,x_0\,r_0} \dots\dots\dots\dots\dots\dots\dots\dots (7.)$$

The horizontal external pressure between B and any point, C, is equal to the pressure of a liquid of the specific gravity w on a vertical plane X F with its upper edge immersed to the depth x, so that its amount is

$$w\int_{x}^{x_1} x\,d\,x = w \cdot \frac{x_1^2 - x^2}{2};$$

and this is balanced by the horizontal component T · cos i of the thrust of the arch at C; whence follows the equation

$$\frac{x_1^2 - x^2}{2} = x_0\,r_0 \cdot \cos i; \dots\dots\dots\dots\dots\dots (8.)$$

which gives for the value of any vertical ordinate,

$$x = \sqrt{(x_1^2 - 2\,x_0 r_0 \cdot \cos i)} = \sqrt{\left\{ x_0^2 + 2\,x_0 r_0 \,(1 - \cos i) \right\}}$$
$$= \sqrt{\left(x_0^2 + 4\,x_0 r_0 \cdot \sin^2 \frac{i}{2}\right)} \ldots\ldots\ldots\ldots\ldots (9.)$$

Let x, x', be any two vertical ordinates. Then from equation 8 it follows that

$$x'^2 - x^2 = 2\,x_0 r_0 \,(\cos i - \cos i') \ldots\ldots\ldots\ldots (10.)$$

or, *the difference of the squares of two ordinates varies as the difference of the cosines of the respective inclinations of the arc at their lower ends.*

From equation 9 is deduced the following expression of the inclination in terms of the vertical ordinate :—

$$2 \sin^2 \cdot \frac{i}{2} = 1 - \cos i = 1 - \sqrt{\frac{1}{1 + \frac{d x^2}{d y^2}}} = \frac{x^2 - x_0^2}{2\,x_0 r_0} \ldots (11.)$$

The various properties of the figure of the hydrostatic arch expressed by the preceding equations are thus summed up in one formula :—

$$x_0 r_0 = x r = \int_0^{y_1} x\,d y = \frac{\int_0^y x\,d y}{\sin i} = \frac{x_1^2 - x_0^2}{2} = \frac{x_1^2 - x^2}{2 \cos i} \ldots (12.)$$

To obtain expressions for the horizontal co-ordinate y, whose maximum value is the half-span y_1, and also for the lengths of arcs of the curve, it is necessary to use elliptic functions.

[The reader who has not studied elliptic functions may here pass at once to Article 184.]

In the use of elliptic functions the notation employed will be that of Legendre; and the classes of functions referred to will be those called by that author the *first* and *second* kind respectively, and tabulated by him in the second volume of his treatise.

Let θ denote a constant angle, called the *modulus* of the functions; ϕ, a variable angle called the *amplitude*; then an elliptic function of the first kind is expressed by

$$\mathrm{F}\ (\theta, \phi) = \int_0^{\phi} \frac{d\,\phi}{\sqrt{(1 - \sin^2 \theta \cdot \sin^2 \phi)}} \ldots\ldots\ldots\ldots (a.)$$

and an elliptic function of the second kind is expressed by

$$\mathrm{E}\ (\theta, \phi) = \int_0^{\phi} \sqrt{(1 - \sin^2 \theta \cdot \sin^2 \phi)}\, d\,\phi \ldots\ldots\ldots\ldots (b.)$$

The values of those functions, when the upper limit of integration is $\varphi = \frac{\pi}{2}$, or 90 degrees, are called *complete* functions, and denoted respectively by

$$\mathrm{F}_1\,(\theta) \text{ and } \mathrm{E}_1\,(\theta). \quad \text{.......................} (c.)$$

In order to apply those functions to the case of the hydrostatic arch, let the amplitude be half the supplement of the inclination of the curve; that is, let

$$\varphi = 90^\circ - \frac{i}{2}; \quad \text{.......................} (d.)$$

so that at D, $\varphi = 0$, at B, $\varphi = 45^\circ$, and at A, $\varphi = 90^\circ$. Let the vertical ordinate and radius of curvature at the point D be denoted respectively by X, R; then

$$\left.\begin{array}{l} \mathrm{X} = \sqrt{(x_0^2 + 4\,r_0\,x_0)}\,; \text{ and} \\ \mathrm{R\,X} = r\,x = r_0\,x_0\,; \end{array}\right\} \quad \text{...............} (13.)$$

for the modulus θ take an angle such that

$$\sin^2\theta = \frac{4\,\mathrm{R}}{\mathrm{X}} = \frac{4\,r_0\,x_0}{x_0^2 + 4\,r_0\,x_0} \quad \text{.................} (e.)$$

Then equation 9, the expression for the vertical ordinate, becomes

$$x = \sqrt{\left(x_0^2 + 4\,x_0 r_0 \cdot \sin^2\frac{i}{2}\right)} = \mathrm{X} \cdot \sqrt{\left(1 - \sin^2\theta \cdot \sin^2\varphi\right)}. \; (14.)$$

The values of this for the points B and A are respectively

$$x_1 = \mathrm{X}\sqrt{\left(1 - \frac{\sin^2\theta}{2}\right)}\,;\; x_0 = \mathrm{X}\sqrt{\left(1 - \sin^2\theta\right)}$$
$$= \mathrm{X} \cdot \cos\theta \quad \text{.........................} (14\text{ A.})$$

Introducing the above value of x into equation 5, we obtain for the area between O A and any other vertical ordinate,

$$\int^y x\,d\,y = x_0 r_0 \cdot \sin i = 2\,\mathrm{X\,R} \cdot \cos\varphi\,\sin\varphi$$
$$= \frac{\mathrm{X}^2 \cdot \sin^2\theta}{2} \cdot \cos\varphi\,\sin\varphi \quad \text{..................} (15.)$$

The value of this expression for the point B is

$$\int_0^{y_1} x\,d\,y = x_0\,r_0 = \mathrm{X\,R} = \frac{\mathrm{X}^2 \cdot \sin^2\theta}{4} \quad \text{.......} (15\text{ A.})$$

Now differentiate the area (15) with respect to the amplitude φ, and divide by x; this gives

$$\frac{d\,y}{d\,\varphi}=\mathrm{X}\cdot\frac{\sin^2\theta}{2}\cdot\frac{\cos^2\varphi-\sin^2\varphi}{\sqrt{(1-\sin^2\theta\cdot\sin^2\varphi)}}$$

$$=-\mathrm{X}\cdot\left\{\frac{2-\sin^2\theta}{2\sqrt{(1-\sin^2\theta\cdot\sin^2\varphi)}}-\sqrt{(1-\sin^2\theta\sin^2\varphi)}\right\}\text{...(16.}$$

This differential being integrated between the limits $\varphi=90°$, which corresponds to $y_0=0$, and $\varphi=90°-\frac{i}{2}$, which corresponds to the required value of y, gives

$$y=\mathrm{X}\cdot\left\{\left(1-\frac{\sin^2\theta}{2}\right)\Big(\mathrm{F}_1(\theta)-\mathrm{F}(\theta,\varphi)\Big)-\mathrm{E}_1(\theta)+\mathrm{E}(\theta,\varphi)\right\}\text{(17.)}$$

For the point B, this gives for the *half-span of the arch*

$$y_1=\mathrm{X}\cdot\left\{\left(1-\frac{\sin^2\theta}{2}\right)\Big(\mathrm{F}_1(\theta)-\mathrm{F}(\theta,45°)\Big)-\mathrm{E}_1(\theta)+\mathrm{E}(\theta,45°)\right\}\text{(18.)}$$

Let s denote the length of any arc of the curve, A C, commencing at the crown. Then

$$s=\int_0^i r\,d\,i=2\int_\varphi^{90°} r\,d\,\varphi\text{..................(19.)}$$

The value of the radius of curvature r in terms of the modulus and amplitude is

$$r=\frac{r\,x}{x}=\frac{\mathrm{X}\cdot\sin^2\theta}{4\sqrt{(1-\sin^2\theta\cdot\sin^2\varphi)}};\text{............(20.)}$$

and this being introduced into the integral (19), gives for the arc A C,

$$s=\frac{\mathrm{X}\cdot\sin^2\theta}{2}\left\{\mathrm{F}_1(\theta)-\mathrm{F}(\theta,\varphi)\right\}\text{............(21.)}$$

The length of the semi-arch A B is

$$s=\frac{\mathrm{X}\cdot\sin^2\theta}{2}\left\{\mathrm{F}_1(\theta)-\mathrm{F}(\theta,45°)\right\}\text{..........(22.)}$$

Such are the formulæ expressing the geometrical properties of the hydrostatic arch. Numerical results can easily be computed from them by the aid of Legendre's tables of the functions F and E.

The relation between the thrust of the arch, the specific gravity of the load, and the modulus is given by the equation

$$\mathrm{T}=w\,r\,x=\frac{w\,\mathrm{X}^2\cdot\sin^2\theta}{4}=\frac{w\,(x_0^2+4\,r_0\,x_0)\sin^2\theta}{4}\text{...(23.)}$$

184. **Geostatic Arches.**—It is proposed, by the term "*Geostatic Arch,*" to denote a linear arch of a figure suited to sustain a pressure similar to that of earth, which (as will be shown in Section 3 of this Chapter) consists, in a given vertical plane, of a pair of conjugate pressures, one vertical, as in Article 125 of Part I., and proportional to the depth below a given plane, horizontal or sloping, and the other parallel to the horizontal or sloping plane, and bearing to the vertical pressure a certain constant ratio, depending on the nature of the material, and other circumstances to be explained in the sequel. In what follows, the horizontal or sloping plane will be called the *conjugate plane,* and ordinates parallel to its line of steepest declivity, when it slopes, or to any line in it, when it is horizontal, *conjugate ordinates.* The intensity of the vertical pressure will be estimated per unit of area of the *conjugate plane;* and the pressure parallel to the line of steepest declivity of that plane, when it slopes, or to any line in it, when it is horizontal, will be called the *conjugate pressure,* and its intensity will be estimated per unit of area of a vertical plane.

Let the origin of co-ordinates be taken at a point in the conjugate plane vertically above the crown of the proposed arch; let x' denote the vertical co-ordinate of any point, and y' the conjugate co-ordinate. Let j be the angle of inclination of the conjugate plane to the horizon. Let w' be the weight of unity of volume of the material to which the pressure is due, and whose upper surface is at the conjugate plane. Then the intensity of the vertical pressure at a given depth x', according to Theorem I. of Article 125, is

$$p'_x = w' x' \cdot \cos j; \qquad (1.)$$

and that of the conjugate pressure

$$p'_y = c^2 p'_x = c^2 w' x' \cdot \cos j; \qquad (2.)$$

c^2 being a constant ratio, expressed in the form of a square, for a reason which will afterwards appear.

Conceive a hydrostatic arch, whose vertical and horizontal co-ordinates are x and y, and which is subjected to the pressure of a material whose weight per cubic foot is

$$w = c\, w' \cos j \qquad (3.)$$

Then at any given point in that hydrostatic arch, whose depth below the surface is $x = x'$, we shall have for the intensities of the vertical and horizontal pressures

$$p_x = p_y = w\, x = c\, w' x' \cdot \cos j = c\, p'_x = \frac{p'_y}{c} \qquad (4.)$$

Now let the figure of an arch be *transformed* from that of the hydrostatic arch by parallel projection, in such a manner that the vertical co-ordinate of any point in the new arch shall be the same with that of the corresponding point in the hydrostatic arch, and that the *conjugate co-ordinate* of any point in the new arch shall bear to the *horizontal co-ordinate* of the corresponding point in the hydrostatic arch the constant ratio c; that is to say, let

$$x' = x;\ y' = c\,y \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5.)$$

The total *vertical* and *horizontal* pressures on the arc between two given points in the hydrostatic arch are respectively

$$\mathrm{P}_x = \int p_x\,d\,y;\ \mathrm{P}_y = \int p_y\,d\,x \quad \ldots\ldots\ldots\ldots\ldots\ldots (6.)$$

The total *vertical* and *conjugate* pressures on the arc between the two corresponding points in the new arch are respectively

$$\mathrm{P}'_x = \int p'_x\,d\,y';\ \mathrm{P}'_y = \int p'_y\,d\,x'; \quad \ldots\ldots\ldots\ldots\ldots (7.)$$

and if into these two expressions we introduce the values of p'_x, p'_y, $d\,x'$, and $d\,y'$, deduced from equations 4 and 5, viz. :—

$$p'_x = \frac{p_x}{c};\ p'_y = c\,p_y;\ d\,x' = d\,x;\ d\,y' = c\,d\,y;$$

we find the following relations between the total vertical and horizontal pressures in a given arc of the hydrostatic arch, and the total vertical and conjugate pressures on the corresponding arc of the transformed arch,

$$\mathrm{P}'_x = \mathrm{P}_x;\ \mathrm{P}'_y = c\,\mathrm{P}_y; \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8.)$$

being the same with the relations which, according to equation 5, exist between the co-ordinates respectively parallel to the pressures in question. Therefore the transformed arch is a parallel projection of the original arch under forces represented by lines which are the corresponding parallel projections of the lines representing the forces acting on the original arch: therefore it is in equilibrio. The conclusions of the preceding investigation may be summed up in the following

THEOREM. *A geostatic arch, transformed from a hydrostatic arch by preserving the vertical co-ordinates, and substituting for the horizontal co-ordinates, conjugate co-ordinates, either horizontal or inclined, and altered in a given ratio, sustains vertical and conjugate pressures, the ratio of the intensity of the conjugate pressure to that of the vertical pressure being the square of the ratio of the conjugate co-ordinates to the original horizontal co-ordinates.*

This transformation is exactly analogous to that of a circular arch into an elliptic arch, in Articles 180, 181.

Let T_0 be the thrust, horizontal or inclined as the case may be, at the crown of a geostatic arch, and T_1 the vertical thrust at the points where the arch is vertical, which in this, as in other cases, is the vertical load of the semi-arch; then

$$T_0 = c\,T_1 \text{.........................} (10.)$$

All the equations relative to the *co-ordinates* of a hydrostatic arch, given in Article 183, are made applicable to a geostatic arch, by substituting x' for x, and $\frac{y'}{c}$ for y. This principle, however, is applicable to *co-ordinates* only, and not to angles of inclination, radii of curvature, nor lengths of arcs. The modulus θ, and amplitude φ, are therefore to be considered as functions, not of inclinations, nor of radii of curvature, but of vertical ordinates; that is to say, let x_0 be the least vertical ordinate at the crown, x_1 the vertical tangent ordinate, and X the greatest vertical ordinate at the loop (which are the same in both kinds of arch), then

$$\left.\begin{array}{l} \theta = \text{arc cos}\,\dfrac{x_0}{\mathrm{X}} = \text{arc cos}\,\dfrac{x_0}{\sqrt{2\,x_1^2 - x_0^2}}; \\[2ex] \varphi = \text{arc sin}\,\dfrac{\sqrt{\mathrm{X}^2 - x'^2}}{\mathrm{X}\cdot\sin\theta} = \text{arc sin}\,\sqrt{\left\{\dfrac{\mathrm{X}^2 - x'^2}{\mathrm{X}^2 - x_0^2}\right\}}; \end{array}\right\} (11.)$$

and $\frac{y'}{c}$ is the same function of θ and φ for a geostatic arch, that y is for a hydrostatic arch.

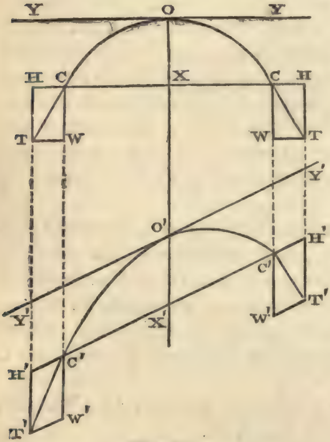

Fig. 91.

185. **Stereostatic Arch.**—This term is employed to denote a linear arch sustaining the pressure of a material in which, at any given point, there are a pair of conjugate pressures, one vertical, and the other in a fixed direction, horizontal or inclined, but not bearing to each other any constant proportion, nor following any invariable law as to their intensities, except that of being of the same intensity throughout each plane which is conjugate to the vertical pressure,—a condition which involves the symmetrical distribution of the vertical load on either side of a vertical axis traversing the crown of the arch.

The principal questions which arise respecting any stereostatic arch are comprehended under the following

Problem. *Given, the mode of distribution of the vertical pressure, and the figure of the arch; required, the mode of distribution of the conjugate pressure necessary in order to produce equilibrium, and also, the thrust at each point of the arch.*

Case 1. *When the direction of the conjugate pressure is horizontal.* This case is represented by the upper diagram in fig 91. Let O, the crown of the arch, be taken as the origin of co-ordinates; let O X be vertical and Y O Y horizontal. Both the figure of the arch and the forces acting on it are symmetrical on either side of the vertical axis O X. Let p_0 denote the intensity of the vertical pressure at the point O, and r_0 the radius of curvature of the arch at that point. Then because at the point O the pressure is normal to the arch, the horizontal thrust along the arch at that point is

$$T_0 = p_0 \, r_0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

Let C be any point in the arch, whose co-ordinates are $O X = x$, $X C = y$, and let

$$i = \text{arc cotan} \frac{d y}{d x}$$

be the inclination of the arch at C to the horizon. Let P_x denote the vertical load on the arc between O and C.

From C draw the vertical line $\overline{C W}$ to represent P_x, and the tangent C T forming the diagonal of the rectangle C W T H. Then $\overline{C T}$ will represent the thrust along the arch at C, and $\overline{C H}$ the horizontal component of that thrust; and if this be different from T_0, the difference must be made up by means of the horizontal pressure applied to the arch between O and C. To express this symbolically, let P_y be the amount of that horizontal pressure, and T the thrust $\overline{C T}$ along the arch at C; then

$$T = \frac{P_x}{\sin i} = P_x \cdot \text{cosec}\, i = P_x \cdot \frac{d s}{d x} \ldots\ldots\ldots\ldots (2.)$$

(where $d s$ denotes the increment of the arc O C).

The horizontal component $\overline{C H}$ of this thrust is

$$T \cdot \cos i = P_x \cdot \text{cotan}\, i = P_x \cdot \frac{d y}{d x};$$

consequently the horizontal pressure which must be applied to the arch between O and C to maintain equilibrium is

$$P_y = T_0 - P_x \cdot \text{cotan}\, i = T_0 - P_x \cdot \frac{d y}{d x}; \ldots\ldots\ldots (3.)$$

and if this equation be fulfilled at every point of the arch, it will be balanced.—Q. E. I.

When P_y is positive, it represents *inward pressure*, such as may arise from the resistance of the materials of the *spandril* of an arch to compression. When P_y is negative, it represents *outward pressure*, such as may arise from the resistance to compression of a portion of material situated below the crown of the ideal linear arch O C, or *tension*, such as may arise from tenacity in the spandril, and in the materials connecting it with the arch.

The *intensity* of the horizontal pressure is found by taking two points in the arch indefinitely near to each other, and finding the ratio which the portion of the horizontal pressure applied between them bears to the difference of their vertical ordinates. Let the intensity required be denoted by p_y; then

$$p_y = \frac{d\mathrm{P}_y}{dx} = -\frac{d\,(\mathrm{P}_x \cdot \operatorname{cotan} i)}{dx} = -\frac{d}{dx}\left(\mathrm{P}_x \frac{dy}{dx}\right) \ldots (4.)$$

(This equation comprehends the cases already considered in Article 168, of a cord under vertical loads, or an arch whose figure is that of such a cord inverted; for in that case, $\mathrm{P}_x = \mathrm{T}_0 \tan i$, and $\mathrm{P}_x \operatorname{cotan} i = \mathrm{T}_0 = \text{constant}$, so that $p_y = 0$.)

If it be required to express the intensity of the horizontal pressure in terms of that of the vertical pressure, let the latter intensity be

$$p_x = \frac{d\mathrm{P}_x}{dy};$$

then

$$p_y = -\frac{d}{dx}\left(\frac{dy}{dx}\int p_x\,dy\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5.)$$

Restricted Case. Let the arch have a *horizontal extrados*, at the height a above the crown O, and let the vertical pressure be due to the weight of material below that extrados; then

$$p_0 = w\,a\,;\; p_x = w\,(a + x);$$

and the vertical load becomes

$$\mathrm{P}_x = \int p_x\,dy = w \int (a + x)\,dy; \ldots\ldots\ldots\ldots (6.)$$

being proportional to the area between the intrados and extrados, and the vertical ordinates at O and C.

Example. Let the linear arch be part of a circle of the radius r, with a horizontal extrados at the distance $r + a$ from its centre.

In this case it is convenient to express all the variables in terms of the inclination i of the arch. Thus we have

$$\left.\begin{aligned} x &= r(1-\cos i); \\ y &= r\cdot\sin i; \\ dx &= r\cdot\sin i\, di; \\ dy &= r\cdot\cos i\, di. \end{aligned}\right\} \dots\dots\dots\dots\dots\dots\dots(7.)$$

It is also useful to make $a = m\,r$, m being the ratio which the depth of load at the crown bears to the radius. Then we have for the thrust at O,

$$T_0 = m\,w\,r^2; \dots\dots\dots\dots\dots\dots\dots\dots\dots(8.)$$

and for the vertical load between O and C,

$$P_x = w\int(a+x)\,dy = w\,r^2\int(m+1-\cos i)\cos i\,di$$
$$= w\,r^2\left\{(1+m)\sin i - \frac{\cos i\sin i}{2} - \frac{i}{2};\right\} \dots\dots(9.)$$

which value being introduced into equation 4, gives for the intensity of the horizontal pressure

$$p_y = \frac{d\,P_y}{dx} = -\frac{d\,(P_x \operatorname{cotan} i)}{dx} = -\frac{1}{r\sin i}\cdot\frac{d\,(P_x \operatorname{cotan} i)}{di}$$
$$= -\frac{w\,r}{\sin i}\cdot\frac{d}{di}\left\{(1+m)\cos i - \frac{\cos^2 i}{2} - \frac{i\cos i}{2\sin i}\right\}$$
$$= w\,r\left(1+m-\cos i - \frac{i-\cos i\sin i}{2\sin^3 i}\right) \dots\dots(10.)$$

The value of the horizontal pressure itself is given by introducing the values of T_0 and P_x from equations 8 and 9 into equation 3, and is as follows :—

$$P_y = w\,r^2\left\{m-(1+m)\cos i + \frac{\cos^2 i}{2} + \frac{i\cos i}{2\sin i}\right\} \dots(11.)$$

The horizontal component of the thrust of the arch at C is given by the equation

$$T\cos i = T_0 - P_y = w\,r^2\left\{(1+m)\cos i - \frac{\cos^2 i}{2} - \frac{i\cos i}{2\sin i}\right\} (12.)$$

When $i = 0$, that is, for the crown of the arch, p_y takes the following value :—

$$w\,r\left(m - \frac{1}{3}\right),$$

so that for every circular linear arch in which the depth of load at the crown, $m\,r$, is less than *one-third* of the radius, p_y has *negative* values at and near the crown, showing that *outward* horizontal pressure or tension is required to preserve equilibrium. In such cases, there is a certain value of the angle i for which $p_y = 0$. At the point where this takes place, P_y consequently attains a *negative maximum*, and the horizontal component $T \cdot \cos i$ of the thrust along the arch attains a *positive maximum*, greater than T_0, because of P_y being negative. Let this point be called C_0, and let the inclination of the arch at it be denoted by i_0. This angle must satisfy the transcendental equation

$$1 + m - \cos i_0 - \frac{i_0 - \cos i_0 \sin i_0}{2 \sin^3 i_0} = 0, \ldots\ldots\ldots (13.)$$

and can therefore be found by approximation only. As a first approximation, may be taken

$$i_0 = \text{arc} \cdot \cos \cdot \frac{3\,m + 1}{2};$$

and then by successive substitutions, nearer and nearer approximations may be found.

Supposing i_0 to have been thus determined to a sufficient degree of accuracy, its substitution for i in the equation 12 will give the maximum value of the horizontal component of the thrust of the arch.

By expanding or contracting the horizontal dimensions of a circular arch, it can be transformed into an elliptic arch, which will be balanced under forces deduced from those applied to the circular arch according to the principles explained in Articles 180, 184. In adapting the equations from 7 to 13 inclusive to an elliptic arch, it is to be observed that i represents *not* the inclination of the elliptic arch itself at a given point, but that of the circular arch from which the elliptic arch is derived at the corresponding point.

CASE 2. *When the direction of the conjugate pressure is inclined.* This case is represented in the lower diagram of fig. 91. The inclined axis of co-ordinates, $Y' O' Y'$, is taken parallel to the direction of the conjugate pressure, and touching the arch at the point O', which is now its crown. Each double ordinate of the arch, $C' X' C' = 2\,y'$, is bisected by the vertical axis, on either side of which the vertical load is symmetrically distributed.

Let j denote the inclination of the conjugate pressure to the horizon. Construct a parallel projection of the given arch, like the upper diagram of the figure, having its vertical ordinates equal to those of the distorted arch, and its horizontal ordinates less in the ratio $\cos j : 1$; conceive it to be under a vertical load, of equal amount to that on the distorted arch, and similarly distributed; determine the horizontal pressures required to keep it in equilibrio; then will the proper projection of those pressures keep the distorted arch in equilibrio.

The relations amongst the co-ordinates of the two arches, and the amounts and magnitudes of the vertical and conjugate pressures, are as follows, quantities relating to the distorted arch being distinguished by accented letters :—

$$\left.\begin{array}{l} x' = x;\ y' = y \sec j; \\ \mathrm{P}'_x = \mathrm{P}_x;\ \mathrm{T}'_0 = \mathrm{T}_0 \sec j;\ \mathrm{P}'_y = \mathrm{P}_y \sec j; \\ p'_x = p_x \cos j;\ p'_y = p_y \sec j. \end{array}\right\} \ldots (14.)$$

Let H′ denote the *conjugate component* of the thrust of the distorted arch at any point C′; then we have

$$\mathrm{H}' = \mathrm{T}'_0 - \mathrm{P}'_y = (\mathrm{T}_0 - \mathrm{P}_y) \sec j; \ldots\ldots\ldots\ldots (15.)$$

and if T′ be the thrust along the distorted arch at C′, then

$$\mathrm{T}' = \sqrt{(\mathrm{P}'^2_x + \mathrm{H}'^2 \pm 2\,\mathrm{H}'\,\mathrm{P}'_x \cdot \sin j)} \ldots\ldots\ldots\ldots (16.)$$

the positive or negative sign being used according as the point C′ is at the depressed or the elevated side of the arch.

186. **Pointed Arches.**—If a linear arch, as in fig. 92, consists of two arcs, B C, C B, meeting in a point at C, it is necessary to equilibrium that there should be concentrated at the point C a load equal to that which would have been distributed over the two arcs A C, C A, extending from the point C to the respective crowns, A, A, of the curves of which two portions form the pointed arch.

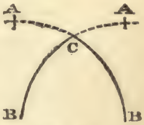

Fig. 92.

187. **Total Conjugate Thrust of Linear Arches.**—The total conjugate thrust of an arch is the conjugate component, horizontal or inclined, as the case may be, of the entire pressure exerted between one semi-arch and its abutment, whether directly, at the point from which the arch springs, or above that point, through the material of the spandril.

When a linear arch is of such a figure as to be balanced under a load of which the pressure is wholly vertical (as in the case described in Article 174), that is to say, when its figure is that in which a cord would hang, loaded with the same weight distributed in the same manner, its conjugate thrust is exerted simply at the point from which it springs, and is equal to the conjugate component of the thrust along the arch, which is a constant quantity throughout its whole extent.

When an arch springs vertically from its abutments, the point of springing sustains the vertical load of the semi-arch only; and the conjugate thrust is exerted wholly through the spandril.

In other cases, the conjugate thrust is exerted partly at the point of springing and partly through the spandril.

THEOREM. *The amount of the conjugate thrust is equal to the conjugate component of the thrust along the arch at the point where that component is a maximum;* for at that point, as appears from the reasoning of Article 185, the intensity of the *conjugate pressure* between the arch and its spandril is nothing: it is, therefore, entirely below that point that the conjugate thrust, whether through the spandril or at the point of springing, is exerted; and consequently the amount of that thrust must be equal to the maximum conjugate component of the thrust along the arch, which is balanced by it. The point of the arch where the conjugate component of the thrust along it is a maximum, is called the *point of rupture,* for reasons which will afterwards appear. It may be at the crown; or it may be in a lower position, to be determined by solving the equation formed by making the intensity of the conjugate pressure between the arch and spandril, as found by the method of Article 185, equal to nothing; that is,

$$p_y = \frac{d\,\mathrm{P}_y}{d\,x} = -\frac{d}{d\,x}\left(\mathrm{P}_x\,\frac{d\,y}{d\,x}\right) = 0 \text{............}(1.)$$

This equation having been solved so as to give the position of the point of rupture, the corresponding value of P_x, being the vertical load supported at that point, is to be computed; and then the conjugate thrust is given by the equation

$$\mathrm{H}_0 = \text{max. value of } \mathrm{P}_x \cdot \frac{d\,y}{d\,x} \text{..................}(2.)$$

(Where the conjugate pressures, as is generally the case, are horizontal, $\frac{d\,y}{d\,x} = \text{cotan}\, i$; and the value of i, the inclination of the arch, which fulfils equation 1, is called the *angle of rupture*).

When the point of rupture is the crown of the arch (as in hydrostatic and geostatic arches), equation 2 gives no result, because of P_x vanishing and $\frac{d\,y}{d\,x}$ increasing indefinitely; but it has already been shown by other methods that in this case, where the conjugate pressures are *horizontal*—

$$\mathrm{H}_0 = \mathrm{T}_0 = p_0 r_0; \text{.........................}(3.)$$

p_0 being the intensity of the vertical load, and r_0 the radius of curvature; but in order to form an equation which shall be applicable whether the conjugate pressures and co-ordinates are horizontal or inclined, the above equation must be converted into one expressed in terms of the co-ordinates; that is to say,

$$\mathrm{H}_0 = \mathrm{T}_0 = \frac{\frac{d\,\mathrm{P}_x}{d\,y}}{\frac{d}{d\,y} \cdot \frac{d\,x}{d\,y}} \text{ (for } y = 0) = \frac{p_0}{\frac{d^2 x}{d\,y^2}} \text{ (for } y = 0) \text{...(4.)}$$

For rectangular co-ordinates $\frac{d^2x}{dy^2} = \frac{1}{r_0}$ at the crown of the arch, so that equation 4 is converted into equation 3.

Thus far as to finding the *amount* of the conjugate thrust. To find the *position of its resultant*, that is to say, the depth of its line of action below the conjugate co-ordinate plane, we must conceive it to act against a vertical plane, extending from the depth of the point of rupture below the conjugate co-ordinate plane, down to the depth of the point of springing below that plane, and find, by the methods of Article 89, the vertical co-ordinate of the *centre of pressure* of the plane so acted upon. That is to say, let x_0 denote the depth of the point of rupture, and x_1 that of the point of springing below the conjugate co-ordinate plane; p_y the intensity of the conjugate pressure between the arch and spandril at any point between those points, and

$$\mathrm{H}_1 = \mathrm{H}_0 - \int_{x_0}^{x_1} p_y\,d\,x, \text{..................(5.)}$$

the conjugate component of the thrust of the arch at the point of springing; also, let x_{H} be the depth of the resultant conjugate thrust below the conjugate co-ordinate plane; then

$$x_{\mathrm{H}} = \frac{\int_{x_0}^{x_1} x\,p_y \cdot d\,x + \mathrm{H}_1\,x_1}{\mathrm{H}_0} \text{................(6.)}$$

Example I. *Circular arch under uniform normal pressure of intensity*, p. 183 (Art. 179).

Here $p_x = p_y = p$; and the point of rupture is at the crown, the horizontal thrust is

$$\mathrm{H}_0 = \mathrm{T} = p\,r \text{........................(7.)}$$

Let the crown be taken for origin of co-ordinates, so that $x_0 = 0$.

CASE 1. *Semicircle.* Here $x_1 = r$; $\mathrm{H}_1 = 0$; and

$$x_{\mathrm{H}} = \frac{\frac{1}{2}\,p\,x_1^2}{p\,r} = \frac{r}{2} \text{......................(8.)}$$

CASE 2. *Segment.* Inclination at springing, i_1. Here $x_1 = r(1 - \cos i)$; $H_1 = p\,r \cdot \cos i$; and

$$x_H = \frac{\frac{1}{2}\,p\,x_1^2 + p\,r\,x_1 \cdot \cos i}{p\,r}$$

$$= r\,\left(\tfrac{1}{2}\,(1 - \cos i)^2 + \cos i\,(1 - \cos i)\right) = \frac{r}{2} \cdot \sin^2 i \ldots (9.)$$

Example II. *Semi-elliptic arch, under conjugate uniform vertical and horizontal pressures* (Art. 180). Let $a = x_1$ be the *rise*, or vertical semi-axis; $c\,a$ the horizontal semi-axis, or *half-span;* and let the origin of co-ordinates be at the crown. Then $p_y = c^2\,p_x$; and we have

$$H_0 = T_0 = a\,p_y = c^2\,a\,p_x = c\,P_x;\ x_H = \frac{a}{2} \ldots\ldots (10.)$$

Example III. *Semi-elliptic distorted arch, with conjugate uniform vertical and oblique pressures* (Art. 181). The vertical and conjugate semidiameters, or *rise* and *inclined half-span*, being denoted by a and $c\,a$ respectively, the equations 10 apply to this case also.

Example IV. *Hydrostatic arch* (Art. 183). The origin of co-ordinates being taken, as in the article referred to, at the point of the extrados vertically above the crown, we have $p_y = p_x = w\,x$,

$$H_0 = T_0 = w \cdot \frac{x_1^2 - x_0^2}{2};\ H_1 = 0;\ \text{and}$$

$$x_H = \frac{w \cdot \int_{x_0}^{x_1} x^2 \cdot d\,x}{H_0} = \frac{2}{3} \cdot \frac{x_1^3 - x_0^3}{x_1^2 - x_0^2} \ldots\ldots\ldots\ldots (11.)$$

Example V. *Geostatic arch, with horizontal or inclined extrados* (Art. 184). Here $p_x = w\,x \cdot \cos j$; $p_y = c^2\,p_x = c^2\,w\,x \cdot \cos j$; $H_0 = T_0 = c\,P_x = c^2\,w \cos j \cdot \frac{x_1^2 - x_0^2}{2}$; and consequently

$$x_H = \frac{2}{3} \cdot \frac{x_1^3 - x_0^3}{x_1^2 - x_0^2}, \ldots\ldots\ldots\ldots\ldots\ldots\ldots (12.)$$

as in the last example.

Example VI. *Semicircular arch with horizontal extrados.* In this case the angle of rupture i_0 is to be determined by means of equation 13 of Article 185; and thence, by equation 12 of the same Article, is to be found H_0. The springing being vertical, we have $i_1 = 90°$; $H_1 = 0$. Let the crown of the arch be taken as origin; then $x = r\,(1 - \cos i)$, $d\,x = r \cdot \sin i \cdot d\,i$, and equation 6 of the present Article becomes

$$x_{\mathrm{H}} = \frac{r^2}{\mathrm{H}_0} \cdot \int_{i0}^{90^\circ} p_y \sin i \, (1 - \cos i) \cdot d\, i; \ldots\ldots\ldots (13.)$$

Example VII. *Circular segmental arch with horizontal extrados.* Let i_1 be the inclination of the arch at the springing, P_1 the total vertical load; then

$$\mathrm{H}_1 = \mathrm{P}_1 \operatorname{cotan} i_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14.)$$

Let i_0 be determined as in the last example.

CASE 1. $i_0 >$ or $= i_1$. In this case $\mathrm{H}_0 = \mathrm{H}_1$, and the conjugate thrust is simply the single horizontal force H_1 at the point of springing.

CASE 2. $i_0 < i_1$. Find H_0 as in the last example, and let the origin of co-ordinates be at the crown; then

$$x_1 = r\,(1 - \cos i_1); \text{ and we have}$$

$$x_{\mathrm{H}} = \frac{1}{\mathrm{H}_0} \left\{ r^2 \int_{i0}^{i1} p_y \sin i\,(1 - \cos i) \cdot d\,i + r\,\mathrm{H}_1 (1 - \cos i_1) \right\} (15.)$$

188. **Approximate Hydrostatic and Geostatic Arches.**—The subject of elliptic functions is so seldom studied, and complete tables of them are so scarce, that it is useful to possess a method of finding the proper proportions of hydrostatic and geostatic arches (Articles 183, 184) to a degree of approximation sufficient for practical purposes, using algebraic functions alone.

Such a method is founded on the fact that a hydrostatic arch approaches nearly to the figure of a semi-elliptic arch of the same height, and having its maximum and minimum radii of curvature in the same *proportion.*

Let x_0, x_1, as in Article 183, be the depth of load of a hydrostatic arch at the crown and springing respectively; r_0, r_1, its radii of curvature at those points; $a = x_1 - x_0$, its rise; y_1 its half-span, given in Article 183 by means of elliptic functions.

Suppose a semi-elliptic arch to be drawn, having the same rise, a, with the hydrostatic arch; let r'_0, r'_1, be its radii of curvature at the crown and springing, whose *proportion to each other* is the same with that of the radii of the hydrostatic arch; that is to say,

$$\frac{r'_1}{r'_0} = \frac{r_1}{r_0} = \frac{x_0}{x_1}.$$

Let b be the half-span of this semi-ellipse. Then because the cubes of the semi-axes of an ellipse are to each other inversely as the radii of curvature at the respective extremities of the semi-axes, we have

$$b = a \cdot \sqrt[3]{\frac{r'_0}{r'_1}} = (x_1 - x_0) \cdot \sqrt[3]{\frac{x_1}{x_0}} \ldots\ldots\ldots\ldots (1.)$$

A rough approximation to the half-span of the hydrostatic arch is found by making $y_1 = b$; but this, in the cases which occur in practice, is too great by an excess which varies between $\frac{1}{15}$ and $\frac{1}{30}$, and is about $\frac{1}{20}$ on an average. Hence we may take, as a *first approximation* whose utmost error in practice is about $\frac{1}{60}$, and whose average error is about $\frac{1}{120}$, the following formula, giving the *half-span* in terms of the *depths of load* at the crown and springing:—

$$y_1 = \frac{19}{20}(x_1 - x_0) \cdot \sqrt[3]{\frac{x_1}{x_0}} \text{..................} (2.)$$

Suppose the *rise* a and *half-span* y_1 of a proposed hydrostatic arch to be given, and that it is required to find the depths of load; equation 2 gives us, as an approximation,

$$\frac{x_1}{x_0} = \left(\frac{20\, y_1}{19\, a}\right)^3;$$

and because $x_1 - x_0 = a$, we have

$$x_1 = a \cdot \frac{\left(\frac{20\, y_1}{19\, a}\right)^3}{\left(\frac{20\, y_1}{19\, a}\right)^3 - 1}; \quad x_0 = a \cdot \frac{1}{\left(\frac{20\, y_1}{19\, a}\right)^3 - 1} \text{..........} (3.)$$

A *closer approximation* is given by the equations

$$\left.\begin{aligned} y_1 &= b - \frac{b^2}{30\, a}; \\ b &= y_1 + \frac{y_1^2}{30\, a}; \\ x_1 = a \cdot \frac{b^3}{b^3 - a^3}&; \quad x_0 = a \cdot \frac{a^3}{b^3 - a^3}. \end{aligned}\right\} \text{.........} (4.)$$

A semicircular or semi-elliptic arch may have its conjugate thrust approximately determined, by considering it as an *approximate geostatic arch*, as follows:—

Let there be given, the half-span of the arch in question, horizontal or inclined, as the case may be, y_2, the depths of load at its crown and springing, x_0, x_1, and the vertical load at the springing, P_1. Determine, by equation 2 or equation 4, the span y_1 of a *hydrostatic arch* for the depths of load x_0, x_1, and let

$$\frac{y_2}{y_1} = c, \text{..............................} (5.)$$

be the ratio of the half-span of the actual arch to that of the hydrostatic arch.

The actual arch may now be conceived as an approximation to a geostatic arch, transformed from the hydrostatic arch by preserving its vertical ordinates and load, and altering its conjugate ordinates and thrust in the ratio c. The conjugate thrust of a hydrostatic arch being equal to the load, we have, as an approximation to the conjugate thrust of the given semi-elliptic or semicircular arch,

$$H_0 = c\,P_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.)$$

Section 3.—*On Frictional Stability.*

189. **Friction** is that force which acts between two bodies at their surface of contact, and in the direction of a tangent to that surface, so as to resist their sliding on each other, and which depends on the force with which the bodies are pressed together.

There is also a kind of resistance to the sliding of two bodies upon each other, which is independent of the force with which they are pressed together, and which is analogous to that kind of strength which resists the division of a solid body by *shearing*,—that is, by the sliding of one part upon another. This kind of resistance is called *adhesion.* It will not be considered in the present section.

Friction may act either as a means of giving stability to structures, as a means of transmitting motion in machines, or as a cause of loss of power in machines. In the present section it is to be considered in the first of those three capacities only.

190. **Law of Solid Friction.**—The following law respecting the friction of solid bodies has been ascertained by experiment :—

The friction which a given pair of solid bodies, with their surfaces in a given condition, are capable of exerting, is simply proportional to the force with which they are pressed together.

If the bodies be acted upon by a lateral force tending to make them slide on each other, then so long as the lateral force is not greater than the amount fixed by this law, the friction will be equal and opposite to it, and will balance it.

There is a limit to the exactness of the above law, when the pressure becomes so intense as to crush or grind the parts of the bodies at and near their surface of contact. At and beyond that limit the friction increases more rapidly than the pressure ; but that limit ought never to be attained in a structure.

From the law of friction it follows, that the friction between two bodies may be computed by multiplying the force with which they are pressed together by a constant co-efficient which is to be determined by experiment, and which depends on the nature of the bodies and the condition of their surfaces: that is to say, let N denote the pressure, f the *co-efficient of friction*, and F the force of friction, then

$$F = f N.$$

191. **Angle of Repose.**—Let A A, in fig. 93, represent any solid body, B B a portion of the surface of another body, with which A A is in contact throughout the plane surface of contact e E. Let $\overline{PC}$ represent the amount, direction, and position of the resultant of a force by which A A is urged *obliquely* towards B B, so that C is the *centre of pressure* of the surface of contact e E. (Art. 89.)

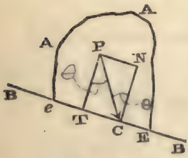

Fig. 93.

Let $\overline{PC}$ be resolved into two rectangular components: one, $\overline{NC}$, normal to the plane of contact, and pressing the bodies together: the other, $\overline{TC}$, tangential to the plane of contact, and tending to make the bodies slide on each other. Let the total force $\overline{PC}$, be denoted by P, its normal component by N, and its tangential component by T; and let the angle of obliquity T P C or P C N be denoted by θ, so that

$$\left.\begin{aligned} N &= P \cdot \cos\theta, \\ T &= P \cdot \sin\theta = N \cdot \tan\theta. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (1.)$$

Then so long as the tangential force T is not greater than fN, it will be balanced by the friction, which will be equal and opposite to it; but the friction cannot exceed fN; so that if T be greater than this limit, it will be no longer balanced by the friction, but will make the bodies slide on each other. Now the condition, that T shall not exceed fN, is equivalent to the condition, that $\frac{T}{N}$, or $\tan\theta$, shall not exceed f.

Hence it follows, *that the greatest angle of obliquity of pressure between two planes which is consistent with stability, is the angle whose tangent is the co-efficient of friction.*

This angle is called the *angle of repose*, and is denoted by φ. It is the steepest inclination of a plane to the horizon, at which a block of a given substance will remain in equilibrio upon it; for if P represents the weight of the body A A, so that P C is vertical, and $\theta = \varphi$, then φ is the inclination of B B to the horizon.

The relations between the friction, the normal pressure, and the total pressure, when the obliquity is equal to the angle of repose, are given by the following equations:—

$$F = T = fN = N \cdot \tan \varphi = P \cdot \sin \varphi = \frac{fP}{\sqrt{1+f^2}} \ldots (2.)$$

192. **Table of Co-efficients of Friction and Angles of Repose.**— Very extensive tables of the co-efficients of friction of different materials used in construction are published in the works of General Morin of the French Artillery, and have been reprinted in various treatises. The following is a *condensed table* compiled from General Morin's tables and from other authorities, giving those constants, and also the reciprocal, $\frac{1}{f} = \operatorname{cotan} \varphi$, for the materials of structures, arranged in a few comprehensive classes. Its practical utility is equal to that of the more voluminous and detailed tables from which it has been condensed:—

	f	φ	$\frac{1}{f}$
Dry masonry and brick-work,....................	0·6 to 0·7	31° to 35°	1·67 to 1·43
Masonry and brickwork, with damp mortar,.....	0·74	$36^{\circ}\frac{1}{2}$	1·35
Timber on stone,..........	about 0·4	22°	2·5
Iron on stone, ...	0·7 to 0·3	35° to $16^{\circ}\frac{2}{3}$	1·43 to 3·33
Timber on timber, ...	0·5 to 0·2	$26^{\circ}\frac{1}{2}$ to $11^{\circ}\frac{1}{3}$	2 to 5
Timber on metals,.........	0·6 to 0·2	31° to $11^{\circ}\frac{1}{3}$	1·67 to 5
Metals on metals,.........	0·25 to 0·15	14° to $8^{\circ}\frac{1}{2}$	4 to 6·67
Masonry on dry clay,....	0·51	27°	1·96
Masonry on moist clay,..	0·33	$18^{\circ}\frac{1}{4}$	3
Earth on earth,............	0·25 to 1·0	14° to 45°	4 to 1
Earth on earth, dry sand, clay, and mixed earth,	0·38 to 0·75	21° to 37°	2·63 to 1·33
Earth on earth, damp clay, ...	1·0	45°	1
Earth on earth, wet clay,	0·31	17°	3·23
Earth on earth, shingle and gravel,..............	0·81 to 1·11	39° to 48°	1·23 to 0·9

193. **Frictional Stability of Plane Joints.**—In a structure composed of a number of pieces connected only by touching each other at plane surfaces (as is the case in masonry and brickwork), it is necessary to stability that the obliquity of the pressure should at no joint exceed the angle of repose.

In structures of masonry, this condition can almost always be complied with by suitably placing the joints.

Both this and other principles depending on the effect of friction in promoting the stability of masonry, will be considered in subsequent sections.

194. **Frictional Stability of Earth.**[*]—A structure of earth, whether produced by excavation or by embankment, preserves its figure at first partly by means of the friction between its grains, and partly by means of their mutual cohesion or tenacity; which latter force is considerable in some kinds of earth, such as clay, especially when moist. It is by its tenacity that a bank of earth is enabled to stand with a vertical face, or even an overhanging face, for a few feet below its upper edge; whereas friction alone, as will afterwards appear, would make it assume an uniform slope.

But the tenacity of earth is gradually destroyed by the action of air and moisture, and of the changes of the weather; so that its friction is the only force which can be relied upon to produce permanent stability. In the present investigation, therefore, the stability of a mass of earth, or of shingle or gravel, or of any other material consisting of separate grains, will be treated as arising wholly from the mutual friction of those grains, and not from any adhesion amongst them.

Previous researches on this subject are based (so far as I am acquainted with them) on some mathematical artifice or assumption, such as Coulomb's "Wedge of Least Resistance." Researches so based, although leading to true solutions of many special problems, are both limited in the application of their results, and unsatisfactory in a scientific point of view. I propose, therefore, to investigate the mathematical theory of the frictional stability of a granular mass, without the aid of any artifice or assumption, and from the following sole

PRINCIPLE. *The resistance to displacement by sliding along a given plane in a loose granular mass, is equal to the normal pressure exerted between the parts of the mass on either side of that plane, multiplied by specific constant.*

The specific constant is the *co-efficient of friction* of the mass, and is the tangent of the *angle of repose*. Let p_n denote the normal pressure per unit of area of the plane in question; q the resistance to sliding (per unit of area also); φ the angle of repose; then the symbolical expression of the above principle is as follows:—

$$\frac{q}{p_n} = \tan \varphi \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.)$$

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[*] This and the ensuing Articles of the present section are to a great extent abridged from a paper "On the Stability of Loose Earth" in the *Philosophical Transactions* for 1856-7.

This principle forms the basis of every investigation of the stability of earth. The peculiarity of the present investigation consists in its deducing the laws of that stability from the above principle alone, without the aid of any other special principle. It will in some instances be necessary to refer to Mr. Moseley's "Principle of the Least Resistance;" but this must be regarded not as a special principle, but as a general principle of statics.

In a granular mass, any plane whatsoever may be considered as a *plane joint*, in the sense in which that term has been employed in Article 193; and hence, and from the principle already stated, follows,

THEOREM I. *It is necessary to the stability of a granular mass, that the direction of the pressure between the portions into which it is divided by any plane should not at any point make with the normal to that plane an angle exceeding the angle of repose.*

From what has been already proved, respecting internal stress, in Part I., Chap. V., Sect. 3, and especially in Articles 108 to 112 inclusive, it is evident, that the plane at any point in a mass, on which the obliquity of the pressure is greatest, is perpendicular to the plane which contains the axes of greatest and least pressure, —the pressure of greatest obliquity being parallel to that plane of greatest and least pressure.

The relations amongst the intensities of the pressures in a solid mass, which are parallel to one plane, as represented by the "Ellipse of Stress," have been investigated in Article 112. The present case, of a mass of earth, is one in which a limit to the *greatest obliquity* is assigned; viz., that it shall not exceed the angle of repose, φ. The relation between that greatest obliquity and the greatest and least pressures, has been found in Article 112, Problem III., Case 1, equation 6, viz.:—

$$\theta_1 = \text{arc} \cdot \sin \frac{p_1 - p_2}{p_1 + p_2};$$

p_1 being taken to represent the greatest, and p_2 the least pressure, and θ_1 the greatest obliquity of pressure. By Theorem I. we have

$$\theta_1 \leq \varphi;$$

(where $\leq$ means, "*less than or equal to;*" that is, "*not greater than*"). Hence follows the following equation:—

$$\frac{p_1 - p_2}{p_1 + p_2} = \sin \theta_1 \leq \sin \varphi; \quad\ldots\ldots\ldots\ldots\ldots\ldots(2.)$$

or in words,

THEOREM II. *At each point in a mass of earth, the ratio of the difference of the greatest and least pressures to their sum cannot exceed the sine of the angle of repose.*

Another symbolical expression of this Theorem is as follows:—

$$\frac{p_1}{p_2} \leqq \frac{1 + \sin \varphi}{1 - \sin \varphi} \ldots\ldots\ldots\ldots\ldots\ldots (2\text{A.})$$

When the directions of any pair of conjugate pressures in the plane of greatest and least pressure in a mass of earth are given, the limits of the ratio which the intensities of those pressures bear to each other are given by the solution of Problem V. of Article 112, equation 27. In that equation, make $\hat{n\,r} = \theta$, the common obliquity of the pair of conjugate pressures, and let θ_1 represent the greatest *actual* obliquity of pressure in the mass, which must not exceed φ; then p, as before, being the greater conjugate pressure, and p' the less, we obtain the following proposition:—

THEOREM III. *The following is the expression of the condition of the stability of a mass of earth, in terms of the ratio of a pair of conjugate pressures in the plane of greatest and least pressures:—*

$$\frac{p}{p'} = \frac{\cos \theta + \sqrt{(\cos^2 \theta - \cos^2 \theta_1)}}{\cos \theta - \sqrt{(\cos^2 \theta - \cos^2 \theta_1)}} \leqq \frac{\cos \theta + \sqrt{(\cos^2 \theta - \cos^2 \varphi)}}{\cos \theta - \sqrt{(\cos^2 \theta - \cos^2 \varphi)}} \ldots (3.)$$

195. **Mass of Earth with Plane Surface.**—Although the preceding principles can be applied to a mass of earth with a surface of any figure, their most useful application is to a mass bounded above by a plane surface, either horizontal or sloping. For such a mass, the three Theorems of Article 125 are true, and may be summed up as follows:—the pressure on a plane parallel to the upper plane surface (which may be called a *conjugate* plane) is vertical, and proportional to the depth:—the pressure on a vertical plane is parallel to the upper plane surface, and conjugate to the vertical pressure:—the state of stress at a given depth is uniform.

Let w be the weight of an unit of volume of the earth; x the depth of a given conjugate plane below the surface; θ the inclination of that conjugate plane; then the intensity of the vertical pressure on that conjugate plane is

$$p_x = w\,x \cdot \cos \theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

The *limits* of the intensity p_y of the conjugate pressure, parallel to the direction of steepest declivity (when the surface slopes) on a vertical plane, at the same depth x below the surface, are deduced from the equation 3 of Article 194, by considering, that this conjugate pressure may be either the greater or the less of the pair of pressures the limits of whose ratio are given by that equation; so that if we use the symbol

$$a \lesseqgtr b \pm c$$

to signify, "a is not greater than $b + c$, and not less than $b - c$," we obtain the following result:—

$$p_y \lesseqgtr w\,x \cdot \cos\theta \cdot \frac{\cos\theta \pm \sqrt{(\cos^2\theta - \cos^2\varphi)}}{\cos\theta \mp \sqrt{(\cos^2\theta - \cos^2\varphi)}} \ldots\ldots (2.)$$

When the plane surface is horizontal, so that $\cos\theta = 1$, equations **1** and 2 become

$$p_x = w\,x;\ p_y \lesseqgtr w\,x \cdot \frac{1 \pm \sin\varphi}{1 \mp \sin\varphi}; \ldots\ldots\ldots\ldots\ldots (3.)$$

as might have been inferred from Theorem II. of Article 194.

When $\theta = \varphi$, or *when the slope is the angle of repose*, the limits of the intensity of the conjugate pressure coincide, and it has but one value, viz.:—

$$p_y = w\,x \cdot \cos\varphi = p_x \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4.)$$

For all values of θ greater than φ, equation 2 becomes impossible; which shows what is otherwise evident, that the angle of repose is the steepest possible slope.

There is a third pressure which may be denoted by p_z, in a direction perpendicular to the first two, p_x and p_y; that is, horizontal, and perpendicular to the vertical plane in which the declivity is steepest; but the intensity of that third pressure will be considered in a subsequent Article. It is of secondary importance in practice, seeing that walls for the support of sloping banks of earth are generally placed so as to resist the pressure of the earth in the direction of steepest declivity.

With the exception of equation 4, the equations of the present Article give only the *limits* of the intensity of the conjugate pressure parallel to the steepest declivity. To find the exact intensity of that pressure, it is necessary to have recourse to a statical principle, first discovered by Mr. Moseley, which is stated in the following Article.

196. **Principle of Least Resistance.**—THEOREM. *If the forces which balance each other in or upon a given body or structure be distinguished into two systems, called respectively* active *and* passive, *which stand to each other in the relation of cause and effect, then will the passive forces be the least which are capable of balancing the active forces, consistently with the physical condition of the body or structure.*

For the passive forces being caused by the application of the active forces to the body or structure, will not increase after the active forces have been balanced by them; and will therefore not increase beyond the least amount capable of balancing the active forces.—Q. E. D.

197. **Earth Loaded with its own Weight.**—In a mass of earth loaded with its own weight only, the gravitation of the earth causes the vertical pressure, the vertical pressure causes a tendency to spread laterally, and the tendency to spread causes the conjugate pressure; therefore the vertical and conjugate pressures stand to each other in the relation of cause and effect, or active and passive respectively; therefore the intensity of the conjugate pressure is the least which is consistent with the conditions of stability given in Articles 194 and 195.

Applying this principle to the equations of Article 195, relative to a bank with a plane upper surface, they become the following:—

Vertical pressure (as before), $p_x = w\,x \cos\theta$ .... (1.)

Conjugate pressure parallel to steepest declivity :—
General case,

$$p_y = w\,x \cdot \cos\theta \, . \, \frac{\cos\theta - \surd\left(\cos^2\theta - \cos^2\varphi\right)}{\cos\theta + \surd\left(\cos^2\theta - \cos^2\varphi\right)}; \ldots\ldots (2.)$$

Horizontal surface, $\theta = 0$, $\cos\theta = 1$; $p_x = w\,x$;

$$p_y = w\,x \cdot \frac{1 - \sin\varphi}{1 + \sin\varphi}; \ldots\ldots\ldots\ldots\ldots\ldots (3.)$$

"Natural slope," $\theta = \varphi$,

$$p_y = p_x = w\,x \cdot \cos\varphi \ldots\ldots\ldots\ldots\ldots\ldots (4.)$$

The third pressure p_z is found in the following manner. Being perpendicular to the plane of p_x and p_y, it must be a *principal pressure* (Arts. 107, 109). Being a passive force, it must have the least intensity consistent with stability, and must therefore be equal to the least pressure in the plane of p_x and p_y.

The greatest and least stresses, or principal pressures, in that plane, are to be found by means of Problem IV. of Article 112, case 3, from the pair of conjugate pressures p_x, p_y, whose obliquity is θ. Let p_1 be the greatest, and p_2 the least principal pressure; then in equations 19 and 20 of Art. 112, for

$$p, p', \widehat{n\,r}, p_x, p_y,$$

we are to substitute respectively,

$$p_x, p_y, \theta, p_1, p_2,$$

giving the following results :—

$$\frac{p_1 + p_2}{2} = \frac{p_x + p_y}{2 \cos \theta} = \frac{w\, x \cdot \cos \theta}{\cos \theta + \sqrt{(\cos^2 \theta - \cos^2 \varphi)}} \ldots\ldots (5.)$$

$$\frac{p_1 - p_2}{2} = \sqrt{\left\{ \frac{(p_x + p_y)^2}{4 \cos^2 \theta} - p_x\, p_y \right\}} = \frac{w\, x \cdot \cos \theta \cdot \sin \varphi}{\cos \theta + \sqrt{(\cos^2 \theta - \cos^2 \varphi)}}; \; (6.)$$

and consequently,

$$\text{Greatest pressure, } p_1 = \frac{w\, x \cdot \cos \theta \cdot (1 + \sin \varphi)}{\cos \theta + \sqrt{(\cos^2 \theta - \cos^2 \varphi)}}; \ldots\ldots\ldots\ldots (7.)$$

$$\text{Least pressure, } p_2 = p_z = \frac{w\, x \cdot \cos \theta\, (1 - \sin \varphi)}{\cos \theta + \sqrt{(\cos^2 \theta - \cos^2 \varphi)}} \ldots\ldots\ldots (8.)$$

The axis of greatest pressure lies in the acute angle between the direction of greatest declivity and the vertical; and its inclination to the horizon, which may be denoted by ψ, is given by the following formula, deduced from equation 17 of Article 112, by making the proper substitutions:—

$$\cos 2\, \psi = \frac{2\, p_z \cos \theta - p_1 - p_2}{p_1 - p_2};$$

from which is easily deduced,

$$\psi = \frac{1}{2} \left\{ \theta + \text{arc} \cdot \sin \frac{\sin \theta}{\sin \varphi} \right\} \ldots\ldots\ldots\ldots\ldots (9.)$$

In using this formula, the arc sin $\frac{\sin \theta}{\sin \varphi}$ is to be taken as greater than a right angle.

The following are the results of the equations 7, 8, 9, for the extreme cases:—

$$\left.\begin{array}{c} \textit{Horizontal surface, } \theta = 0; \\ p_1 = w\, x = p_x; \\ p_2 = p_z = w\, x \cdot \dfrac{1 - \sin \varphi}{1 + \sin \varphi} = p_y; \\ \psi = 90^\circ \text{, or the axis of greatest pressure is vertical.} \end{array}\right\} (10.)$$

$$\left.\begin{array}{c} \textit{Natural Slope, } \theta = \varphi; \\ p_1 = w\, x\, (1 + \sin \varphi); \\ p_2 = p_z = w\, x\, (1 - \sin \varphi); \\ \psi = \dfrac{1}{2}\, (\theta + 90^\circ) \text{, or the axis of greatest pressure bisects} \\ \text{the angle between the slope and the vertical.} \end{array}\right\} (11.)$$

198. Pressure of Earth against a Vertical Plane.—In fig. 94, let O X represent a vertical plane in or in contact with a mass of earth whose upper surface Y O Y is either horizontal or inclined at any angle θ, and is cut by the vertical plane in a direction perpendicular to that of steepest declivity. It is required to find the pressure exerted by the earth against that vertical plane, *per unit of breadth*, from O down to X, at a depth $\overline{\mathrm{O\,X}} = x$ beneath the surface, and the direction and position of the resultant of that pressure.

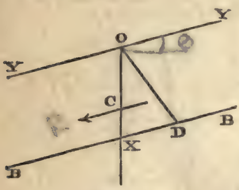

Fig. 94.

The *direction* of that resultant is already known to be parallel to the declivity Y O Y.

Let B B be a plane traversing X, parallel to Y O Y. In that plane take a point D, at such a distance X D from X, that the weight of a prism of earth of the length X D and having an *oblique* base of the area unity in the plane O X, shall represent the intensity of the conjugate pressure per unit of area of a vertical plane at the depth X. Draw the straight line O D; then will the ordinate, parallel to O Y, drawn from O X to O D at any depth, be the length of an oblique prism, whose weight, per unit of area of its oblique base, will be the intensity of the conjugate pressure at that depth. Let O D X be a triangular prism of earth of the thickness unity; the weight of that prism will be the *amount* of the conjugate pressure sought, and a line parallel to O Y, traversing its centre of gravity, and cutting O X in the *centre of pressure* C, will be the *position* of the resultant of that pressure. The depth $\overline{\mathrm{O\,C}}$ of that centre of pressure beneath the surface is evidently two-thirds of the total depth $\overline{\mathrm{O\,X}}$.

To express this symbolically, make

$$\overline{\mathrm{X\,D}} = \frac{p_y}{w \cdot \cos\theta} = x \cdot \frac{p_y}{p_x} = x \cdot \frac{\cos\theta - \sqrt{(\cos^2\theta - \cos^2\varphi)}}{\cos\theta + \sqrt{(\cos^2\theta - \cos^2\varphi)}}; \ldots (1).$$

(by equation 2 of Article 197);

en the amount of the conjugate pressure, or weight of the prism X D, is

$$\mathrm{P}_y = \int_0^x p_y \cdot dx = \frac{p_y}{p_x} \int_0^x p_x\, dx$$

$$= \frac{w\,x^2}{2} \cdot \cos\theta \cdot \frac{p_y}{p_x} = \frac{w\,x^2}{2} \cdot \cos\theta \cdot \frac{\cos\theta - \sqrt{(\cos^2\theta - \cos^2\varphi)}}{\cos\theta + \sqrt{(\cos^2\theta - \cos^2\varphi)}}; \ldots (2.)$$

and the centre of pressure is given by the equation

$$\overline{\mathrm{O\,C}} = \frac{2x}{3} \text{..........................} (3.)$$

In the extreme cases, equation 2 takes the following forms :—

For a horizontal surface; $\theta = 0$;

$$\mathrm{P}_y = \frac{w\,x^2}{2} \cdot \frac{1 - \sin\varphi}{1 + \sin\varphi} \text{......................} (4.)$$

For a surface sloping at the angle of repose; $\theta = \varphi$;

$$\mathrm{P}_y = \frac{w\,x^2}{2} \cdot \cos\varphi. \text{.......................} (5.)$$

The principles of this Article serve to determine the pressure of earth against retaining walls, as will afterwards be shown.

199. **Supporting Power of Earth Foundations.**—The two preceding Articles refer to the case in which the conjugate pressure at a given depth is caused solely by the vertical pressure due to the weight of earth above that point, and is therefore, in virtue of the "principle of least resistance," the least conjugate pressure consistent with the weight of the vertical column of earth in question.

But the conjugate pressure may be increased beyond that least amount, by the application of the pressure of an external body; for example, the weight of a building founded on the earth. In this case, the conjugate pressure will be the *least* which is consistent with the vertical pressure due to the weight of the *building;* and if that conjugate pressure does not exceed the *greatest* conjugate pressure consistent (according to equation 2, 3, or 4 of Article 195) with the weight of the *earth* above the same stratum on which the building rests, the mass of earth will be stable.

The most important case in practice is that in which the surface of the ground is horizontal; so that the intensity of the vertical pressure due to the weight of the earth is $w\,x$; x being the depth of the base of the foundation of the building below the surface of the earth.

In this case, the *greatest* horizontal pressure, at the depth x, consistent with stability, as given by equation 3 of Article 195, is as follows :—

$$p_x = w\,x \cdot \frac{1 + \sin\varphi}{1 - \sin\varphi} \text{..................} (1.)$$

The greatest intensity of vertical pressure consistent with this horizontal pressure is

$$p' = p_y \cdot \frac{1 + \sin \varphi}{1 - \sin \varphi} = w\,x \left(\frac{1 + \sin \varphi}{1 - \sin \varphi}\right)^2; \qquad (2.)$$

and this is *the greatest intensity of pressure, consistent with stability, of a building founded on a horizontal stratum of earth at the depth* x, *the angle of repose being* φ.

If A be the area of the foundation of the building, $w\,x\,A$ will be the weight of earth displaced by it; and if the pressure of the building on its base be uniformly distributed, $p'\,A$ will be the weight of the building; so that

$$\frac{p'}{w\,x} = \left(\frac{1 + \sin \varphi}{1 - \sin \varphi}\right)^2 \qquad (3.)$$

is the *limit of the ratio in which the weight of a building exceeds the weight of earth displaced by it,* when the pressure is uniformly distributed over the base.

If the pressure of the building be not uniformly distributed over the base, its *greatest* intensity must not exceed that given by equation 2, and its *least* intensity must not fall short of $w\,x$. This condition determines the greatest *inequality of distribution* of the pressure of a building which is consistent with the stability of a given kind of earth. The most useful and frequent example of this case is that in which the base is rectangular, and the intensity of the pressure increases at an uniform rate from one edge to the opposite edge of the rectangle, being an *uniformly varying stress* (Articles 91, 92, 94). In this case, let p_0 denote the mean intensity of the pressure of the building, b the breadth of its base in the direction along which the pressure varies, anc $c\,b$ the *utmost deviation of the centre of pressure of the base from it centre of figure,* consistent with the stability of the earth whic supports it; then

$$p_0 = \frac{p' + w\,x}{2} = w\,x \cdot \frac{1 + \sin^2 \varphi}{(1 - \sin \varphi)^2}; \qquad (4.$$

$$c = \frac{p' - w\,x}{6\,(p' + w\,x)} = \frac{\sin \varphi}{3\,(1 + \sin^2 \varphi)} \qquad (5$$

200. **Abutting Power of Earth.**—If a vertical plane surfac of some body which is pressed horizontally, such as a buttres or a retaining wall, abuts or presses horizontally against a horiz al layer of earth, of the depth x, the limit of the resistance w ch that layer is capable of opposing to the horizontal thrust o he vertical plane is determined by the *greatest* horizontal pre re consistent with the stability of [illegible] th. Hence the amou of

that horizontal resistance, per unit of horizontal breadth of the vertical abutting plane, is given by the equation

$$P_y = \frac{w x^2}{2} \cdot \frac{1 + \sin \varphi}{1 - \sin \varphi}.$$

[illegible]

The *centre of resistance* is at $\frac{2x}{3}$ below the surface of the earth.

201. **Table of Examples** of the results of the formulæ in Articles 197, 198, 199, and 200.

φ	0°	15°	30°	45°	60°
$\frac{90° - \varphi}{2}$	45°	$37°\frac{1}{2}$	30°	$22°\frac{1}{2}$	15°
$f = \tan \varphi$	0	0·268	0·577	1·000	1·732
$\frac{1}{f} = \text{cotan}\, \varphi$	∞	3·732	1·732	1·000	0·577
$\sin \varphi$	0	0·259	0·500	0·707	0·866
$\frac{1 - \sin \varphi}{1 + \sin \varphi}$	1	0·588	$0·33\dot{3}$	0·172	0·072
$\frac{1 + \sin \varphi}{1 - \sin \varphi}$	1	1·700	3·000	5·826	13·924
$\cos \varphi$	1	0·966	0·866	0·707	0·500
$\cos^2 \varphi$	1	0·933	0·750	0·500	0·250
$\left(\frac{1 - \sin \varphi}{1 + \sin \varphi}\right)^2$	1	0·346	0·111	0·0295	0·0052
$\left(\frac{1 + \sin \varphi}{1 - \sin \varphi}\right)^2$	1	2·890	9·000	33·94	193·8
$\frac{1 + \sin^2 \varphi}{(1 - \sin \varphi)^2}$	1	1·945	5·000	17·47	97·4
$\frac{\sin \varphi}{3(1 + \sin^2 \varphi)}$	0	0·081	$0·13\dot{3}$	0·157	0·165

REMARK. The column headed 0° is applicable to *liquids*.

202. **Frictional Tenacity or Bond of Masonry and Brickwork.**—The overlapping or breaking of the joints, commonly called the *bond,* in masonry and brickwork, has three objects—first, to distribute the vertical load which rests on each stone or brick over two or three of the stones or bricks of the course next below, and so to produce a more nearly uniform distribution of the load than would otherwise take place; secondly, to enable the structure to resist forces tending to break it by *shearing,* or sliding of one part on another, in a vertical plane; and thirdly, to enable it to resist forces tending to tear it asunder horizontally.

For masonry and brickwork laid either dry, or in common mortar which has not had time to acquire practically appreciable tenacity, the resistance to horizontal tension mentioned above as the third object of the bond, is due to the mutual friction of the overlapping portions of the beds or horizontal faces of the stones or bricks, and may be called "*frictional tenacity.*" The amount of the frictional tenacity at any horizontal joint is the product of the vertical load upon the portion of that joint where two blocks of stone or brick overlap each other, into the co-efficient of friction, which, as stated in the table of Article 192, is about 0·74.

Let fig. 94 A represent a portion of a wall with a horizontal top A; and let it be required to determine the frictional tenacity at a horizontal joint B, whose depth below A is x, the intensity of that tenacity per unit of area of a vertical plane at B, and the aggregate tenacity of the wall from A down to B, with which it is capable of resisting a force tending to tear it into two parts by separation at the serrated dark line which extends from A to B in the figure.

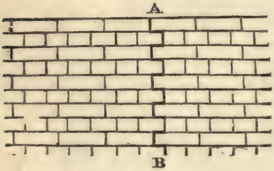

Fig. 94 A.

Let w be the weight of an unit of volume of the material of the wall; b the length of the overlap at each joint; t the thickness of the wall. Then

$$w\,b\,t\,x$$

is the vertical pressure on the overlapping portions of the stones or bricks at B, and consequently, if f be the co-efficient of friction, the amount of frictional tenacity for the joint B is

$$f\,w\,b\,t\,x. \dots\dots\dots\dots\dots\dots\dots\dots\dots (1.)$$

The intensity of that tenacity per unit of area of a vertical plane is found by dividing its amount by the area of a vertical section of one course of stones or bricks. Let h be the depth of a course; then $h\,t$ is the area of its vertical section; and the intensity of the frictional tenacity of the joint immediately below is

$$\frac{f\,w\,b\,x}{h} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.)$$

Let n be the number of courses from A down to B. Then the value of x for the uppermost course is $= h$, and for the lowest course, $= n\,h$; and the mean value of x is $\frac{n+1}{2} \cdot h$; so that the mean tenacity per course is

$$\frac{n+1}{2}\,f\,w\,b\,t\,h;$$

and the mean intensity,

$$\frac{n+1}{2} \cdot f\,w\,b.$$

Hence the amount of the aggregate frictional tenacity of the wall, from A down to B, is

$$n \cdot \frac{n+1}{2} \cdot f\,w\,b\,t\,h = \frac{f\,w\,b\,t\,(x^2 + h\,x)}{2\,h} \ldots\ldots\ldots (3.)$$

From the equations 2 and 3 it is obvious that the frictional tenacity of masonry and brickwork is increased by increasing the ratio $\frac{b}{h}$ which the length of the overlap bears to the depth of a course. This may be effected either by increasing the length of the stones or bricks (to which the overlap bears a definite proportion, depending on the style of bond adopted), or by diminishing their depth; but to both those expedients there is a limit fixed by the liability of stones and bricks to break across when the length exceeds the depth in more than a certain ratio, which for brick and stone of ordinary strength is about 3.

For *English bond* (as in fig. 94 A), consisting of a course of *stretchers* (or bricks laid lengthwise), and a course of *headers* (or bricks laid crosswise), alternately,—and also for *Flemish bond*, in which each course consists of alternate headers and stretchers, the overlap b is one-fourth of the length, or about three-fourths of the depth, of a brick. The value of $\frac{b}{h}$ is therefore $\frac{3}{4}$; but to allow for irregularities of figure and of laying in the bricks, it is safe to make it $\frac{2}{3}$ in the formulæ. Substituting this in equations 2 and 3, and making $f = \frac{3}{4}$, we find for the intensity of the frictional tenacity, *where one-half of the face of the wall consists of ends of headers*,

$$\frac{w\,x}{2};\ldots\ldots(4.)$$

and for the amount from the top of the wall down to the depth x,

$$\frac{wt\,(x^2 + hx)}{4}\ldots\ldots(5.)$$

The tenacity of the wall in the direction of its thickness, which resists the separation of its front and back portions by splitting, is often as important as its longitudinal tenacity, and sometimes more so. Where one-half of the face, as in fig. 94 A, consists of ends of headers, the overlap of each course in the direction of the thickness is generally one-half of the length of a brick instead of one quarter; so that $\frac{b}{h}$ is to be made $= \frac{4}{3}$ instead of two-thirds. Hence in this case, the *transverse frictional tenacity* (as it may be called) is *double* of the longitudinal frictional tenacity, its intensity at the depth x being

$$w\,x,\ldots\ldots(6.)$$

and its amount from the top of the wall down to the depth x, for a length of wall denoted by l,

$$\frac{w\,l\,(x^2 + h\,x)}{2}\ldots\ldots(7.)$$

In a brick wall consisting *entirely of stretchers*, as in fig. 94 B, the *longitudinal tenacity* is double of that of the wall in fig. 94 A, where one-half of the face consists of ends of headers. But that increased longitudinal tenacity is attained by a total sacrifice of transverse tenacity, when the wall is more than half a brick thick. In brickwork, therefore, in which the longitudinal is of more importance than the transverse tenacity (as is the case in furnace chimneys), a sufficient amount of transverse tenacity is to be preserved by having courses of headers at intervals. The effects of this arrangement are computed as follows:—

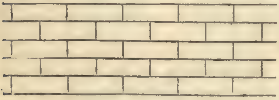

Fig. 94 B.

Let s be the number of courses of stretchers for each course of headers; so that $\frac{1}{s+1}$ of the face of the wall consists of ends of headers, and $\frac{s}{s+1}$ of sides of stretchers.

Let L denote the intensity of the longitudinal frictional tenacity, and T that of the transverse frictional tenacity, at the depth x. The following table represents the values of those intensities in the extreme cases :—

s	$\frac{1}{s+1}$	$\frac{s}{s+1}$	L	T
1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{w\,x}{2}$	$w\,x$
∞	0	1	$w\,x$	0

Now, in intermediate cases, the longitudinal tenacity will vary nearly as the proportion of sides of stretchers in the face of the wall $\frac{s}{s+1}$, and the transverse tenacity as the proportion of ends of headers; whence we have the following formulæ for the intensities :—

$$L = \frac{s}{s+1} \cdot w\,x; \text{.......................} (8.)$$

$$T = \frac{2}{s+1}\, w\,x \text{.......................} (9.)$$

Consequently, for the aggregate tenacities down to a given depth x, when the length of the wall is l, and its thickness t, we have

$$\textit{Longitudinal,}\ \frac{s}{4\,(s+1)} \cdot w\,t\,(x^2 + h\,x); \text{.........} (10.)$$

$$\textit{Transverse,}\ \frac{1}{2\,(s+1)} \cdot w\,l\,(x^2 + h\,x) \text{...........} (11.)$$

To make the longitudinal and transverse frictional tenacities of equal intensity, we should have $s = 2$, or two courses of stretchers for one course of headers. This makes

$$L = T = \frac{2\,w\,x}{3} \text{......................} (12.)$$

In round factory chimneys, it is usual to make $s = 4$; and then we have

$$L = \frac{4}{5} \cdot w\,x;\ T = \frac{2}{5} \cdot w\,x \text{...............} (13.)$$

The preceding formulæ are applicable not only to brickwork, but to ashler masonry in which the proportions of the dimensions of the stones are on an average nearly the same with those of bricks.

The formulæ 9 and 11 may also be used to find the *transverse tenacity of a rubble wall*, if $\frac{1}{s+1}$ be taken to represent the *proportion of the face of the wall which consists of the ends of squared headers or bond stones, connecting the front and back of the wall together.*

The principles of the present Article may be relied on as a means of *comparing* one piece of masonry or brickwork with another, so far as their security depends on the horizontal tenacity produced by the friction of the courses. But inasmuch as the *absolute numerical results* have been arrived at by an indirect process, from the tangent of the angle of repose of masonry and brickwork laid with damp mortar, these results are to be considered as uncertain, and as requiring direct experiments for their verification or correction. No such experiments have yet been made.

203. **Friction of Screws, Keys, and Wedges.**—The pieces of structures in timber and metal are often attached together by the aid of keys or wedges, or of screws. The stability of those fastenings arises from friction, and requires for its maintenance that the obliquity of the pressure between the wedge or key and its seat, or between the thread of the screw and that of its nut, shall not exceed the smallest value of the angle of repose of the materials.

204. **Friction of Rest and Friction of Motion.**—For some substances, especially those whose surfaces are sensibly indented by a moderate pressure, such as timber, the friction between a pair of surfaces which have remained for some time at rest, relatively to each other, is somewhat greater than that between the same pair of surfaces when sliding on each other. This excess, however, of the *friction of rest* over the *friction of motion*, is instantly destroyed by a slight vibration; so that the *friction of motion* is alone to be relied on as giving stability to a structure. In Article 192, accordingly, the co-efficients of friction and angles of repose in the table relate to the *friction of motion*, where there is any sensible difference between it and the *friction of rest.*

SECTION 4.—*On the Stability of Abutments and Vaults.*

205. **Stability at a Plane Joint.**—The present section relates to the stability of structures composed of blocks, such as stones or bricks, touching each other at joints, which are plane surfaces, capable of exerting pressure and friction, but not tension.

The conclusions of the present section are applicable to structures of masonry or brickwork, uncemented, or laid in ordinary mortar; for although ordinary mortar sometimes attains in the course of years a tenacity equal to that of limestone, yet, when fresh, its tenacity is too small to be relied on in practice as a means of resisting tension at the joints of the structure; so that a structure of masonry or brickwork, requiring, as it does, to possess stability while the mortar is fresh, ought to be designed on the supposition, that the joints have no appreciable tenacity. The mortar adds somewhat to the *frictional stability*, as has already been stated in the table of Article 192, and thus contributes indirectly to the *frictional tenacity* described in Article 202.

There are kinds of *cement* whose tenacity becomes at once equal to that of brick, or even to that of stone. So far as the joints are cemented with such kinds of cement, a structure is to be considered as *one piece*, and its safety is a question of strength.

A plane joint which has no tenacity is incapable of resisting any force, except a pressure, whose *centre of stress* falls within the joint, and whose obliquity does not exceed the angle of repose.

If the resistance of the material of the blocks which meet at the joint to a crushing force were infinitely great, it would be sufficient for stability that the centre of pressure should fall anywhere within the joint, how close soever to the edge; but for the actual materials of construction, it is necessary that the centre of pressure should not be so near the nearest edge of the joint as to produce a pressure at that edge sufficiently intense to injure the material. Hence it appears that the exact determination of the limiting position of the centre of pressure at a plane joint is, strictly speaking, a question relating to the strength of materials. Nevertheless, an approximation to that position can be deduced from an examination of the examples which occur in practice, without having recourse to an investigation founded on the theory of the strength of materials. Some of the most useful results of such an examination are expressed as follows:—

Let q denote the ratio which the distance of the *centre of pressure* of a given plane joint from its *centre of figure* bears to the diameter or breadth of the same joint, measured along the straight line which traverses its centre of pressure and centre of figure; so that if t be that diameter, $q\,t$ shall be the distance of the centre of pressure from the centre of figure. Then the ratio q is found in practice to have the following values:—

In *retaining walls* designed by British engineers,... $\frac{3}{8}$, or 0·375.

In *retaining walls* designed by French engineers,... $\frac{3}{10}$, or 0·3.

In the *abutments of arches*, in *piers* and *detached buttresses*, and in *towers* and *chimneys* exposed to the pressure of the wind, it has been found by experience to be advisable so to limit the deviation of the centre of pressure from the centre of figure, that the maximum intensity of the pressure, supposing it to be an *uniformly varying* pressure (see Article 94), shall not exceed the *double* of the mean intensity. As in Article 94, let P be the total pressure; S the area of the joint; let $\frac{P}{S} = p_0$ be the mean intensity of the pressure, which is also the intensity at the centre of figure of the joint, and at each point in a neutral axis traversing that centre of figure; let x be the perpendicular distance of any point from that axis, and let the pressure at that point be $p = p_0 + a\,x$, so that if x_1 be the greatest positive distance of a point at the edge of the joint from the neutral axis, the maximum pressure will be

$$p_1 = p_0 + a\,x_1.$$

Now, by the condition stated above, $p_1 = 2\,p_0$, and, consequently,

$$a = \frac{p_1 - p_0}{x_1} = \frac{p_0}{x_1} = \frac{P}{x_1 S} \text{.....................} (1.)$$

If the diameter of the joint is bisected by the centre of figure, and if x_0 (as in Article 94) be the distance of the centre of pressure from the neutral axis, we shall have

$$q = \frac{x_0}{2\,x_1};$$

$\frac{aI}{P}$

and by inserting in this equation the value of x_0, as given by equation 4 of Article 94, and having regard to the value of a, as given by equation 1 of this Article, we find

$$q = \frac{a\,I}{2\,P\,x_1} = \frac{I}{2\,S\,x_1^2}; \text{.....................} (2.)$$

an expression whose value depends wholly on the figure of the joint—that is, of the transverse section of the abutment, pier, buttress, tower, or chimney.

Referring to the table at the end of Article 95 for the values of the moment of inertia I, the following results are obtained for joints of different figures. In each case in which there is any difference in the values of q for different directions, the deviation of the centre of pressure is supposed to take place in that direction in which the *greatest* deviation is admissible—that is to say, at right angles to the neutral axis for which I is a maximum; so that if h be the diameter in that direction, $x_1 = \frac{h}{2}$.

FIGURE OF BASE.	I	S	q
I. Rectangle— Length, ...h Breadth,..................b	$\frac{h^3 b}{12}$	$h b$	$\frac{1}{6}$
II. Square— Side,.......................h	$\frac{h^4}{12}$	h^2	$\frac{1}{6}$
III. Ellipse— Longer axis, ...h Shorter axis, ...b	$\frac{\pi h^3 b}{64}$	$\frac{\pi h b}{4}$	$\frac{1}{8}$
IV. Circle— Diameter, ...h	$\frac{\pi h^4}{64}$	$\frac{\pi h^2}{4}$	$\frac{1}{8}$
V. Hollow rectangle— Outside dimensions,...h, b Inside dimensions,...h', b'	$\frac{h^3 b - h'^3 b'}{12}$	$h b - h' b'$	$\frac{h^3 b - h'^3 b'}{6 h^2 (h b - h' b')}$
VI. Hollow square— Outside dimensions, ...h Inside dimensions, ...h'	$\frac{h^4 - h'^4}{12}$	$h^2 - h'^2$	$\frac{h^2 + h'^2}{6 h^2}$
VII. Circular ring— Diameter, Outside, ...h Do. Inside,........h'	$\frac{\pi (h^4 - h'^4)}{64}$	$\frac{\pi (h^2 - h'^2)}{4}$	$\frac{h^2 + h'^2}{8 h^2}$

$\frac{\pi r^4}{4}$

When the solid parts of the hollow square and of the circular ring are very thin, the expressions for q in Examples VI. and VII. become approximately equal to the following :—

VIII. Hollow square,..$q = \frac{1}{3}$;

IX. Circular ring,..$q = \frac{1}{4}$;

which values are sufficiently accurate for practical purposes when applied to square and round factory chimneys.

The conditions of stability of a block supported upon another block at a plane joint may be thus summed up :—

Referring to fig. 93, Article 191, let A A represent the upper block, B B part of the lower block, e E the joint, C its centre of pressure, $\overline{\mathrm{P\,C}}$ the resultant of the whole pressure distributed over the joint, whether arising from the weight of the upper block, or from forces applied to it from without. Then the conditions of stability are the following :—

I. *The obliquity of the pressure must not exceed the angle of repose,* that is to say,

$$\angle \mathrm{PCN} \leqq \varphi \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.)$$

II. *The ratio which the deviation of the centre of pressure from the centre of figure of the joint bears to the length of the diameter of the joint traversing those two centres, must not exceed a certain fraction, whose value varies, according to circumstances, from one-eighth to three-eighths,* that is to say,

$$\frac{\frac{1}{2}\,\overline{e\mathrm{E}} - \overline{\mathrm{CE}}}{\overline{e\mathrm{E}}} \leqq q \ldots\ldots\ldots\ldots\ldots\ldots (4.)$$

The first of these conditions is called that of *stability of friction,* the second, that of *stability of position.*

206. **Stability of a Series of Blocks; Line of Resistance; Line of Pressures.**—In a structure composed of a series of blocks, or of a series of courses so bonded that each may be considered as one block, which blocks or courses press against each other at plane joints, the two conditions of stability must be fulfilled at each joint.

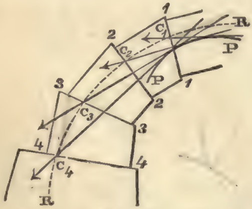

Fig. 95.

Let fig. 95 represent part of such a structure, 1, 1, 2, 2, 3, 3, 4, 4, being some of its plane joints.

Suppose the centre of pressure C_1 of the joint 1, 1, to be known, and also the amount and direction of the pressure, as indicated by the arrow traversing C_1. With that pressure combine the weight of the block 1, 2, 2, 1, together with any other external force which may act on that block; the resultant will be the total pressure to be resisted at the joint 2, 2, will be given in magnitude, direction, and position, and will intersect that joint in the centre of pressure C_2. By continuing this process there are found the centres of pressure C_3, C_4, &c., of any number of successive joints, and the directions and magnitudes of the resultant pressures acting at those joints.

The magnitude and position of the resultant pressure at any joint whatsoever, and consequently the centre of pressure at that joint, may also be found simply by taking the resultant of all the forces which act on one of the parts into which that joint divides the structure, precisely as in the "*method of sections*" already described in its application to framework, Article 161.

The centres of pressure at the joints are sometimes called *centres of resistance.* A line traversing all those centres of resistance, such as the dotted line R, R, in fig. 95, has received from Mr. Moseley the name of the "*line of resistance;*" and that author has also shown how in many cases the equation which expresses the form of that line may be determined, and applied to the solution of useful problems.

The straight lines representing the resultant pressures may be all parallel, or may all lie in the same straight line, or may all intersect in one point. The more common case, however, is that in which those straight lines intersect each other in a series of points, so as to form a polygon. A curve, such as P, P, in fig. 95, touching all the sides of that polygon, is called by Mr. Moseley the "*line of pressures.*"

The properties which the line of resistance and line of pressures must have, in order that the conditions of stability may be fulfilled, are the following :—

To insure stability of position, *the line of resistance must not deviate from the centre of figure of any joint by more than a certain fraction* (q) *of the diameter of the joint, measured in the direction of deviation.*

To insure stability of friction, *the normal to each joint must not make an angle greater than the angle of repose with a tangent to the line of pressures drawn through the centre of resistance of that joint.*

207. **Analogy of Blockwork and Framework.**—The point of intersection of the straight lines representing the resultant pressures at any two joints of a structure, whether composed of blocks or of bars, must be situated in the line of action of the resultant of the entire load of the part of the structure which lies between the two joints; and those three resultants must be proportional to the three sides of a triangle parallel to their directions.

Hence the polygon formed by the intersections of the lines representing the pressures at the successive joints in fig. 95, is analogous to a polygonal frame; for the sides of that polygon represent the directions of resistances, which sustain loads acting through its angles, as in the instances of framework described in Articles 150, 151, 153, and 154, and represented in fig. 75. A structure of blocks is especially analogous to an open polygonal frame, like those in Articles 151 and 154, represented by fig. 75, with the piece E omitted because of the absence of ties.

The question of the stability of a structure composed of blocks with plane joints may therefore be solved in the following manner :—

(1.) Determine and lay down on a drawing of the structure the line of action and the magnitude of the resultant of the external forces applied to each block, including its own weight. Either one or two of those resultants, as the case may be, will be the supporting force or forces.

(2.) Draw a *polygon of external forces*, like that in fig. 75* or 75**. Two contiguous sides of that polygon will represent the external forc acting on the two extreme blocks of the series, of which one may be a supporting pressure and the other a load, or both may be supporting pressures. In either case their intersection gives the point O, from which radiating lines are to be drawn to the angles of the polygon of external forces, to represent the directions and magnitudes of the resistances of the several joints.

(3.) Draw a polygon having its angles on the lines of action of the external forces, as laid down in step (1.) of the process, and its sides parallel to the radiating lines of step (2). This polygon will represent the *equivalent polygonal frame* of the given structure, and will have a side corresponding to each joint; and each side of the polygon (produced if necessary) will cut the corresponding plane joint in its *centre of pressure*, and will show the direction of the resultant pressure at the joint.

Then if each centre of pressure falls within the proper limits of position, and the direction of each resultant pressure within the proper limits of obliquity, as prescribed in Article 205, the structure will be balanced; and the conditions of stability will be fulfilled under variations of the distribution of the load, which will be the greater, the greater is the diameter of each joint; for every increase in the diameters of the joints increases the limits within which the figure of the equivalent polygonal frame may vary, and every variation of that figure corresponds to a variation in the distribution of the load.

208. **Transformation of Blockwork Structures.**—THEOREM. *If a structure composed of blocks have stability of position when acted on by forces represented by a given system of lines, then will a structure whose figure is a parallel projection of the original structure have stability of position when acted on by forces represented by the corresponding parallel projection of the original system of lines; also, the centres of pressure and the lines representing the resultant pressures at the joints of the new structure will be the corresponding projections of the centres of pressure and the lines representing the resultant pressures at the joints of the original structure.*

For the relative volumes, and consequently the relative weights, of the several blocks of which the structure is composed, are not altered by the transformation; and if those weights in the new structure be represented by lines, parallel projections of the lines representing the original lines, and if the other forces applied externally to the pieces of the new structure be represented by the corresponding parallel projections of the lines representing the corresponding forces applied to the pieces of the original structure, then will each external force acting on the new structure be the parallel projection of a force acting on the corresponding point of the original structure; therefore the resultant pressures at the joints of the new structure, which balance the external forces, will be represented by the parallel projections of the lines representing the resultant pressures at the corresponding joints in the original structure; therefore (Article 62, Proposition I.), the centres of pressure, where those resultants cut the joints, will divide the diameters of the joints in the same ratios in the new and in the original structures; therefore if the original structure have stability of position, the new structure will also have stability of position.

This is the extension, to a structure composed of blocks, of the *principle of the transformation of structures*, already proved for frames in Article 166, and for cords and linear arches in Article 177.

209. **Frictional Stability of a Transformed Structure.**—The question, whether the new structure obtained by transformation will possess *stability of friction*, is an independent problem, to be solved by determining the obliquity of each of the transformed pressures relatively to the joint at which it acts.

Should the pressure at any joint in the transformed structure prove to be too oblique, frictional stability can in most cases be secured, without appreciably affecting the stability of position, by altering the angular position of the joint, without shifting its centre of figure, until its plane lies sufficiently near to a normal to the pressure as originally determined.

210. **Structure not Laterally Pressed.**—If fig. 96 represents a structure consisting of a single series of blocks, or courses, separated by plane joints, and has no lateral pressure applied to it from without, then the centre of resistance at any one of those joints, such as D E, is simply the point C where that joint is intersected by a vertical let fall from the centre of gravity G of the part of the structure A B E D which lies above that joint; and the conditions of stability are,—that no joint shall be inclined to the horizon at an angle steeper than the angle of repose,—and that the point C shall not at any joint approach the edge of the joint within a distance bearing a certain proportion to the diameter of the joint.

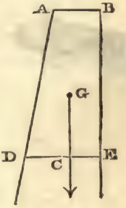

Fig. 96.

211. **The Moment of Stability** of a body or structure supported at a given plane joint is the moment of the couple of forces which must be applied in a given vertical plane to that body or structure in addition to its own weight, in order to transfer the centre of resistance of the joint to the limiting position consistent with stability. The applied couple usually consists of the thrust of a frame, or an arch, or the pressure of a fluid, or of a mass of earth, against the structure, together with the equal, opposite, and parallel, but not directly opposed, resistance of the joint to that lateral force.

The moment of stability may be different according to the position of the axis of the applied couple.

The moment of that couple is determined in the following manner :—

Conceive a line to pass through all the limiting positions of the centre of resistance of the joint, so as to enclose a space beyond which that centre must not be found.

The product of the weight of the structure into the horizontal distance of a point in this line from a vertical line traversing the centre of gravity of the structure is the MOMENT OF STABILITY *of the structure, when the applied thrust acts in a vertical plane parallel to that horizontal distance, and tends to overturn the structure in the direction of the given point in the line limiting the position of the centre of resistance;* for that, according to Article 41, is the moment of the couple, which, being combined with a single force equal to the weight of the structure, transfers the line of action of that force parallel to itself through a distance equal to the given horizontal distance of the centre of resistance from the centre of gravity of the structure.

To express this symbolically, let t be the length of the diameter of the joint where it is cut by the vertical plane traversing the centre of gravity of the structure and parallel to the applied thrust; let j be the inclination of that diameter to the horizon; let $q\,t$ be the distance of the given limiting centre of resistance from the middle point of that diameter, and $q'\,t$ the distance from the same middle point to the point where the diameter is cut by the vertical line through the centre of gravity of the structure, and let W be the weight of the structure. Then the moment of stability is

$$\mathrm{W}\,(q \pm q')\,t \cos j; \ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

the sign $\left\{ \begin{matrix} + \\ - \end{matrix} \right\}$ being used according as the centre of resistance, and the vertical line through the centre of gravity, lie towards $\left\{ \begin{matrix} \text{opposite sides} \\ \text{the same side} \end{matrix} \right\}$ of the middle of the diameter.

Let h denote the height of the structure above the middle of the plane joint which is its base, b the breadth of that joint in a direction perpendicular or conjugate to the diameter t, and w the weight of an unit of volume of the material. Then we shall have

$$\mathrm{W} = n \cdot w\,h\,b\,t, \ldots\ldots\ldots\ldots\ldots\ldots (2.)$$

where n is a *numerical factor* depending on the *figure* of the structure, and on the angles which the dimensions, h, b, t, make with each other; that is, the angles of obliquity of the co-ordinates to which the figure of the structure is referred. Introducing this value of the weight of the structure into the formula 1, we find the following value for the moment of stability :—

$$n\,(q \pm q')\cos j \cdot w \cdot h\,b\,t^2 \ldots\ldots\ldots\ldots\ldots\ldots(3.)$$

This quantity is divided by points into three factors, viz. :—

(1.) $n\,(q \pm q')\cos j$, a *numerical factor*, depending on the *figure* of the structure, the *obliquities* of its co-ordinates, and the *direction* in which the applied force tends to overturn it.

(2.) w, the specific gravity of the material.

(3.) $h\,b\,t^2$, a geometrical factor, depending on the dimensions of the structure.

Now the first factor is the same in all structures having figures of the same class, with co-ordinates of equal obliquity, and exposed to similarly applied external forces; that is say, to all structures whose figures, together with the lines of action of the applied forces, are *parallel projections of each other, with co-ordinates of equal obliquity;* hence for any set of structures which fulfil that condition, the moments of stability are proportional to

I. The specific gravity of the material;
II. The height;
III. The breadth;
IV. The *square* of the thickness; that is, of the dimension of the base which is parallel to the vertical plane of the applied force.

212. **Abutments Classed.**—In the title of the present section, the word "abutment" is used in an extended sense, to denote every structure, which by its stability of position and of friction, sustains some pressure which *abuts* or acts laterally against it. The structures comprehended under this definition may be classed as follows :—

I. *Buttresses,* which sustain the thrust of a frame or a rib, at one or more definite points.

II. *Towers and chimneys,* which sustain the lateral pressure of the wind, uniformly or almost uniformly distributed, and liable to act in every horizontal direction.

III. *Dams* for sustaining the lateral pressure of water, and *retaining walls* for sustaining that of earth—the intensity of the pressure being proportional to the depth beneath the surface.

IV. *Arch abutments,* which resemble both buttresses and retaining walls, and whose properties will be treated of after those of stone and brick arches shall have first been considered with reference to the stability at their joints.

213. **Buttresses in General.**—Let fig. 97 represent a vertical section of a buttress, against which a strut, rib, or piece of framework abuts at C, exerting a given force P in a given direction C A. In order that the buttress may be stable, it must fulfil

the conditions of stability at each of its bed-joints. Let D E be one of those joints.

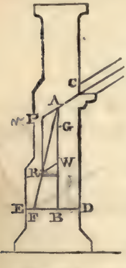

Fig. 97.

Should several pressures abut against the buttress, the force P acting in the line C A may be held to represent the resultant of all the forces which are applied above the particular joint D E under consideration.

Let G be the centre of gravity of that part of the buttress which is above the joint D E, and let W denote the weight of the same part. Through G draw the vertical line A G B, cutting the direction of the lateral thrust in A, and the joint D E in B; make $\overline{\mathrm{A\,W}} = \mathrm{W}$, $\overline{\mathrm{A\,P}} = \mathrm{P}$; complete the parallelogram A P R W; then $\overline{\mathrm{A\,R}}$ will represent the resultant of all the forces which act on the part of the buttress above the joint D E, and to which the resultant of the resistance at that joint must be equal and directly opposed. A R being produced, cuts D E in F, the centre of resistance of that joint, which must not fall beyond a certain prescribed limit, that the condition of stability of position may be fulfilled. In order that the condition of stability of friction may be fulfilled, the angle A F B must not be less than the complement of the angle of repose.

The most convenient mode of expressing this problem algebraically depends on the circumstances of the particular case. The following example is that which is most frequent and useful in practice; viz., when the inner face C D of the buttress is vertical, and the joint D E horizontal.

In this case, let the point of application of the lateral force, C, be taken for the origin of co-ordinates. Let

i denote the angle of inclination of the applied lateral pressure to the horizon;—

$x = \overline{\mathrm{C\,D}}$, the depth of the joint in question below C;—

$y_0 = \overline{\mathrm{B\,D}}$, the horizontal distance of the centre of gravity of the part of the buttress above that joint from the inner face;—

$y = \overline{\mathrm{D\,F}}$, the horizontal distance of the centre of resistance of the joint from its inner edge.

The resultant resistance, which acts through F in the direction F A, may be resolved into two components, respectively parallel, equal, and opposite to the weight W and applied force P. The couple of forces W is right-handed, and has the arm $\overline{\mathrm{F\,B}} = y - y_0$. The couple of forces P is left-handed, and has for its arm the perpendicular distance of F from the line of action C A of the applied force, viz.:—

$$x \cos i - y \sin i.$$

The former of those couples tends to maintain the stability of the buttress: the latter tends to overturn it. Equating their magnitudes, we obtain for the expression of the condition of stability of position the following :—

$$W(y-y_0) = P(x \cos i - y \sin i) \text{.............} (1.)$$

From this fundamental equation the solutions of various problems may be deduced, of which the following are examples :—

I. The buttress and the lateral force being given, to find the centre of resistance at a given joint.

$$y = \frac{W y_0 + P x \cos i}{W + P \sin i} \text{.....................} (2.)$$

This is the equation of the "line of resistance."

The condition of stability is expressed in terms of y thus—

$$y \leqq \left(q + \frac{1}{2}\right) t \text{........................} (3.)$$

II. The relation between the weight and the dimensions of the part of the buttress under consideration being given as in equations 2 and 3 of Article 211, it is required to find the least thickness at the joint D E consistent with stability.

For this purpose we must substitute for $W(y-y_0)$ in equation 1 of this Article its limit; that is to say, the *moment of stability*, as expressed in equation 3 of Article 211; and for y we must substitute its limiting value in terms of the thickness, as given by equation 3 of this Article. Thus we obtain the following equation :—

$$n(q+q')\,w\,h\,b\,t^2 = P\left(x \cos i - \left(q + \frac{1}{2}\right) t \sin i\right) \text{......} (4.)$$

To simplify the form of this quadratic equation, make

$$\frac{P x \cos i}{n(q+q')\,w\,h\,b} = A, \quad \frac{\left(q+\frac{1}{2}\right) P \sin i}{2\,n(q+q')\,w\,h\,b} = B;$$

then equation 4 becomes

$$t^2 = A - 2Bt,$$

the solution of which is

$$t = \sqrt{A + B^2} - B \text{.......................} (5.)$$

In detached buttresses, it is in general desirable to give q the value assigned by equation 2 of Article 205, for the reason there stated.

III. To find the obliquity of the pressure at the joint D E, we have the equation

$$\tan \angle \mathrm{FAB} = \frac{\mathrm{P} \cos i}{\mathrm{W} + \mathrm{P} \sin i} \ldots\ldots\ldots\ldots\ldots\ldots (6.)$$

As the resultant of the resistance at each joint must act in a line traversing the point A, the locus of that point is the "*line of pressures,*" defined in Article 206.

The greatest obliquity of pressure occurs at that joint which is immediately below the point of abutment C. Let W_0, therefore, denote the weight of material above that joint, and the condition of stability of friction will be given by the equation

$$\frac{\mathrm{P} \cos i}{\mathrm{W}_0 + \mathrm{P} \sin i} \leqq \tan \varphi. \ldots\ldots\ldots\ldots\ldots\ldots (7.)$$

214. **Rectangular Buttress.**—In a rectangular buttress, the breadth b and thickness t are constant; and if h_0 be taken to denote the height of the top of the buttress above the point C,

$$h = h_0 + x$$

will be its height above a given joint. Also, because the centre of gravity of the portion above any bed-joint is vertically above the centre of the joint, $q' = 0$, and $y_0 = \frac{1}{2} t$; and because

$$\mathrm{W} = w\, h\, b\, t,$$

$n = 1$.

These values being substituted in equations 2, 4, 5, and 7 of Article 213, give the following results:—

Equation of the line of resistance—

$$y = \frac{\frac{1}{2} w (h_0 + x)\, b\, t^2 + \mathrm{P}\, x \cos i}{w (h_0 + x)\, b\, t + \mathrm{P} \sin i} \ldots\ldots\ldots\ldots\ldots (1.)$$

The least thickness compatible with stability (x_1 being the depth of the base of the wall below C) is found by making

$$\mathrm{A} = \frac{\mathrm{P}\, x_1 \cos i}{q\, w (h_0 + x_1)\, b}; \quad \mathrm{B} = \frac{\left(q + \frac{1}{2}\right) \mathrm{P} \sin i}{2\, q\, w (h_0 + x_1)\, b}.$$

whence follows

Eq 1. A quadratic Solve for

$$t = \sqrt{A + B^2} - B = \sqrt{\left\{ \frac{P\, x_1 \cos i}{q\, w\, (h_0 + x_1)\, b} + \left(\frac{\left(q + \frac{1}{2}\right) P \sin i}{2\, q\, w\, (h_0 + x_1)\, b} \right)^2 \right\}} - \frac{\left(q + \frac{1}{2}\right) P \sin i}{2\, q\, w\, (h_0 + x_1)\, b} \ldots\ldots (2.)$$

The least volume of material above the level of the point C which is compatible with stability of friction, is given by making

$$\frac{P \cos i}{w\, h_0\, b\, t + P \sin i} = \tan \varphi,$$

Eq. 1, Art 2

that is to say,

$$h_0\, b\, t = \frac{P}{w} \left(\frac{\cos i}{\tan \varphi} - \sin i \right) = \frac{P}{w} \cdot \frac{\cos (\varphi + i)}{\sin \varphi} \ldots\ldots\ldots (3.)$$

The equation 1 of the line of resistance is that of a rectangular hyperbola traversing the point A (which is in this case invariable), and having a vertical asymptote, whose distance from the inner face of the buttress is

$$\frac{t}{2} + \frac{P \cos i}{w\, b\, t} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4.)$$

being the limit which y continually approaches, but never attains, as the depth x increases without limit.

As the depth x increases without limit, the thickness required for the wall approaches the following limit:—

$$t = \sqrt{\left(\frac{P \cos i}{q\, w\, b} \right)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5.)$$

which depends on the horizontal component of the lateral force alone.

Supposing this value to be adopted for the thickness of the buttress, in order that it may be stable, how deep soever the base may be below the point C,—then to insure stability of friction, the height of the top above C must have the following value:—

$$h_0 = q\, t \cdot \frac{\cos (\varphi + i)}{\sin \varphi \cos i} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.)$$

Instead of the rectangular mass $h_0\, b\, t$, there may be substituted a *pinnacle* of the same volume, and of any figure.

215. **Towers and Chimneys** are exposed to the lateral pressure of the wind, which, without sensible error in practice, may be assumed to be horizontal, and of uniform intensity at all heights above the ground.

The surface exposed to the pressure of the wind by such structures is usually either flat, or cylindrical, or conical, and differing very little from the cylindrical form. Octagonal chimneys, which are occasionally erected, may be treated as sensibly circular in plan. The inclination of the surface of a tower or chimney to the vertical is seldom sufficient to be worth taking into account in determining the pressure of the wind against it.

The greatest intensity of the pressure of the wind against a flat surface directly opposed to it hitherto observed in Britain, has been 55 lbs. per square foot; and this result, obtained by observations with anemometers, has been verified by the effects of certain violent storms in destroying factory chimneys and other structures.

In any other climate, before designing a structure intended to resist the lateral pressure of wind, the greatest intensity of that pressure should be ascertained, either by direct experiment, or by observation of the effects of the wind on previous structures.

The total pressure of the wind against the side of a cylinder is about one-half of the total pressure against a diametral plane of that cylinder.

Let fig. 98 represent a chimney, square or circular, and let it be required to determine the conditions of stability of a given bed-joint D E.

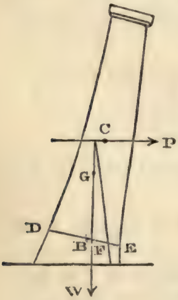

Fig. 98.

Let S denote the area of a diametral vertical section of the part of the chimney above the given joint, and p the greatest intensity of pressure of the wind against a flat surface. Then the total pressure of the wind against the chimney will be sensibly

$$\left.\begin{aligned} \mathrm{P} &= p\,\mathrm{S} \text{ for a square chimney;} \\ \mathrm{P} &= p\,\frac{\mathrm{S}}{2} \text{ for a round chimney;} \end{aligned}\right\} \ldots(1.)$$

and its resultant may, without appreciable error, be assumed to act in a horizontal line through the *centre of gravity of the vertical diametral section*, C. Let H denote the height of that centre above the joint D E; then the moment of the pressure is

$$\left.\begin{aligned} \mathrm{H\,P} &= \mathrm{H}\,p\,\mathrm{S} \text{ for a square chimney;} \\ \mathrm{H\,P} &= \frac{\mathrm{H}\,p\,\mathrm{S}}{2} \text{ for a round chimney;} \end{aligned}\right\} \ldots\ldots\ldots\ldots(2.)$$

and to this the *least moment of stability* of the portion of the chimney above the joint D E, as determined by the methods of Article 211, should be equal.

For a chimney whose axis is vertical, the moment of stability is the same in all directions. But few chimneys have their axes exactly vertical; and the least moment of stability is obviously that which opposes a lateral pressure acting in that direction toward which the chimney leans.

Let G be the *centre of gravity of the part of the chimney* which is above the joint D E, and B a point in the joint D E vertically below it; and let the line $\overline{\mathrm{D\,E}} = t$ represent the diameter of that joint which traverses the point B. Let q', as in former examples, represent the ratio which the deviation of B from the middle of the diameter D E bears to the length t of that diameter.

Let F be the limiting position of the centre of resistance of the joint D E, nearest the edge of that joint towards which the axis of the chimney leans, and let q, as before, denote the ratio which the deviation of that centre from the middle of the diameter D E bears to the length t of that diameter.

Then, as in equation 3 of Article 211, the least moment of stability is denoted by

$$\mathrm{W} \cdot \overline{\mathrm{B\,F}} = (q - q')\,\mathrm{W}\,t \ldots\ldots\ldots\ldots\ldots\ldots(3.)$$

The value of the co-efficient q is determined by considering the manner in which chimneys are observed to give way to the pressure of the wind. This is generally observed to commence by the opening of one of the bed-joints, such as D E, at the windward side of the chimney. A crack thus begins, which extends itself in a zig-zag form diagonally downwards along both sides of the chimney, tending to separate it into two parts, an upper leeward part, and a lower windward part, divided from each other by a fissure extending obliquely downwards from windward to leeward. The final destruction of the chimney takes place, either by the horizontal shifting of the upper division until it loses its support from below, or by the crushing of a portion of the brickwork at the leeward side, from the too great concentration of pressure on it, or by both those causes combined; and in either case the upper portion of the structure falls in a shower of fragments, partly into the interior of the portion left standing, and partly on the ground beside its base.

It is obvious that in order that the stability of a chimney may be secure, no bed-joint ought to tend to open at its windward edge; that is to say, there ought to be some pressure at every point of each bed-joint, except the extreme windward edge, where the intensity may diminish to nothing; and this condition is fulfilled with sufficient accuracy for practical purposes, by assuming the pressure to be an uniformly varying pressure, and so limiting the position of the centre of pressure F, that the intensity at the leeward edge E shall be double of the mean intensity.

It has already been shown, in Article 205, what values this condition assigns to the co-efficient q for different forms of the bed-joints. Chimneys in general consist of a hollow shell of brickwork, whose thickness is small as compared with its diameter; and in that case it is sufficiently accurate for practical purposes to give to q the following values :—

$$\left.\begin{array}{l} \text{For square chimneys, } q = \frac{1}{3}; \\ \text{For round chimneys, } q = \frac{1}{4}. \end{array}\right\} \dots\dots\dots\dots\dots (4.)$$

The following general equation, between the moment of stability and the moment of the external pressure, expresses the condition of stability of a chimney :—

$$\mathrm{H\,P} = (q - q')\,\mathrm{W}\,t \dots\dots\dots\dots\dots\dots\dots\dots (5.)$$

This becomes, when applied to square chimneys,

$$\mathrm{H}\,p\,\mathrm{S} = \left(\frac{1}{3} - q'\right)\mathrm{W}\,t;$$

and when applied to round chimneys,

$$\frac{\mathrm{H}\,p\,\mathrm{S}}{2} = \left(\frac{1}{4} - q'\right)\mathrm{W}\,t. \dots\dots\dots\dots (6.)$$

The following approximate formulæ, deduced from these equations, are useful in practice :—

Let B be the *mean* thickness of brickwork above the joint D E under consideration, and b the thickness to which that brickwork would be reduced, if it were spread out flat upon an area equal to the external area of the chimney. That reduced thickness is given with sufficient accuracy by the formula

$B - \frac{4B}{4t}$

$$b = \mathrm{B}\left(1 - \frac{\mathrm{B}}{t}\right) \dots\dots\dots\dots\dots\dots\dots\dots (7.)$$

but in most cases the difference between b and B may be neglected.

Let w be the weight of an unit of volume of brickwork; being, on an average, about 112 lbs. per cubic foot, or, if the bricks are dense, and laid very closely, with thin layers of mortar in the joints, from 115 to 120 lbs. per cubic foot. Then we have, very nearly,

$$\left.\begin{array}{l}\text{for square chimneys, } W = 4\,w\,b\,S;\\ \text{for round chimneys, } W = 3\cdot 14\,w\,b\,S;\end{array}\right\}\dots\dots\dots\dots(8.)$$

which values being substituted in the equation 6, give the following formulæ :—

$$\left.\begin{array}{l}\text{For square chimneys, } H\,p = \left(\frac{4}{3} - 4\,q'\right)\cdot w\,b\,t;\\[2ex] \text{For round chimneys, } H\,p = \left(1\cdot 57 - 6\cdot 28\,q'\right)w\,b\,t;\end{array}\right\}\dots\dots(9.)$$

These formulæ serve two purposes; first, when the greatest intensity of the pressure of the wind, p, and the external form and dimensions of a proposed chimney are given, to find the mean reduced thickness of brickwork, b, required above each bed-joint, in order to insure stability; and secondly, when the dimensions and form and the thickness of the brickwork of a chimney are given, to find the greatest intensity of pressure of wind which it will sustain with safety.

The shell of a chimney consists of a series of divisions, one above another, the thickness being uniform in each division, but diminishing upwards from division to division. The bed-joints between the divisions, where the thickness of brickwork changes (including the bed-joint at the base of the chimney), have obviously less stability than the intermediate bed-joints; hence it is only to the former set of joints that it is necessary to apply the formulæ. To illustrate the application of the formulæ, a table is given in the Appendix, showing the dimensions and figure, and the stability against the wind, of the great chimney of the works of Messrs. Tennant and Company, at St. Rollox, near Glasgow, which was erected from the designs of Messrs. Gordon and Hill, and is, with the exception of the spire of Strasburg, the Great Pyramid, and the spire of St. Stephen's at Vienna, the most lofty building in the world.

216. **Dams or Reservoir-Walls** of masonry are intended to resist the direct pressure of water. A dam, when a current of water falls over its upper edge, becomes a *weir*, and requires protection for its base against the undermining action of the falling stream. Such structures are not considered in the present Article, which is confined to walls for resisting the pressure of water only.

In fig. 99, let E D represent a horizontal bed-joint of a reservoir-wall, which wall has a plane surface O D exposed to the pressure of the contained water, whose upper surface is the horizontal plane OY. Consider a vertical layer of the wall of the length unity, sustaining the pressure of a vertical layer of water of the length unity also. Then from Articles 89 and 124 it appears, that the total pressure exerted against that layer of the wall is equal to the weight of the triangular prism of water O D K, right angled at D, whose thickness is unity, and whose side $\overline{DK}$ is equal to the depth of the joint DE beneath the surface OY; and it also appears, that the resultant of that pressure acts in the line H C, being a perpendicular upon O D from the centre of gravity H of the prism of water; so that $\overline{CD} = \frac{\overline{OD}}{3}$. Let G be the centre of gravity of the vertical layer of masonry above D E, and G B W a vertical line drawn through it; produce H C, cutting that vertical line in A; take $\overline{AW}$ to represent the weight of the layer of masonry, and $\overline{AP}$ to represent the pressure of the layer of water; complete the parallelogram A P R W; $\overline{AR}$ will represent the total pressure on the joint D E for each unit of length of the wall, and F, where that line cuts D E, will be the centre of resistance of that joint, which must fall within the limits consistent with stability of position, while at the same time the angle A F D must not be less than the complement of the angle of repose.

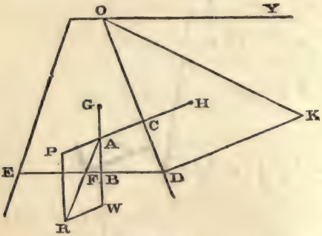

Fig. 99.

To treat this case algebraically, let x denote the depth of D beneath the surface of the water, w' the weight of an unit of volume of water, and j the inclination of O D to the vertical. Then the pressure of the vertical layer of water is

$OD = x \sec j$

vol $= x^2 \sec j$

$$P = \frac{w' x^2}{2} \cdot \sec j, \dots\dots\dots\dots\dots\dots\dots (1.)$$

its centre C being at the depth $\frac{2}{3} x$.

This force, together with the equal and opposite oblique component of the resistance of the joint D E at F, constitute a couple tending to overturn the wall, whose arm is the perpendicular distance of F from C P; that is to say,

$$\overline{CD} - \overline{FD} \cdot \sin j.$$

Now $\overline{\mathrm{C\,D}} = \frac{x \cdot \sec j}{3}$, and if, as before, we make $\overline{\mathrm{E\,D}} = t$, $\overline{\mathrm{F\,D}} = \left(q + \frac{1}{2}\right) t$; consequently we have for the arm of the couple in question,

$$\frac{x \cdot \sec j}{3} - \left(q + \frac{1}{2}\right) t \cdot \sin j,$$

which being multiplied by the pressure, gives the moment of the overturning couple; and this being made equal to moment of stability of the wall, we obtain the following equation :—

$$\mathrm{W} \cdot \overline{\mathrm{F\,B}} = \mathrm{W}(q \pm q')\,t = \frac{w' x^3}{6} \cdot \sec^2 j - w' x^2 t \left(\frac{q}{2} + \frac{1}{4}\right) \tan j \ldots (2.)$$

When the inner face of the wall is vertical, $\sec j = 1$, and $\tan j = 0$; and the above equation becomes

$$\mathrm{W}\,(q \pm q')\,t = \frac{w' x^3}{6} \ldots\ldots\ldots\ldots\ldots\ldots (2\ \text{A.})$$

To obtain a convenient general formula for comparing walls of similar figures but different dimensions, let n, as in Article 211, denote the ratio of the area of the vertical section of the wall to that of the circumscribed rectangle, so that if w be the weight of an unit of volume of masonry, the weight of the vertical layer of masonry under consideration is

$$\mathrm{W} = n\,w\,h\,t,$$

where h is the depth of the joint $\overline{\mathrm{D\,E}}$ below the top of the wall. Then equations 2 and 2 A take the following forms :—

$$n\,(q \pm q')\,w\,h\,t^2 = \frac{w' x^3}{6} \sec^2 j - w' x^2 t \left(\frac{q}{2} + \frac{1}{4}\right) \tan j ; \ldots\ldots (3.)$$

$$n\,(q \pm q')\,w\,h\,t^2 = \frac{w' x^3}{6} ; \ldots\ldots\ldots\ldots\ldots\ldots (3\ \text{A.})$$

—equations analogous to equation 4 of Article 213. To obtain a formula suitable for computing the requisite thickness of wall t, let

$$\frac{w' x^3 \cdot \sec^2 j}{6\,n\,(q \pm q')\,w\,h} = \mathrm{A};$$

$$\frac{w' x^2 \left(\frac{q}{2} + \frac{1}{4}\right) \tan j}{2\,n\,(q \pm q')\,w\,h} = \mathrm{B};$$

then

$$t^2 = \mathrm{A} - 2\,\mathrm{B}\,t;$$

which quadratic equation being solved, gives

$$t = \sqrt{\mathrm{A} + \mathrm{B}^2} - \mathrm{B}\,; \dots\dots (4.)$$

or for a wall with a vertical inner face, for which $\mathrm{B} = 0$,

$$t = \sqrt{\mathrm{A}}. \dots\dots (4\text{ A.})$$

In most cases which occur in practice, the surface of the water O Y either is, or may occasionally be, at or near the level of the top of the wall, so that h may be made $= x$. In such cases, let

$$\frac{\mathrm{A}}{x^2} = \frac{w' \sec^2 j}{6\,n\,(q \pm q')\,w} = a,$$

$$\frac{\mathrm{B}}{x} = \frac{w' \left(\frac{q}{2} + \frac{1}{4}\right) \tan j}{2\,n\,(q \pm q')\,w} = b,$$

and we have

$$\frac{t^2}{x^2} = a - 2\,b\,\frac{t}{x},$$

which being solved, gives

$$\frac{t}{x} = \sqrt{a + b^2} - b\,; \dots\dots (5.)$$

and for a wall with a vertical inner face,

$$\frac{t}{x} = \sqrt{a} = \sqrt{\left(\frac{w'}{6\,n\,(q \pm q')\,w}\right)} \dots\dots (5\text{ A.})$$

The vertical and horizontal components of the pressure of the water are respectively

$$\text{Vertical, } \mathrm{P} \sin j = \frac{w' x^2}{2} \tan j,$$

$$\text{Horizontal, } \mathrm{P} \cos j = \frac{w' x^2}{2};$$

Consequently the condition of *stability of friction* at the joint D E is given by the equation

$$\frac{\mathrm{P} \cos j}{\mathrm{W} + \mathrm{P} \sin j} = \frac{w' x^2}{2\,\mathrm{W} + w' x^2 \tan j} \leqq \tan \varphi \dots\dots (6.)$$

If the ratio $\frac{t}{x}$ has been determined by means of equation 5, then we have

$$\mathrm{W} = n\,w\,x\,t = n\,w\,x^2 \cdot \frac{t}{x};\ldots\ldots\ldots\ldots\ldots\ldots(7.)$$

so that by cancelling the common factor x^2, equation 6 is brought to the following form:—

$$\frac{w'}{2\,n\,w\,\frac{t}{x} + w' \tan j} \leqq \tan \varphi \ldots\ldots\ldots\ldots\ldots\ldots(8.)$$

Example I. *Rectangular Wall.*—In this case $n = 1$; $q' = 0$; $j = 0$; consequently,

$$a = \frac{w'}{6\,q\,w};\ b = 0;$$

equation 5 A becomes

$$\frac{t}{x} = \sqrt{a} = \sqrt{\frac{w'}{6\,q\,w}};\ldots\ldots\ldots\ldots\ldots\ldots(9.)$$

and equation 8,

$$\frac{w'}{2\,w\sqrt{\frac{w'}{6\,q\,w}}} = \sqrt{\frac{3\,q\,w'}{2\,w}} \leqq \tan \varphi;\ldots\ldots\ldots(10.)$$

but it is unnecessary to attend in practice to this last equation, which is fulfilled for the greatest values of q that ever occur.

Example II. *Triangular Wall*, with the apex at O.

In this case $\frac{t}{x}$ is the same for every horizontal joint; so that if the thickness be just sufficient for stability at any joint, it will be just sufficient for stability at every other joint. A reservoir-wall whose vertical section is triangular, may therefore be said to be *of uniform stability.*

The value of n for a triangle is $\frac{1}{2}$. With respect to the value of q', that case will be considered in which the inner face of the wall is vertical, so that $q' = \frac{1}{6}$, $j = 0$.

Then by equation 5 A,

$$\frac{t}{x} = \sqrt{a} = \sqrt{\left\{\frac{w'}{3\left(q+\frac{1}{6}\right)w}\right\}}\ ;\text{.........}(11.)$$

and by equation 8

$$\frac{w'}{w\frac{t}{x}} = \sqrt{\left(3\left(q+\frac{1}{6}\right)\frac{w'}{w}\right)} \leqq \tan\varphi\text{.........}(12.)$$

This last equation fixes a limit to the value of q, independently of the distribution of the pressure on each bed-joint, viz.:—

$$q \leqq \frac{w}{3w'} \cdot \tan^2\varphi - \frac{1}{6}\text{..................}(13.)$$

The insertion of this value of q in equation 11 gives

$$\frac{t}{x} = \frac{w'}{w \cdot \tan\varphi}\text{.....................}(14.)$$

The value of $\tan\varphi$ for *masonry* being about 0·74, w being on an average 144 lbs. and w' 62·4 lbs. per cubic foot, the limit of q is found to be

$$0{\cdot}421 - 0{\cdot}167 = 0{\cdot}254, \text{ or } \frac{1}{4}, \text{ nearly,}$$

and that of $\frac{t}{x}$, by equation 14, is

$$0{\cdot}585.$$

For *brickwork*, $\tan\varphi$ is about the same as for masonry, and w is 112 lbs. per foot, nearly; hence the limit of q is

$$0{\cdot}327 - 0{\cdot}167 = 0{\cdot}16, \text{ or } \frac{1}{6}, \text{ nearly,}$$

while that of $\frac{t}{x}$ is 0·75.

Example III. *Triangular Wall with Vertical Axis.*—When the wall stands on a soft foundation, it may be desirable in some cases so to form it, that the centre of resistance F shall be at the middle of each joint, and shall also be vertically beneath the centre of gravity of the part of the wall above the joint. In this case, the point of intersection A of the lines of action of the pressure and weight must also fall in the middle of each joint. To fulfil these conditions, the vertical section of the wall should be an isosceles triangle, the outer and inner faces forming equal angles j on opposite sides of the vertical axis of the wall, and the angle j should be such that a straight line perpendicular to O D at C shall bisect the base; that is to say,

$$\frac{t \sin j}{2} = \frac{x \sec j}{3};$$

but

$$\frac{t}{2x} = \tan j,$$

$\sin^2 + \cos^2 = 1$

whence we have

$\frac{t \sin j}{2x} = \frac{\sec j}{3}$

substitute $\tan j$ $= \frac{t}{2x}$ & reduce

$$\left.\begin{aligned} &\sin^2 j = \frac{1}{3}; \ \cos^2 j = \frac{2}{3}; \\ &\tan j = \frac{t}{2x} = \sqrt{\frac{1}{2}} = 0{\cdot}707; \\ \text{and} \quad &j = 35^\circ \frac{1}{4}; \end{aligned}\right\} \text{.........(15.)}$$

$\sqrt{\frac{1}{3}}$

so that the base of the wall is to its height as the diagonal to the side of a square.

Equation 8 in this case becomes

$$\frac{w'}{w\sqrt{2} + w'\sqrt{\frac{1}{2}}} = \frac{w'\sqrt{2}}{2w + w'} \leqq \tan \phi \text{ ....(16.)}$$

This condition is always fulfilled so far as the frictional stability of one course of masonry on another is concerned. As the object, however, of giving the wall the figure now in question, is to distribute the pressure uniformly over a soft foundation, let it be supposed that its base rests on a material for which $\tan \phi = \frac{1}{4}$. Then we must have

$$\frac{w'\sqrt{2}}{2w + w'} \leqq \frac{1}{4};$$

and consequently

$$w \geqq 2\left(\sqrt{2} - \frac{1}{4}\right) w' = 2{\cdot}33\, w' = 145 \text{ lbs. per cubic foot;}$$

and unless the masonry be of this weight per cubic foot, its friction on a horizontal base, of a material for which $\tan \phi = \frac{1}{4}$, will not be of itself sufficient to resist the thrust of the water.

217. **Retaining Walls.**—Figs. 100 and 101 represent vertical sections of retaining walls against which banks of earth abut. In

each figure a vertical layer of the masonry and of the earth is supposed to be considered, whose length is unity. D E is the base

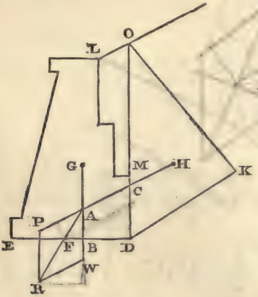

Fig. 100.

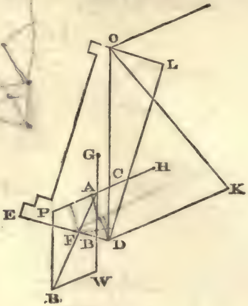

Fig. 101.

of the layer of masonry, F the centre of resistance of that base, B a point vertically below G, the centre of gravity of the weight which rests on that base, $\overline{AW}$ a line representing that weight, $\overline{AP}$ a line representing the thrust of the earth; $\overline{AR}$, the diagonal of the parallelogram A P R W, is a line representing the resultant pressure at the base D E, and cutting that base in the centre of resistance F.

In each figure, D O is a vertical plane traversing the inner edge D of the base of the wall, and cutting the plane of the surface of the bank in O. In fig. 100, the whole of the wall lies in front of that vertical plane; so that the weight, represented by $\overline{AW}$ (or by W simply), which rests on the base D E, consists of the weight of the masonry *together with the weight of the mass of earth, if any* (represented by O L M), *which is vertically above that base;* and G is the common centre of gravity of the compound mass of masonry and earth, which is situated in front of the plane O D.

In fig. 101, on the other hand, a part of the masonry, represented by D L O, lies *behind* the plane O D. If the prism D L O consisted of earth, its weight would be supported by the earth beneath it; therefore the earth beneath that prism exerts a pressure vertically upwards sufficient to sustain the weight of a prism of earth of a volume equal to that of the prism of masonry; therefore the weight represented by A W (or by W simply) which rests on the base D E, consists of the weight of the masonry in the vertical layer of the wall, *less* the weight of earth which would fill D L O; and G is the common centre of gravity of the masonry E D O which lies before the plane O D, and of the prism D L O, considered as having a specific gravity equal to the *excess of the specific gravity of masonry above that of earth.*

p. 219

It has already been shown in Article 198, that the pressure of the earth against the vertical plane O D (which pressure is parallel to the surface of the bank, and represented by A P, or by P simply), is equal to the weight of the prism of earth O D K, in which D K, parallel to the surface of the bank, is equal to the vertical depth O D multiplied by the ratio of the conjugate pressures at a point,

$$\frac{p_y}{p_x} = \frac{\cos\theta - \sqrt{(\cos^2\theta - \cos^2\varphi)}}{\cos\theta + \sqrt{(\cos^2\theta - \cos^2\varphi)}},$$

which ratio depends on the slope θ of the bank, and angle of repose φ, and that the resultant of that pressure traverses C, at the height

$$\overline{\mathrm{C\,D}} = \frac{x}{3}$$

above D. For the sake of brevity (w' being the weight of unity of volume of the earth), let

$$w' \cos\theta \frac{p_y}{p_x} = w_1; \qquad a$$

then equation 2 of Article 198 becomes

$$\mathrm{P} = \frac{w_1 x^2}{2} \text{..............................(1.)}$$

This force has to be multiplied, as in previous Articles, by the perpendicular distance of F from C P, to give the moment of the couple which tends to overturn the wall. Let t be the thickness $\overline{\mathrm{DE}}$, and i the angle of inclination of D E to the horizon; then the arm of the couple in question is

$$\left(\frac{x}{3} - \left(q + \frac{1}{2}\right) t \sin i\right) \cos\theta - \left(q + \frac{1}{2}\right) t \cdot \cos i \cdot \sin\theta$$

$$= \frac{x\cos\theta}{3} - \left(q + \frac{1}{2}\right) t \cdot \sin(\theta + i);$$

which being multiplied by the force P, and equated to the moment of stability of the weight which rests on the base D E, gives the following condition of stability of position:—

$$\mathrm{W}(q \pm q')\, t \cdot \cos i = \frac{w_1 x^3 \cos\theta}{6} - \frac{w_1 x^2 t}{2}\left(q + \frac{1}{2}\right) \sin(\theta + i) \text{...(2.)}$$

Now suppose (as in Article 211 and elsewhere) that W bears a definite ratio n to the weight $w\,x\,t \cdot \cos i$ of a rectangle of masonry whose height is $\overline{\mathrm{O\,D}} = x$, and its breadth the horizontal distance of E from O D, $t\cos i$; then the first side of equation 2, being the moment of stability, becomes as follows:—

$$n\,(q \pm q')\,w\,x\,t^2\cos^2 i.$$

Divide both sides of the equation by

$$n\,(q \pm q')\,w\,x^3\cos^2 i,$$

and for brevity's sake, let

$$\frac{w_1 \cdot \cos\theta}{6\,n\,(q \pm q')\,w\cos^2 i} = a;$$

$$\frac{w_1\left(q+\frac{1}{2}\right)\sin(\theta+i)}{4\,n\,(q \pm q')\,w\cos^2 i} = b;$$

then

$$\frac{t^2}{x^2} = a - 2\,b\,\frac{t}{x} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.)$$

and consequently

$$\frac{t}{x} = \sqrt{a+b^2} - b \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4.)$$

The inclination of the resultant A R to the vertical is given by the equation

$$\tan \angle \mathrm{W\,A\,R} = \frac{\mathrm{P}\cos\theta}{\mathrm{W}+\mathrm{P}\sin\theta} \ldots\ldots\ldots\ldots\ldots (5.)$$

When the base D E is horizontal, this should not exceed the tangent of the angle of repose. When that base is inclined at the angle i, the condition of frictional stability is thus expressed :—

$$\angle \mathrm{W\,A\,R} - i \leqq \varphi'; \ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.)$$

φ' being the angle of repose of the foundation of the wall.

The object of giving the base of the wall an inclined position is to diminish the obliquity of the pressure on it, and so to enable the condition of frictional stability to be fulfilled.

The values adopted for q in practice vary from $\frac{3}{10}$ to $\frac{3}{8}$.

218. **Rectangular Retaining Walls.**—In a vertical rectangular wall, $n = 1$, $q' = 0$, $i = 0$; so that, in equations 3 and 4 of Article 217,

$$\left.\begin{aligned} a &= \frac{w_1\cos\theta}{6\,q\,w}; \\ b &= \frac{w_1\left(q+\frac{1}{2}\right)\sin\theta}{4\,q\,w}. \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots (1.)$$

Example I. When the surface of the bank is horizontal, so that $\theta = 0$, then

$$w_1 = w' \frac{1 - \sin \varphi}{1 + \sin \varphi}, \; b = 0,$$

See a p 257

and

$$\frac{t}{x} = \sqrt{a} = \sqrt{\left\{ \frac{w'(1 - \sin \varphi)}{6\,q\,w\,(1 + \sin \varphi)} \right\}} \quad \dots\dots\dots (2.)$$

Also

$$W = w\,x^2 \cdot \frac{t}{x};$$

$P = \frac{w_1 x^2}{2}$

so that equation 5 of Article 217 becomes

$$\tan \angle WAR = \frac{P}{W} = \frac{w_1\,x^2}{2\,w\,x^2 \cdot \frac{t}{x}} = \frac{w_1\,x}{2\,w\,t}$$

$$= \sqrt{\left\{ \frac{3\,q\,w'(1 - \sin \varphi)}{2\,w\,(1 + \sin \varphi)} \right\}}$$

$$\leqq \tan \varphi' \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (3.)$$

If the material on which the wall rests is the same with that of the bank, we may assume $\varphi' = \varphi$; in which case, by squaring equation 3, and attending to the fact that [illegible]

$$\tan^2 \varphi = \frac{\sin^2 \varphi}{1 - \sin^2 \varphi} = \left(\frac{\sin \varphi}{1 - \sin \varphi} \right)^2 \cdot \frac{1 - \sin \varphi}{1 + \sin \varphi},$$

we obtain the equation

$$\frac{3\,q\,w'}{2\,w} \leqq \left(\frac{\sin \varphi}{1 - \sin \varphi} \right)^2 \dots\dots\dots\dots\dots\dots\dots (4.)$$

Assuming that the specific gravity of the earth is four-fifths of that of the masonry, or $\frac{w}{w'} = \frac{5}{4}$, we find that this equation is fulfilled for the ordinary value of q, $\frac{3}{8}$, so long as φ exceeds 27°.

Example II. When the surface of the bank slopes at the angle of repose φ, then $w_1 = w' \cos \varphi$, and

$w' \cos \theta$ [illegible] $= w_1$

$$a = \frac{w' \cos^2 \varphi}{6\,q\,w};$$

See Eq 1.

$$b = \frac{\left(q + \frac{1}{2}\right) w' \cos \varphi \sin \varphi}{4\,q\,w};$$

so that equation 4 of Article 217 becomes

$$\frac{t}{x} = \cos\varphi\left\{\sqrt{\left(\frac{w'}{6\,q\,w} + \frac{(q+\frac{1}{2})^2\,w'^2\sin^2\varphi}{16\,q^2\,w^2}\right)} - \frac{(q+\frac{1}{2})\,w'\sin\varphi}{4\,q\,w}\right\} \quad (5.)$$

219. **Trapezoidal Walls.**—In fig. 102, let E Q represent the vertical face of a rectangular wall, suited to sustain the thrust of a given bank, and let F be the centre of resistance of the base. Make $\overline{\mathrm{QN}} = 3\ \overline{\mathrm{EF}} = 3\left(\frac{1}{2} - q\right)t$; then the centre of gravity g of the triangular prism of masonry E Q N will be vertically above the centre of resistance F; therefore if that prism be removed, so as to reduce the cross section of the wall to a trapezoid with a sloping face E N, the position of the centre of resistance F will not be altered, and the wall will still fulfil the condition of stability of position, the thickness t being determined as for a rectangular wall. If $q = \frac{3}{8}$, the thickness of the wall at the summit will be $\frac{5}{8}$ of the thickness at the base. The face of the wall is said to *batter;* the rate of the batter being the ratio $\frac{\overline{\mathrm{QN}}}{\overline{\mathrm{EQ}}} = 3\left(\frac{1}{2} - q\right)\frac{t}{x}$.

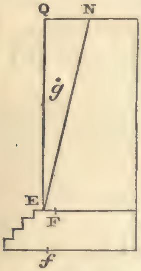

Fig. 102.

As the vertical component of the pressure on the base of the wall is diminished by this change, the obliquity of that pressure will be increased; and it may in some cases be necessary to make the base slope backwards, as in fig. 101.

220. **Battering Walls of Uniform Thickness.**—When a wall for supporting a *horizontal-topped* bank is of uniform thickness, and has a sloping or curved face, as in figs. 103 and 104, its moment of stability may be determined with a degree of accuracy sufficient for practical purposes, in the following manner:—

Let E Q in each figure represent the vertical face of a rectangular wall of the same height x and thickness t with the proposed wall, and let g be the centre of gravity of that rectangular wall. Then

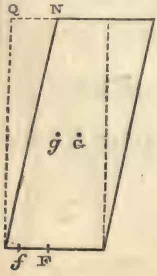

Fig. 103.

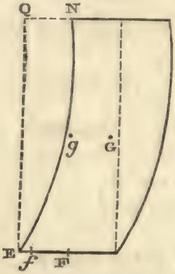

Fig. 104.

$$W \cdot q t = q w x t^2$$

will be its moment of stability per unit of length.

Divide the area E Q N included between the vertical face E Q and the face of the proposed wall, E N, by the height x. Then

$$q' t = \overline{g\,G} = \frac{\mathrm{E\,Q\,N}}{x}, \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.)$$

will be the distance of the centre of gravity G of the sloping or curved wall from that of the rectangular wall; and the change of figure will increase the stability in the ratio $q + q' : q$; that is to say, the moment of stability will now be

$$\mathrm{W}\,(q + q')\,t = (q + q')\,w\,x\,t^2 \ldots\ldots\ldots\ldots\ldots\ldots(2.)$$

If E N is a straight line (fig. 103),

$$q'\,t = \frac{\overline{\mathrm{Q\,N}}}{2}; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.)$$

If E N is a parabolic arc,

$$q'\,t = \frac{2\,\overline{\mathrm{Q\,N}}}{3}; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4.)$$

a formula which is also sensibly correct when E N is an arc of a circle.

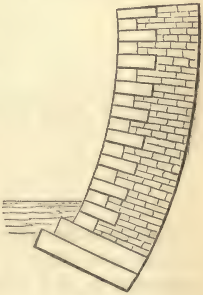

Fig. 105.

Walls with a "curved batter" are usually built as shown in fig. 105, with the bed-joints perpendicular to the face of the wall. This diminishes the obliquity of the pressure on the base.

221. **Foundation Courses of Retaining Walls** have their width increased beyond the thickness of the wall by a series of steps in front, as shown in figs. 102 and 105. The objects of this are, at once to distribute the pressure over a greater area than that of any bed-joint in the body of the wall, and to diffuse that pressure more equally, by bringing the centre of resistance nearer to the middle of the base than it is in the body of the wall. The power of earth to support foundations has already been considered in Article 199.

222. **Counterforts** are projections from the inner face of a retaining wall. A wall and its counterforts, if the bond of the masonry is well preserved, constitute a wall having successive divisions of its length alternately of greater and of less thickness. The moment of stability of a wall with counterforts, per unit of length,

when the wall is well bonded, may be found, with sufficient accuracy for practical purposes, by adding together the moments of stability of one of the parts between two counterforts, and of one of the parts whose thickness is augmented by the addition of a counterfort, and dividing the sum by the joint length of those two parts.

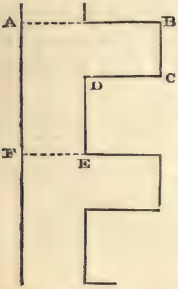

Fig. 106.

For example, let fig. 106 represent a portion of the *plan*, or horizontal section, of a vertical rectangular retaining wall whose height is h, with a row of rectangular counterforts of the same height with the wall. Let $t = \overline{\mathrm{F\,E}}$ be the thickness of a part of the wall between two counterforts, and $b = \overline{\mathrm{E\,D}}$ its length; let $\mathrm{T} = \overline{\mathrm{A\,B}}$ be the thickness of a counterforted part of the wall, including the counterfort, and $c = \overline{\mathrm{B\,C}}$ its length.

The moment of stability of the first part is

$$q\,w\,h\,b\,t^2;$$

and that of the second part,

$$q\,w\,h\,c\,\mathrm{T}^2.$$

Adding together those moments, and dividing their sum by the total length $b + c = \overline{\mathrm{A\,F}}$, the mean moment of stability per unit of length is found to be

$$q\,w\,h \cdot \frac{b\,t^2 + c\,\mathrm{T}^2}{b + c} \dots\dots\dots\dots\dots\dots (1.)$$

This is the same with the moment of stability per unit of length of a wall of the uniform thickness,

$$t_1 = \sqrt{\left\{\frac{b\,t^2 + c\,\mathrm{T}^2}{b + c}\right\}}, \dots\dots\dots\dots\dots\dots (2.)$$

which may be called the *equivalent uniform wall.*

The quantity of masonry in the counterforted wall is to the quantity in the equivalent uniform wall in the ratio

$$b\,t + c\,\mathrm{T} \ : \ (b + c)\,t_1,$$

which is always less than unity; so that there is a saving of masonry (though in general but a small one) by the use of counterforts.

223. **Arches of Masonry.**—An arch of masonry consists of a ring of wedge-formed stones, called *arch-stones* or *voussoirs,* pressing against each other at surfaces called *bed-joints,* which are, or ought

to be, perpendicular or nearly perpendicular to the *soffit*, or internal concave surface of the arch. The outer or convex surface of the ring of arch-stones, which may be either a curved surface parallel to the soffit, or, what is better, a series of steps, sustains the vertical pressure of that part of the load which arises from the weight of materials other than the arch-stones themselves; and that outer surface also exerts in many cases a horizontal or inclined thrust against the *spandrils* and *abutments.* The abutments sustain also the thrust of the lowest voussoirs, vertical or inclined, as the case may be. Sometimes an arch springs at once from the ground, so that its abutments are its foundations.

A wall standing on an arch, in the plane of the arch-ring, is called a *spandril wall.* The arch of a bridge requires a pair of *external spandril walls*, one over each face of the arch; the space between them is filled up to a certain level with solid masonry, and above that level it is sometimes filled with earth or rubbish, and sometimes occupied by a series of *internal spandril walls* parallel to the external spandril walls, and having vacant spaces between them—a mode of construction favourable both to stability and to lightness. In order to form a continuous platform for the roadway, the spaces between the internal spandril walls are sometimes covered with flags of some strong stone (such as slate), and sometimes arched over with small transverse arches. The external spandril walls are the abutments of those arches, and must have stability sufficient to sustain their thrust: when the spandrils are filled with earth or rubbish, the external spandril walls must have stability sufficient to withstand the pressure of the filling material.

In determining the conditions of stability of an arch, it is convenient to consider only a rib, or vertical layer, of arch, abutment, and spandril, of the thickness unity (*e. g.*, one foot). When there are spandril walls with vacant spaces between, an ideal specific gravity is to be adopted for the material of the spandrils, found by supposing the weight of the material of the spandril walls to be uniformly distributed, so as to fill the vacuities; that is to say, let w be the weight of an unit of volume of the material of the walls, $\Sigma \cdot \mathrm{T}$ the sum of the thicknesses of all the walls, and $\Sigma \cdot \mathrm{S}$ the sum of the widths of the spaces between them; then in computations respecting the stability of the arch, the spandrils may be supposed to be completely filled with a material whose weight per unit of volume is

$$w' = w \cdot \frac{\Sigma \cdot \mathrm{T}}{\Sigma \cdot \mathrm{T} + \Sigma \cdot \mathrm{S}} \dots\dots\dots\dots\dots\dots\dots (1.)$$

224. **Line of Pressures in an Arch; Condition of Stability.**—According to the principles explained in Articles 206 and 207, if a

straight line be drawn through each bed-joint of the arch-ring representing the position and direction of the resultant of the pressure at that joint, the straight lines so drawn form a polygon, and each of the angles of that polygon is situated in the line of action of the resultant external force acting on the arch-stone, which lies between the pair of joints to which the contiguous sides of the polygon correspond; so that the polygon is similar to a polygonal frame, loaded at its angles with the forces which act on the arch-stones (their own weight included). A curve inscribed in that polygon, so as to touch all its sides, is the *line of pressures* of the arch. The smaller and the more numerous the arch-stones into which the arch-ring is subdivided, the more nearly does the polygon coincide with the curve; and the curve or line of pressures represents an ideal *linear arch*, which would be balanced under the continuously-distributed forces which act on the real arch under consideration. From the near approach of this linear arch to the polygon whose sides traverse the centres of resistance of the bed-joints, the points where the linear arch cuts those joints may be taken without sensible error for the centres of resistance.

Now in order that the stability of the arch may be secure, it is necessary that no joint should tend to open either at its outer or at its inner edge; and in order that this may be the case, the centre of resistance of each joint should not deviate from the centre of the joint by more than one-sixth of the depth of the joint; that is to say, the centre of resistance should lie within the *middle third* of the depth of the joint; whence follows this

THEOREM. *The stability of an arch is secure, if a linear arch, balanced under the forces which act on the real arch, can be drawn within the middle third of the depth of the arch-ring.*

It has already been stated that the tenacity of fresh mortar is not sufficiently great to be taken into account in determining the stability of masonry; and hence, where cement is not used, all horizontal or oblique conjugate forces which maintain the equilibrium of the arch-ring must be pressures, acting on the arch from without inwards. The linear arch, therefore, is limited in such cases to those forms which are balanced under *pressures from without alone;* that is to say, that the intensity of the horizontal or conjugate pressure, denoted by p_y in Article 185, equation 4, must not at any point be negative.

It is true that arches have stood, and still stand, in which the centres of resistance of joints fall beyond the middle third of the depth of the arch-ring; but the stability of such arches is either now precarious, or must have been precarious while the mortar was fresh.

When tenacity to resist horizontal or oblique tension is given to the spandrils of an arch, and to the joints between them and the arch-stones, by means of cement, hoop-iron bond, iron cramps, or otherwise, the conjugate tension denoted by $-p_y$ must not at any point exceed a safe proportion of that tenacity; that is to say, about one-eighth. By this means stability may be given to arches of seemingly anomalous figures; but such structures are safe on a small scale only.

225. **Angle, Joint, and Point of Rupture.**—The first step towards determining whether a proposed arch will be stable, is to *assume* a linear arch parallel to the intrados or soffit of the proposed arch, and loaded vertically with the same weight, distributed in the same manner. The *size* of this assumed linear arch is a matter of indifference, provided each point in it is considered as subjected to the same forces which act at the *corresponding joint* in the real arch; that is, *the joint at which the inclination of the real arch to the horizon is the same with that of the assumed arch at the given point.*

The assumed arch is next to be treated as a stereostatic arch, according to the method of Article 185; and by equation 4 of that Article is to be determined, either a general expression or a series of values of the intensity p_y of the conjugate pressure, horizontal or oblique, as the case may be, required to keep the arch in equilibrio under the given vertical load. If that pressure is nowhere negative, a curve similar to the assumed arch, drawn through the middle of the arch-ring, will be either exactly or very nearly the line of pressures of the proposed arch; p_y will represent, either exactly or very nearly, the intensity of the lateral pressure which the real arch, tending to spread outwards under its load, will exert at each point against its spandril and abutments; and the thrust along the linear arch at each point will be the thrust of the real arch at the corresponding joint.

On the other hand, if p_y has some negative values for the assumed linear arch, there must be a pair of points in that arch where that quantity changes from positive to negative, and is equal to nothing. The angle of inclination i_0 at that point, called the *angle of rupture*, is to be determined by solving equation 1 of Article 187. The corresponding joints in the real arch are called the *joints of rupture;* and it is below those joints only that conjugate pressure from without is required to sustain the arch.

In fig. 107, let B C A represent one-half of a symmetrical arch, O Y a horizontal axis of co-ordinates in or above the spandril, K L D E an abutment, and C the joint of rupture, found by the method already described. The *point* of rupture, which is the centre of resistance of the joint of rupture, is somewhere within the middle third of the depth of that joint; and from that point down to the springing joint B, the line of pressures is a curve similar to the assumed linear arch, and parallel to the intrados, being kept in equilibrio by the lateral pressure between the arch and its spandril and abutment.

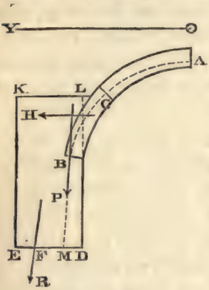

Fig. 107.

From the joint of rupture C to the crown A, the fact that the assumed linear arch would require lateral tension to keep it in equilibrio, shows that the true line of pressures must be a *flatter* curve than the assumed linear arch; the figure of the true line of pressures being determined by the condition, that it shall be a linear arch balanced under vertical forces only; that is to say, that the horizontal component of the thrust along it at each point is a constant quantity, and equal to the horizontal component of the thrust along the arch at the joint of rupture.

That horizontal thrust, denoted by H_0, is found by means of equation 2 of Article 187, and is the horizontal thrust of the entire arch.

[If the arch is distorted, *conjugate thrust* is to be read instead of "*horizontal thrust*," wherever that phrase occurs.]

The only point in the line of pressures above the joints of rupture which it is important to determine, is that which is at the crown of the arch, A; and it is found in the following manner:—

Find the centre of gravity of the load between the joint of rupture C and the crown A; and draw through that centre of gravity a vertical line.

Then if it be possible, from one point in that vertical line, to draw a pair of lines, one parallel to a tangent to the soffit at the joint of rupture, and the other parallel to a tangent to the soffit at the crown, so that the former of those lines shall cut the joint of rupture, and the latter the keystone, in a pair of points which are both within the middle third of the depth of the arch-ring, the stability of the arch will be secure; and if the first point be the point of rupture, the second will be the centre of resistance at the crown of the arch, and the crown of the true line of pressures.

When the pair of points related as above do not fall at opposite limits of the middle third of the arch-ring, their exact positions are to a small extent uncertain; but that uncertainty is of no consequence in practice. Their most probable positions are equi-distant from the middle line of the arch-ring.

Should the pair of points fall beyond the middle third of the arch-ring, the depth of the arch-stones must be increased.

226. **Thrust of an Arch of Masonry.**—The line of pressures, or *equivalent linear arch*, of an arch of masonry, with its point of rup-

ture and total thrust, having been determined by the methods described in the two preceding Articles, the distribution of that thrust, and the line of action of its resultant, are to be found by the methods of Article 187.

227. **Abutments of Arches.**—The abutment of an arch, when it is not simply a foundation, is a buttress, or a wall with or without counterforts, which is bounded, or may be considered as bounded by a vertical face L D (fig. 107) towards the arch.

Two external forces are applied to the abutment of an arch besides its own weight, viz., the vertical load of the half-arch, P, whose resultant acts through B, the centre of resistance of the springing joint, and the thrust H, found in amount and position by methods already referred to, which acts through B also if the angle of rupture is equal to or greater than the inclination of the arch at B; and which, if there is either no joint of rupture, or a joint of rupture above B, is distributed between B and A, or B and C, as the case may be. The resultant of the vertical load and conjugate thrust being taken as the entire pressure applied to the abutment, its conditions of stability and requisite dimensions are to be found by the methods described in Articles 213, 214, and 222.

For the abutment of an arch, as for the arch-ring, the centre of resistance should fall within the middle third of the base, so that the proper value of q is one-sixth.

If the figure of an arch be transformed by parallel projection, the proper figures for the abutments of the new arch are the corresponding parallel projections of the original abutments.

228. **Skew Arches** are of figures derived from those of symmetrical arches by distortion in a horizontal plane. The elevation of the face of a skew arch, and every vertical section parallel to its face, being similar to the corresponding elevation and vertical section of a symmetrical arch, the forces which act in a vertical layer or rib of a skew arch with its abutments, are the same with those which act in an equally thick vertical layer of a symmetrical arch with its abutments, of the same dimensions and figure, and similarly and equally loaded.

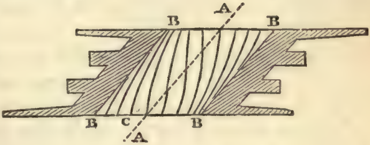

Fig. 108.

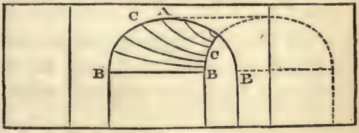

Fig. 109.

Fig. 108 represents a plan of a skew arch, with counterforted abutments. The *angle of skew*, or *obliquity*, is the angle which the axis of the archway, A A, makes with a perpendicular to the face of the arch, B C A B. The span of the archway, "*on the square*," as it is called (that is, the perpendicular distance between the abutments), is less than the span *on the skew*, or parallel to the face of the arch, in the ratio of the cosine of the obliquity to unity. It is the span *on the skew* which is equal to that of the corresponding symmetrical arch.

The best position for the bed-joints of the arch-stones is perpendicular to the thrust along the arch. If, therefore, there be drawn on the soffit of a skew arch, a series of parallel curves, made by the intersections of the soffit with vertical planes parallel to the face of the arch, the best forms for the bed-joints will be a series of curves drawn on the soffit of the arch so as to cut the whole of the former series of curves at right angles, such as C C in figs. 108 and 109. Joints of the best form being difficult to execute, spiral joints are used in practice as an approximation.

229. **Groined Vaults.**—A groined vault, represented in plan, looking upwards, by fig. 110, is formed by the intersection of two archways. The ribs at the edges where the soffits of the archways intersect and interrupt each other, are called the *groins*. The portions of the arches which form the groined vault, properly speaking, abut against the groins; the groins themselves, and the four independent portions of the archways, abut against four buttresses at the corners of the vault. The *crown* of the vault is the point where the groins meet.

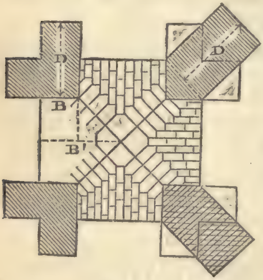

Fig. 110.

The line marked B′ is the length from the crown to the face of one of the archways; and B is the breadth of the portion of one of the buttresses against which that archway abuts, whether directly or through the groin. The thrust due to the length of archway B′ is concentrated upon the breadth of abutment B; its intensity is therefore increased in the ratio $\frac{B'}{B}$; and if t be the thickness which an abutment requires to withstand the thrust of the plain archway, the thickness D required for the buttress, in a direction perpendicular to B, will be

$$D = t\sqrt{\frac{B'}{B}} \dots\dots\dots\dots\dots\dots (1.)$$

At the left-hand side of the figure, the buttresses are compound and rectangular:—at the right-hand side, a single diagonal buttress is opposed to the thrust of each groin, and to the combined thrusts of the two archways which abut against it. The breadth of the diagonal buttress being the *resultant* of the breadths of the compound buttresses, its thickness is simply equal to theirs.

230. **Clustered Arches** are arched ribs, of which several spring from one buttress, as is shown in plan in fig. 111. The thrust against the buttress is the resultant of the thrusts of the ribs; the vertical pressure is the sum of their loads.

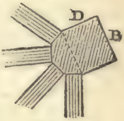

Fig. 111.

231. **Piers of Arches.**—A *pier* is a pillar against which two or more arches abut, in such a manner that their horizontal thrusts balance each other, so that the pier has only to sustain the vertical pressure of the half-arches which rest on it. The piers of a bridge or viaduct are usually oblong walls, of a length equal to that of the soffits of the arches, two of which spring from the opposite sides of each pier. It is customary to make the thickness of a pier, at the springing of the arches, from one-sixth to one-ninth of the span of the arches which it sustains. Mr. Hosking, in his *Treatise on Bridges*, has pointed out, that this thickness is usually greater than is necessary; and that there is in general no reason that the thickness of the pier should be more than is just sufficient to support the rings of arch-stones that spring from it.

If one of two arches which abut against the same pier falls, the other arch, having its thrust unbalanced, usually overthrows the pier, and consequently falls also; so that if a viaduct consists of a series of arches with piers between, the fall of a single arch causes the destruction of the whole viaduct. To lessen the damage caused by accidents of this kind, it is customary in long viaducts, to introduce at intervals what are called *abutment piers*, which have stability sufficient to resist the thrust of a single arch; so that when an arch falls, the destruction is limited to the division of the viaduct between the two nearest abutment piers.

In some important bridges over large rivers, where it has been considered advisable to spare no expense in order to render the structure durable, each pier is an abutment pier.

232. **Open and Hollow Piers and Abutments.**—In some cases the piers and abutments of bridges, in order to save materials, and to diminish the pressure on the foundations, are made with arched openings through them, or with rectangular hollows in their interior. The bottoms of such openings or hollows should be closed, when they are small by courses of large stones, and when they are large by inverted arches, in order that the area of the foundation, over which the pressure is distributed, may be as large as if the building were solid.

The moment of stability of an abutment, with arched openings through it, or hollows in its interior, is less than that of a solid abutment of the same external dimensions, very nearly in the same ratio in which the *moment of inertia of the horizontal section* of the abutment is diminished by means of the vacuities. (See Article 95.)

233. **Tunnels.**—If the depth of a tunnel beneath the surface of the ground is great compared with the height of its archway, the proper form for the line of pressures, which must lie within the middle third of the thickness of its arch, is the elliptic linear arch of Article 180, in which the ratio of the horizontal to the vertical semi-axis is the square root of the ratio of the horizontal to the vertical pressure of the earth, as already shown in Article 180, equation 5, and Article 197, equation 3; that is to say,

$$\frac{\text{horizontal semi-axis}}{\text{vertical semi-axis}} = c = \sqrt{\frac{p_y}{p_x}} = \sqrt{\left(\frac{1 - \sin \varphi}{1 + \sin \varphi}\right)};\ldots(1.)$$

φ being the angle of repose.

If the earth is firm, and little liable to be disturbed, the proportion of the half-span, or horizontal semi-axis, to the rise or vertical semi-axis, may be made *greater* than is given by the preceding equation, and the earth will still resist the additional horizontal thrust; but that proportion should never be made *less* than the value given by the equation, or the sides of the tunnel will be in danger of being forced inwards.

In a drainage tunnel, the entire ellipse may be used as the figure of the arch; but in a railway tunnel, where it is necessary to have a flat floor, the sides and roof of the tunnel comprise in height the upper two-thirds, or three-fourths, of the ellipse, which is closed below by a circular segmental inverted arch of a slight curvature, its depression being one-eighth of its span, or thereabouts. By this mode of construction, the vertical pressure of the sides of the tunnel is concentrated upon foundation courses directly below them, from which they spring. The ratio which the entire width of the tunnel, measured *outside* the masonry or brickwork, bears to the joint width of that pair of foundations, must not exceed the limit of the ratio of the weight of a building to the weight of earth displaced by it, as given by Article 199, equation 3. The inverted arch serves to prevent the foundations of the sides of the tunnel from being forced inwards by the horizontal pressure of the earth.

The *exact* form for the line of pressures in the sides and roof of a tunnel is the *geostatic arch* of Article 184. This principle requires attention when the roof of the tunnel is near the surface. Let x_0 be the depth of the crown of the tunnel, and x_1 that of its greatest horizontal diameter, beneath the surface. From those ordinates as data, design a *hydrostatic arch*, either by the exact method of Article 183, or by the approximate method of Article

188; contract the horizontal co-ordinates of that arch in the ratio $c = \sqrt{\frac{p_y}{p_x}}$, and the result will be the geostatic arch required.

234. **Domes.**—A true dome is a shell of masonry or brickwork, of the figure of a solid of revolution with a vertical axis; that is, it is spherical, spheroidal, conoidal, or conical, and is circular in plan. Its tendency to spread at its base is resisted by the stability of a cylindrical wall, or of a series of buttresses surrounding the base of the dome, or by the tenacity of a metal hoop encircling the base of the dome.

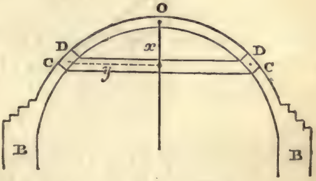

Fig. 112.

The conditions of stability of a dome are ascertained in the following manner:—Let fig. 112 represent a vertical section of a dome, springing from a cylindrical wall B B. The shell of the dome is supposed to be thin as compared with its external and internal dimensions. Let the centre of the crown of the dome, O, be taken as origin of co-ordinates; let x be the depth of any circular joint in the shell, such as C C, below O, and y the radius of that joint. Let i be the angle of inclination of the shell at C to the horizon, and $d\,s$ the length of an elementary arc of the vertical section of the dome, such as C D, whose vertical height is $d\,x$, and the difference of its lower and upper radii $d\,y$: so that

$$\frac{d\,y}{d\,x} = \text{cotan}\, i; \quad \frac{d\,s}{d\,x} = \text{cosec}\, i.$$

Let P_x be the weight of the part of the dome above the circular joint C C. Then the total thrust, in the direction of a set of tangents to the dome, radiating obliquely downwards all round the joint C C, is

$$P_x \cdot \frac{d\,s}{d\,x} = P_x \cdot \text{cosec}\, i;$$

and the total horizontal component of that radiating thrust is

$$P_x \cdot \frac{d\,y}{d\,x} = P_x \cdot \text{cotan}\, i.$$

Let p_y denote the intensity of that horizontal radiating thrust, per unit of periphery of the joint C C; then because the periphery of that joint is $2\,\pi\,y$ ($= 6{\cdot}2832\,y$), we have

$$p_y = \frac{P_x\, \text{cotan}\, i}{2\,\pi\,y} \quad \text{..........................(1.)}$$

It has been shown in Article 179, that if there be an inward radiating pressure upon a ring, of a given intensity per unit of arc, there is a thrust exerted all round that ring, whose amount is the product of that intensity into the radius of the ring. The same proposition is true, substituting an outward for an inward radiating pressure, and a tension all round the ring for a thrust. If, therefore, the horizontal radiating pressure of the dome at the joint C C be resisted by the tenacity of a hoop, the tension at each point of that hoop, being denoted by P_y, is given by the equation

$$P_y = y\, p_y = \frac{P_x \operatorname{cotan} i}{2\,\pi} \text{.....................} (2.)$$

Now conceive the hoop to be removed to the circular joint D D, distant by the arc $d\,s$ from C C, and let its tension in this new position be

$$P_y - d\,P_y.$$

The difference, $d\,P_y$, when the tension of the hoop at C C is the greater, represents a *thrust* which must be exerted all round the ring of brickwork C C D D, and whose *intensity per unit of length of the arc* C D is

$$p_z = \frac{d\,P_y}{d\,s} = \frac{1}{2\,\pi} \cdot \frac{d}{d\,s} (P_x \operatorname{cotan} i) \text{.............} (3.)$$

Every ring of brickwork for which p_z is either nothing, or positive, is stable, independently of the tenacity of cement; for in each such ring there is no tension in any direction.

When p_z becomes *negative*, that is, when P_y has passed its maximum, and begins to diminish, there is *tension* horizontally round each ring of brickwork, which, in order to secure the stability of the dome, must be resisted by the tenacity of cement, or of external hoops, or by the resistance of abutments.

Such is the condition of stability of a dome. The inclination to the horizon of the surface of the dome at the joint where $p_z = 0$, and below which that quantity becomes negative, is the *angle of rupture* of the dome; and the horizontal component of its thrust at that joint, is its total horizontal thrust against the abutment, hoop, or hoops, by which it is prevented from spreading.

A dome may have a circular opening in its crown. Oval arched openings may also be made at lower points, provided at such points there is no tension; and the ratio of the horizontal to the inclined axis of any such opening should be fixed by the equation

$$\frac{\text{horiz. axis}}{\text{inclined axis}} = c = \sqrt{\frac{p_z}{p_y \sec i}} \text{.............} (4.)$$

Example I. *Spherical Dome.*—Uniform thickness, t; weight of material per unit of volume, w; radius, r.

$$x = r(1 - \cos i);\ y = r \sin i;\ ds = r\,di.$$

$$\left.\begin{aligned} &P_x = 2\pi w t r^2 (1 - \cos i); \\ &p_y = \frac{w t r \cos i}{1 + \cos i};\ P_y = \frac{w t r^2 \cos i \sin i}{1 + \cos i}; \\ &p_z = \frac{d P_y}{r\,d i} = w t r \cdot \frac{\cos^2 i + \cos i - 1}{1 + \cos i}. \end{aligned}\right\} \text{........(5.)}$$

The angle of rupture, for which $p_z = 0$, is

$$i_0 = \text{arc} \cdot \cos \frac{\sqrt{5} - 1}{2} = 51^\circ\, 49';\ \text{...............(6.)}$$

and from this angle we obtain, for the horizontal thrust of the dome, per unit of periphery at the joint of rupture,

$$p_y = 0{\cdot}382\, w t r;$$

and for the tension on a hoop to resist that thrust, } ....(7.)

$$P_y = 0{\cdot}3\, w t r^2.$$

Example II. *Truncated Conical Dome* (fig. 113).—Apex, O. Depth of top of dome below apex, x_0; of base of dome, x_1; i, uniform inclination; t, uniform thickness; $y = x$ cotan i.

Then at the base of the dome,

$$\left.\begin{aligned} &P_x = \pi w t \cdot \frac{\cos i}{\sin^2 i} (x_1^2 - x_0^2); \\ &p_y = \frac{w t \cos i}{2 \sin^2 i} \left(x_1 - \frac{x_0^2}{x_1}\right); \\ &P_y = \frac{w t \cos^2 i}{2 \sin^3 i} (x_1^2 - x_0^2); \\ &p_z = w t x_1 \cdot \text{cotan}^2\, i. \end{aligned}\right\} \text{...(8.)}$$

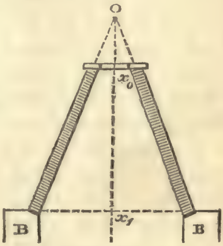

Fig. 113.

p_z being everywhere positive, there is in this dome no joint of rupture.

Example III. *Truncated Conical Dome, supporting on its summit a turret or "lantern," of the weight* L.

$$\left.\begin{aligned} \mathrm{P}_z &= \pi\, w\, t \cdot \frac{\cos i}{\sin^2 i}\,(x_1^2 - x_0^2) + \mathrm{L}\,; \\ p_y &= \frac{w\, t \cos i}{2 \sin^2 i}\left(x_1 - \frac{x_0^2}{x_1}\right) + \frac{\mathrm{L}}{2\,\pi\, x_1}\,; \\ \mathrm{P}_y &= \frac{w\, t \cos^2 i}{2 \sin^3 i}\,(x_1^2 - x_0^2) + \frac{\mathrm{L} \operatorname{cotan} i}{2\,\pi}\,; \\ p_z &= w\, t\, x_1 \cdot \operatorname{cotan}^2 i. \end{aligned}\right\} \text{........(9.)}$$

235. **Strength of Abutments and Vaults.**—The dimensions required in an abutment, arch, or dome, to insure stability, are in most cases sufficient to insure strength also; but instances occur, in which the condition of sufficient strength requires to be independently considered, and it may be convenient here so far to anticipate the subject of strength as to state that condition, viz., that the *intensity* of the thrust in the materials shall at no point exceed a certain limit, found by dividing the resistance of the material to crushing by a number called the *factor of safety.* The factor of safety in existing bridges ranges from 3 or 4 to 50 and upwards. In tunnels it is about 4. Tredgold considers, that in bridges the best value for the factor of safety is about 8 (*Treatise on Masonry*). The resistance of some of the most important materials of masonry to crushing is stated in a table at the end of this volume; but a prudent engineer, who contemplates a great work in masonry, will not trust to tables alone, but will ascertain the strength of the materials at his command by direct experiment.

235 A. **Transformation of Structures in Masonry.**—The principle already stated in Article 126, that to determine the *intensity* of a force in a transformed structure, the *projected line* representing the *amount* of the force must be divided by the *projected area* over which it is distributed, requires special attention in considering the strength of transformed structures of masonry.

To exemplify the application of that principle, conceive a rectangular prism whose dimensions are x, y, z, x being vertical: its volume is $\mathrm{V} = x\,y\,z$. Let w be the weight of unity of volume of the material of which it is composed; and let the weight of the prism be represented by a line parallel to x, of the length W; then

$$\mathrm{W} = w\,x\,y\,z \text{..............................(1.)}$$

The *amount* of an upward vertical pressure on the base of this prism, which balances W, will be represented by a line equal and opposite to W: that is

$$\mathrm{P} = -\,\mathrm{W}; \text{..............................(2.)}$$

and the *intensity* of that pressure will be

$$p = \frac{\mathrm{P}}{y z} = - w x \text{.........................} (3.)$$

Now let there be a parallel projection of this prism, whose dimensions, $x' = a x$, $y' = b y$, $z' = c z$, are oblique to each other. The weight of the new prism will be represented by a line parallel to x', of the length

$$\mathrm{W}' = a \mathrm{W} \text{.............................} (4.)$$

Let

$$\mathrm{C} = 1 - \cos^2 \widehat{y'z'} - \cos^2 \widehat{z'x'} - \cos^2 \widehat{x'y'}$$
$$+ 2 \cos \widehat{y'z'} \cdot \cos \widehat{z'x'} \cdot \cos \widehat{x'y'} \text{..................} (5.)$$

Then the volume of the new prism is

$$\mathrm{V}' = x' y' z' \sqrt{\mathrm{C}} = \mathrm{V} \cdot a b c \sqrt{\mathrm{C}}; \text{............} (6.)$$

consequently the *intensity of its weight* is

$$w' = \frac{\mathrm{W}'}{\mathrm{V}'} = \frac{a \mathrm{W}}{a b c \sqrt{\mathrm{C}} \cdot \mathrm{V}} = \frac{w}{b c \sqrt{\mathrm{C}}} \text{............} (7.)$$

The *area* of the lower surface of the new prism is

$$y' z' \cdot \sin \widehat{y'z'} = y z \cdot b c \sin \widehat{y'z'}; \text{...............} (8.)$$

The *amount* of the stress on that area is

$$- \mathrm{W}' = \mathrm{P}' = a \mathrm{P} = a p y z \text{.................} (9.)$$

being represented by a line P′, which is the projection of P, and parallel to x'.

The *intensity* of this new stress is

$$p' = \frac{\mathrm{P}'}{y' z' \cdot \sin \widehat{y'z'}} = \frac{a p}{b c \cdot \sin \widehat{y'z'}} \text{............} (10.)$$

and if we consider the relation between stress and weight,

$$\mathrm{P}' = - \mathrm{W}',$$

that is,

$$p' y' z' \sin \widehat{y'z'} = - w' x' y' z' \sqrt{\mathrm{C}} \text{.............} (11.)$$

we find

$$p' = \frac{- w' x' \sqrt{\mathrm{C}}}{\sin \widehat{y'z'}}. \text{.......................} (12.)$$

CHAPTER III.

STRENGTH AND STIFFNESS.

SECTION 1.—*Summary of the Theory of Elasticity as applied to Strength and Stiffness.*

236. **The Theory of Elasticity** relates to the laws which connect the stresses, or pressures and tensions, which act at the surface and in the interior of a body, with the alterations of dimensions and figure which the body and its parts simultaneously undergo. That theory, therefore, is the foundation of the principles of the strength and stiffness of materials of construction. The theory of elasticity has many other applications,—to crystallography, to light, to sound, to heat, and to other branches of physics. Its full discussion would of itself require a voluminous work; in the present section, its principles are to be briefly summed in so far as they are applicable to the strength and stiffness of structures.

237. **Elasticity** is the property which bodies possess of occupying, and tending to occupy, portions of space of determinate volume and figure, at given pressures and temperatures, and which, in a homogeneous body, manifests itself equally in every part of appreciable magnitude.

238. An **Elastic Force** is a force exerted between two bodies at their surface of contact, or between two parts into which a body either is divided or is capable of being divided at the surface of actual or ideal separation between those parts. The intensity of an elastic force is stated in *units of weight* per *unit of area* of the surface at which it acts. That kind of force is in fact identical with *stress*, the statical laws of which have already been explained in Part I., Chapter V., Sections 2, 3, and 4, Articles 86 to 126.

239. **Fluid Elasticity.**—The elasticity of a *perfect fluid* is such that its parts resist change of volume only, and not change of figure; whence it follows, that the pressure exerted by a perfectly fluid mass is wholly perpendicular to its surface at every point: principles which form the basis of hydrostatics and hydrodynamics. Fluids are either gaseous or liquid. A *gaseous fluid* is one whose parts (so far as is known by experiment) exert a pressure against

each other and against the vessel containing them, how great soever the volume to which they are expanded. See Arts. 110, and 117 to 124.

240. **Liquid Elasticity.**—The elasticity of a *perfect liquid* resists change of volume only, and differs from that of a gaseous fluid chiefly in this: that the greatest variations of the pressure which it is possible to apply to a liquid mass produce very small variations of its volume.

The *compression* undergone by a liquid mass in consequence of the application of a given pressure over its surface, is measured by the ratio of the diminution of volume produced by the given pressure to the entire volume of the mass: a ratio which is always a very small fraction. The *compressibility* of a given liquid is the compression produced by a unit of elastic pressure; in other words, the ratio of a compression to the pressure producing it. The *modulus* or *co-efficient of elasticity* of a liquid is the ratio of a pressure applied to and exerted by the liquid, to the accompanying compression, and is therefore the reciprocal of the compressibility. The following empirical formula for the compressibility of pure water at any temperature between 32° and 128° Fahrenheit has been deduced from the experiments of M. Grassi (*Comptes Rendus, XIX.; Philos. Mag.*, June, 1851).—*Compressibility per Atmosphere,*

$$= \frac{1}{40\,(\mathrm{T} + 461^\circ) \cdot \mathrm{D}}.$$

T, temperature in degrees of Fahrenheit. D, density of water at that temperature under one atmosphere, the maximum density of water under the pressure of one atmosphere being taken as unity. See Art. 123, equation 5. At the temperature of maximum density, 39·1 Fah., the compressibility of water per atmosphere is 0·00005, and its modulus of elasticity, 20,000 atmospheres, or 294,000 lbs. per square inch.

Compressibilities of some Liquids, per Atmosphere, from M. Grassi's experiments.

Saturated aqueous solution of nitrate of potash,	0·0000306565
Saturated aqueous solution of carbonate of potash,	0·0000303294
Artificial sea water,	0·0000445029
Saturated aqueous solution of chloride of calcium,	0·0000209830
Æther,	0·00011137 to 0·00013073
Alcohol,	0·00008245 to 0·00008587

The compressibility of æther and alcohol increases with the pressure.

241. **Rigidity or Stiffness.**—A *solid* body, besides resisting change of volume like a liquid, possesses also *rigidity*, or the property of

resisting change of figure. As in the case of liquids, the utmost alteration of volume of which a solid body is capable by any pressure which can be applied to it, is always a very small fraction of its entire volume. The stresses at the surface of a solid body or particle are not necessarily normal, but may have any direction, from normal to tangential.

242. **Strain and Fracture.**—In popular language the words *strain* and *stress* are applied indifferently to denote either the system of forces at the surface of a solid body whereby its volume and figure are altered, or the alteration of volume and figure of the body and its parts thereby produced. For the sake of clearness in scientific language, certain authors have recently endeavoured to appropriate the word *strain* to the alterations, of what nature soever, in the volume and figure of a solid body and of its parts, produced by forces applied to it, and the word *stress* as formerly defined. This nomenclature will be used in the present treatise. *Fracture* of a solid occurs when a strain is carried so far as to cause actual division of the solid into parts. The strains and fractures to which a solid, considered as a whole, is subject, may be classified according to the following table. To each kind of strain there corresponds a kind of stress; being the external force which produces that strain, and the equal and opposite force wherewith the solid resists that strain:—

	Strain.	Fracture.
Longitudinal.....	Extension	......Tearing.
	Compression	......Crushing and Cleaving.
Transverse........	Distortion	......Shearing.
	Torsion	......Wrenching.
	Bending	......Breaking across.

243. **Perfect and Imperfect Elasticity. Plasticity.**—A body is said to be *perfectly elastic*, which, if strained at a constant temperature by the application of a stress, recovers its original volume, or volume and figure, when such stress is withdrawn. Deviations from this property constitute *imperfect elasticity*. Gases, and liquids perfectly free from viscosity, are perfectly elastic.

The elasticity of every solid is sensibly perfect when the strain does not exceed a certain limit. This has been proved to be the case even for solids so plastic as moistened clay. For every solid there are limits, which if a strain exceed, *set*, or permanent alteration of volume or figure, is produced, and such *limits* of *elasticity* are less, and often considerably less, than the strains required to produce fracture. It has been proved by Mr. Hodgkinson that these limits depend on the duration of the strain, being less for a long-continued strain than for a brief strain. The *elasticity of volume* in solids is in general much more nearly perfect than the *elasticity of figure.* It is true that the density of many metals is permanently increased by hammering, rolling, and wiredrawing, and that of some other materials by intense pressure (Fairbairn; *Report of the British Association*, 1854); but the stresses which operate during these processes are very great. A body which is capable of undergoing great alterations of figure, and whose elasticity of figure is very imperfect, is a *plastic solid.* The gradations are insensible between plastic solids and *viscous liquids*, in which there is a resistance to change of figure, but no tendency to recover any particular figure.

Rise of temperature, so far as we yet know, increases elasticity of volume in all substances, and at the same time diminishes the amount and the perfection of elasticity of figure, so as to make solids more plastic and liquids less viscous.

244. The **Ultimate Strength** of a solid is the stress required to produce fracture in some specified way. The **Proof Strength** is the stress required to produce the greatest strain of a specific kind consistent with safety; that is, with the retention of the strength of the material unimpaired. A stress exceeding the proof strength of the material, although it may not produce instant fracture, produces fracture eventually by long-continued application and frequent repetition. Strength, whether ultimate or proof, is the product of two quantities, which may be called **Toughness** and **Stiffness.** *Toughness*, ultimate or proof, is here used to denote the greatest strain which the body will bear without fracture or without injury, as the case may be: *stiffness*, which might also be called *hardness*, is used to denote the ratio borne to that strain by the stress required to produce it,—being, in fact, a *modulus of elasticity* of some specified kind. *Malleable* and *ductile* solids have ultimate toughness greatly exceeding their proof toughness. *Brittle* solids have their ultimate and proof toughness equal or nearly equal.

Resilience or **Spring** is the quantity of *mechanical work* required to produce the proof strain, and is equal to the product of that strain, by the *mean stress* in its own direction which takes place during the production of that strain,—such stress being either exactly or nearly equal to one-half of the stress corresponding to the proof strain. Hence the resilience of a solid is exactly or nearly one-half of the product of its proof toughness by its proof strength; in other words, one-half of the product of the square of its proof toughness by its stiffness.

Each solid has as many different kinds of stiffness, toughness, strength, and resilience as there are different ways of straining it, as the following table shows. In that table *pliability* is used as a general term to denote the inverse of *stiffness*:—

Stress.	Strain.	Stiffness.	Pliability.	Fracture.	Strength.
Pull.	Stretching or Extension.	...	Extensibility.	Tearing.	Tenacity.
Thrust.	Squeezing or Compression.	...	Compressibility.	Crushing.	...
Shearing.	Distortion.	...	...	Shearing.	...
Twisting.	Twisting or Torsion.	...	...	Wrenching.	...
Bending.	Bending.	Transverse Stiffness.	Flexibility.	Breaking Across.	...

Those kinds of stiffness and strength which have no single word to designate them, are called *resistance to* the kind of strain or fracture to which they are opposed.

245. **Determination of Proof Strength.**—It was formerly supposed that the proof strength of any material was the utmost stress consistent with perfect elasticity; that is, the utmost stress which does not produce a *set*, as defined in Article 243. Mr. Hodgkinson, however, has proved that a set is produced in many cases by a stress perfectly consistent with safety. The determination of proof strength by experiment is now, therefore, a matter of some obscurity; but it may be considered that the best test known is, the *not producing an* INCREASING SET *by repeated application.*

246. The **Working Stress** on the material of a structure is made less than the proof strength in a certain ratio determined by practical experience, in order to provide for unforeseen contingencies.

247. **Factors of Safety** are of three kinds, viz.:—the ratio in which the *ultimate strength* exceeds the *proof strength*, the ratio in which the *ultimate strength* exceeds the *working stress*, and the ratio in which the *proof strength* exceeds the *working stress*. The following table gives examples of the values of those factors which occur in practice:—

	$\frac{\text{Ult. Strength.}}{\text{Proof Strength.}}$	$\frac{\text{Ult. Strength.}}{\text{Working Stress.}}$	$\frac{\text{Proof Strength.}}{\text{Working Stress.}}$
Strongest steel,	$1\frac{1}{2}$	...	...
Ordinary steel and wr. iron, steady load,	2	3	$1\frac{1}{2}$
" " moving load,	...	4 to 6	2 to 3
Wrought iron boilers,	2	8	4
Cast iron, steady load,	2 to 3	3 to 4	about $1\frac{1}{2}$
" moving load,	...	6 to 8	2 to 3
Timber; average,	3	10	$3\frac{1}{3}$
Stone and brick,	about 2	4 to 10, av. abt. 8	av. about 4

248. **Divisions of the Mathematical Theory of Elasticity.**—The theory of the elasticity of solids has been reduced to a body of *mathematical principles* applicable to those cases in which the strains of the particles of the body are so small, that quantities in the stresses depending on the squares, products, and higher powers of the strains may be neglected without appreciable error, and that, consequently, *Hooke's Law*—"*ut tensio sic vis*"—is sensibly true for all relations between strains and stresses. This condition is fulfilled in nearly all cases in which the stresses are within the limits of proof strength—the exceptions being a few substances, very pliable, and at the same time very tough, such as caoutchouc. The mathematical theory, as thus limited, consists of three parts, viz., the resolution and composition of stresses, the resolution and composition of strains, and the relations between strains and stresses. The resolution and composition of stresses has already been fully discussed in Part I., Chapter V., Section 3.

249. **Resolution and Composition of Strains.**—Let a solid of any figure be conceived to be ideally divided into a number of indefinitely small cubes by three series of planes parallel respectively to three co-ordinate planes. Each such elementary cube is distinguished by means of the distances, x, y, z, of its centre from the three co-ordinate planes. If the solid be strained in any manner, each of the elementary cubical particles will have its dimensions and figure changed, and will become a parallelopiped, which may be right or oblique—its size being conceived to be so small, that the curvature of its faces is inappreciable. The *simple* or *elementary strains* of which a particle, cubical in its free state, is susceptible, are six in number, viz.:—three *longitudinal* or *direct strains*, being the three proportional variations of its linear dimensions, which are elongations when positive, and compressions when negative; and three *transverse strains*, being the three *distortions*, or variations of the angles between its faces from right angles, which are considered as positive or negative according to some arbitrary but fixed rule, and are expressed by the proportions of the arcs subtending them to radius. When the values of those six strains for every particle are expressed by functions of the co-ordinates, x, y, z, the state of strain of the solid is completely expressed mathematically. The six elementary strains, in the cases to which the theory is limited, are very small fractions.

The method of reducing the state of strain of the solid at a given point, as expressed by a system of six elementary strains relatively to one system of rectangular axes, to an equivalent system of six elementary strains relatively to a new system of rectangular axes, is founded on the following theorem. Let α, β, γ, be the longitudinal strains of the dimensions of a given particle along x, y, z,

λ, μ, ν, the distortions of its angles in the planes $y\,z$, $z\,x$, $x\,y$. Conceive the surface of the second order whose equation is

$$\alpha x^2 + \beta y^2 + \gamma z^2 + \lambda y z + \mu z x + \nu x y = 1.$$

Transform this equation so as to refer the same surface to the new axes of co-ordinates; the six co-efficients of the transformed equation will be the elementary strains referred to the new axes. Other ways of resolving strains have been pointed out by Professor W. Thomson, *Cambridge and Dublin Mathematical Journal*, May, 1855.

The sum of the direct strains $\alpha + \beta + \gamma$ represents the cubic dilatation of a particle when positive, and the cubic compression when negative. The state of strain of a transparent body may be ascertained experimentally by its action on polarized light. On this subject experiments have been made by Fresnel, Sir D. Brewster, M. Wertheim, and Mr. Clerk Maxwell.

250. **Displacements.**—Let ξ, η, ζ, be the projections, parallel to x, y, z, respectively, of the *displacement* of a particle in a strained solid from its position when the solid is free, expressed as functions of x, y, z. Then

$$\alpha = \frac{d\,\xi}{d\,x};\quad \beta = \frac{d\,\eta}{d\,y};\quad \gamma = \frac{d\,\zeta}{d\,z};$$

$$\lambda = \frac{d\,\zeta}{d\,y} + \frac{d\,\eta}{d\,z};\quad \mu = \frac{d\,\xi}{d\,z} + \frac{d\,\zeta}{d\,x};$$

$$\nu = \frac{d\,\eta}{d\,x} + \frac{d\,\xi}{d\,y}.$$

251. **Analogy of Stresses and Strains.**—It has been shown in Article 104, that the elastic forces exerted on and by an originally cubical particle, which constitute the state of stress of the solid at the point where that particle is situated, may be resolved into six *elementary stresses*, viz.:—three *normal stresses*, perpendicular respectively to the three pairs of faces, and tending directly to alter the three linear dimensions of the particle—and three pairs of *tangential stresses* acting *along* the double pairs of faces to which they are applied, and tending directly to alter the angles made by such double pairs of faces. To reduce the state of stress at a given point expressed by a system of six elementary stresses referred to one system of rectangular co-ordinates to an equivalent system of elementary stresses referred to a new system of rectangular co-ordinates, equations have been given in Articles 105, 106, 107, 108, 109, and 112. The whole of those equations are virtually comprehended under the following theorem:—Let p_{xx}, p_{yy}, p_{zz}, be the

three normal stresses, and p_{yz}, p_{zx}, p_{xy}, the three tangential stresses; conceive the surface whose equation is

$$p_{xx}x^2 + p_{yy}y^2 + p_{zz}z^2 + 2p_{yz}yz + 2p_{zx}zx + 2p_{xy}xy = 1.$$

Transform this equation so as to refer the same surface to the new set of axes; the six co-efficients of the transformed equation will be the six elementary stresses referred to the new axes. For the complete investigation of this subject, see M. Lamé's *Leçons sur la Théorie mathématique de l'Elasticité des Corps solides*, Paris, 1852. The above equation is transformed into the equation of Article 249 by substituting respectively α, β, γ, λ, μ, ν, for p_{xx}, p_{yy}, p_{zz}, $2p_{yz}$, $2p_{zx}$, $2p_{xy}$; and by making corresponding substitutions in all the equations of Articles 105, 106, 107, 108, 109, and 112, they are made applicable to strains instead of stresses.

252. The **Potential Energy of Elasticity** of an originally cubic particle in a given state of strain is the *work* which it is *capable of performing* in returning from that state of strain to the free state; and is the product of the volume of the particle by the following function:—

$$\mathrm{U} = \frac{1}{2}(\alpha p_{xx} + \beta p_{yy} + \gamma p_{zz} + \lambda p_{yz} + \mu p_{zx} + \nu p_{xy}).$$

This function was first employed by Mr. Green, *Cambridge Transactions*, vol. vii.

253. **Co-efficients of Elasticity.**—According to Hooke's Law, each of the six elementary stresses may, without sensible error, be treated as a linear function of the six elementary strains, each multiplied by a particular *co-efficient* or *modulus of elasticity*. By expressing all the stresses in terms of the strains, the potential energy U is transformed into a homogeneous quadratic function of the six elementary strains, which must have twenty-one terms, and consequently *twenty-one co-efficients*, multiplying respectively the six half-squares and the fifteen binary products of the six elementary strains. The co-efficient of $\frac{1}{2}\alpha^2$ in U is that of α in p_{xx}; the co-efficient of $\alpha\beta$ in U is that of α in p_{yy} and also that of β in p_{xx}; and so on.

254. **Co-efficients of Pliability.**—According to Hooke's Law also, each of the six elementary strains may be treated, without sensible error, as a linear function of the six elementary stresses, so as to transform U to a homogeneous quadratic function of the elementary stresses p_{xx}, &c., having twenty-one terms, and twenty-one co-efficients expressing different kinds of *pliability*. The word "pliability" is here used in an extended sense, to include liability to

alteration of figure of every kind, whether by elongation, linear compression, or distortion.

Co-efficients, whether of elasticity or of pliability, may be thus classified :—*Direct*, or *longitudinal*, when they express relations between longitudinal strains, and normal stresses in the same direction; *lateral*, when they express relations between longitudinal strains, and normal stresses in directions at right angles to the strains; *transverse*, when they express relations between distortions, and tangential stresses in the same direction; *oblique*, when they express any other relations between strains and stresses.

255. **An Axis of Elasticity** is any direction in a solid body, with respect to which some kind of symmetry exists in the relations between strains and stresses. *An axis of direct elasticity* is a direction in a solid body, such that a longitudinal strain in that direction produces a normal stress, and no tangential stress on a plane normal to that direction. Every such axis is a direction of maximum or minimum direct elasticity relatively to the directions adjacent.

By the aid of the calculus of forms, and of an improvement in the geometry of oblique co-ordinates, it has been shown that every homogeneous solid must have *at least three* axes of direct elasticity, which may be rectangular or oblique with respect to each other,—that the number of such axes increases as the symmetry of the action of elastic forces becomes greater,—and that their various possible arrangements correspond exactly with those of the normals to the faces and edges of the various *primitive crystalline forms* (*Phil. Trans.*, 1856–7).

256. In an **Isotropic** or **Amorphous Solid** the action of elastic forces is alike in all directions. Every direction is an axis of elasticity. The co-efficients of oblique elasticity and oblique pliability are all null. The number of different co-efficients of elasticity, and of different co-efficients of pliability, is three. The following notation and equations show their relations to each other :—

Elasticities.

Direct, ………………………… $A = \dfrac{\mathfrak{a} - \mathfrak{b}}{\mathfrak{a}^2 - \mathfrak{a}\,\mathfrak{b} - 2\,\mathfrak{b}^2}$;

Lateral, ………………………… $B = \dfrac{\mathfrak{b}}{\mathfrak{a}^2 - \mathfrak{a}\,\mathfrak{b} - 2\,\mathfrak{b}^2}$;

Transverse, ………………………… $C = \dfrac{A - B}{2}$;

Elasticity of volume, ……………… $\dfrac{1}{\mathfrak{d}} = \dfrac{A + 2\,B}{3}$.

Pliabilities.

Direct,.............................. $\mathfrak{a} = \dfrac{A+B}{A^2+AB-2B^2}$;

(otherwise, the extensibility.)

Lateral,.............................. $\mathfrak{b} = \dfrac{B}{A^2+AB-2B^2}$;

Transverse,........................ $\mathfrak{c} = \dfrac{1}{C} = 2(\mathfrak{a}+\mathfrak{b})$;

Cubic compressibility,........... $\mathfrak{d} = 3\mathfrak{a} - 6\mathfrak{b}$.

257. **Modulus of Elasticity.**—The quantity to which the term "*modulus of elasticity*" was first applied by Dr. Young, is the reciprocal of the extensibility, or longitudinal pliability; that is to say,

$$E = \frac{1}{\mathfrak{a}} = A - \frac{2B^2}{A+B}.$$

This quantity expresses the ratio of the normal stress on the transverse section of a bar of an isotropic solid to the longitudinal strain, *only when the bar is perfectly free to vary in its transverse dimensions,* but not under other circumstances. The values of Young's modulus have been determined experimentally for almost every solid substance of importance, and a table of them is given at the end of the volume.

258. **Examples of Co-efficients.**—The only complete sets of co-efficients of elasticity and pliability which have yet been computed are those for brass and crystal, deduced from the experiments of M. Wertheim (*Annales de Chimie*, 3d series, vol. xxiii.), and are as follows—the unit of pressure being *one pound on the square inch:*—

	Brass.	Crystal.
A	22,224,000	8,522,600.
B	11,570,000	4,204,400.
C	5,327,000	2,159,100.
$\frac{1}{\mathfrak{d}}$	15,121,000	5,643,800.
$\frac{1}{\mathfrak{a}}$	14,300,000	5,746,000.
$\mathfrak{a}$	0·0000000699	0·0000001740.
$\mathfrak{b}$	0·0000000239	0·0000000575.
$\mathfrak{c}$	0·0000001877	0·0000004631.
$\mathfrak{d}$	0·0000000661	0·0000001772.

259. The **General Problem of the Internal Equilibrium of an Elastic Solid** is this :—Given the free form of a solid, the values of its co-efficients of elasticity, the attractions acting on its particles, and the stresses applied to its surface : to find its change of form, and the strains of all its particles. This problem is to be solved, in general, by the aid of an ideal division of the solid (as already described) into molecules rectangular in their free state, and referred to rectangular co-ordinates. For isotropic solids, some particular cases are most readily solved by means of spherical, cylindrical, or otherwise curved co-ordinates. The general equation of internal equilibrium in a solid acted on by its own weight, has already been given in Article 116, equation 2. If, in that equation, the values of the stresses in terms of the strains, expressed, as in Article 250, in terms of the *displacements* of the particles, be introduced, equations are obtained, which being integrated, give the displacements, and consequently the strains and stresses. The general problem is of extreme complexity; but the cases which occur in practice, and to which the remainder of this chapter relates, can generally be solved with sufficient accuracy by comparatively simple approximate methods. Most of those approximate methods are analogous to the "method of sections" described in its application to framework in Article 161. The body under consideration is conceived to be divided into two parts by an ideal plane of section; the forces and couples acting on one of those two parts are computed, and they must be equal and opposite to the forces and couples resulting from the *entire* stress at the ideal sectional plane, which is so found. Then as to the *distribution* of that stress, direct and shearing, some law is assumed, which if not exactly true, is known either by experiment or by theory, or by both combined, to be a sufficiently close approximation to the truth.

Except in a few comparatively simple cases, the strict method of investigation, by means of the equations of internal equilibrium, has hitherto been used only as a means of determining whether the ordinary approximative methods are sufficiently close.

SECTION 2.—*On Relations between Strain and Stress.*

260. **Ellipse of Strain.**—In Articles 249, 251, 252, 253, 254, 256, and 257, of the preceding section, certain general principles respecting the relations amongst strains, and the analogies and other relations between strain and stress, are stated without a detailed demonstration. In the present section the more simple cases of those principles, to which there will be occasion to refer in the sequel, are to be demonstrated.

Let a solid body be supposed to undergo a strain, or small alteration of dimensions and figure, of such a nature that all the displacements of its particles from their original positions are parallel to one plane; and let that plane be represented by the plane of the paper in fig. 114. In the first instance, let the state of strain of the body be uniform throughout; that is, let all parts of the body which originally were equal and similar to each other, continue equal and similar to each other notwithstanding their alteration of dimensions and figure.

Round any centre O, with the radius *unity*, let a circle be traced amongst the particles of the body, B C A F. Because of the uniformity of the strain, this circle will be changed into a parallel projection of a circle; that is, into an ellipse. Let *b c a f* be that ellipse, and O *a* and O*b* its semi-axes, the body being so placed in its strained condition that the central particle O may remain unchanged in position, in order that the circle and ellipse may be the more easily compared. Then the particle which was at A is displaced to *a*, and the particle which was at B is displaced to *b*; and particles which were at points in the circle, such as C and F, are displaced to corresponding points in the ellipse, such as *c* and *f*.

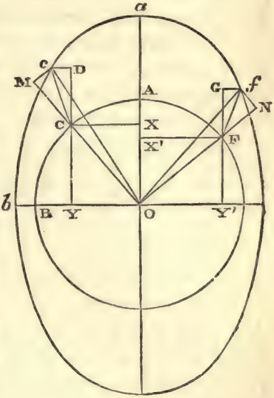

Fig. 114.

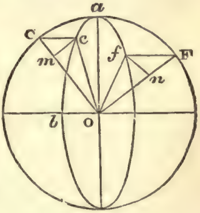

Fig. 115.

In the direction O A, the body has undergone the extension

$$\overline{\mathrm{A}\,a} = \alpha;$$

and in the direction O B, at right angles to O A, the extension

$$\overline{\mathrm{B}\,b} = \beta;$$

and the combination of those two extensions or elementary direct strains, in rectangular directions, constitutes the state of strain of the body parallel to the given plane; that state of strain being completely known, when α, β, and the directions of the pair of rectangular *axes of strain* O A, O B, are known.

One or both of the elementary strains might have been compressive, instead of tensile, in which case one or both of the quantities denoting them would have been negative, to express diminution of size.

A square whose sides are unity, and parallel to O A and O B, being traced amongst the particles of the body in the free state, is converted by the strain into a rectangle whose sides are $1 + \alpha$ and $1 + \beta$, and still parallel to O A and O B.

Let it now be required to express the state of strain of the body with reference to two new rectangular axes, O C and O F, that is to say, to find the alterations of dimensions and figure produced by the strains on a figure originally square, described on O C and O F.

Let $x = \overline{OX}$, $y = \overline{OY}$, be the original co-ordinates of C, and $x' = \overline{OX'}$, $y' = \overline{OY'}$, those of F; and let the angle $AOC = 90° - AOF = \theta$. Then

$$x = \cos\theta = -y'$$

$$y = \sin\theta = x'.$$

Also, let $x + \xi = \overline{YD}$, $y + \eta = \overline{OY} + \overline{Dc}$, be the co-ordinates of c, the new position of C; and let $x' + \xi' = \overline{Y'G}$, $y' + \eta' = \overline{OY'} + \overline{Gf}$, be the co-ordinates of f, the new position of F. Then because of the uniformity of the strain, the *component displacements* ξ, η, ξ', η', have the following values :—

$$\left.\begin{aligned} \xi &= \overline{CD} = \alpha x = \alpha\cos\theta; \\ \eta &= \overline{Dc} = \beta y = \beta\sin\theta; \\ \xi' &= \overline{FG} = \alpha x' = \alpha y = \alpha\sin\theta; \\ \eta' &= \overline{Gf} = \beta y' = -\beta\cos\theta. \end{aligned}\right\} \text{..........(1.)}$$

$\overline{Oc}$ and $\overline{Of}$ are the sides of the oblique parallelogram into which the square on $\overline{OC}$ and $\overline{OF}$ has been transformed by the strain. The relations between the new and the original figure are distinguished into two direct strains and a distortion, in the following manner :—

From c let fall cM perpendicular to O C M; and from f let fall fN perpendicular to O F N. Then

$$\alpha' = \overline{CM} \text{ is the extension of } \overline{OC};$$

$$\beta' = \overline{FN} \text{ is the extension of } \overline{OF};$$

and $\nu' = \overline{cM} + fN$ is the *distortion* or deviation from rectangularity; and the values of those three new elementary strains, relatively to the pair of axes which make the angle θ with the *principal axes* O A, O B, in terms of the *principal elementary stresses*, α, β, are as follows :—

$$\left.\begin{array}{c} \alpha' = \xi \cos\theta + \eta \sin\theta = \alpha \cos^2\theta + \beta \sin^2\theta; \\ \beta' = \xi' \sin\theta - \eta' \cos\theta = \alpha \sin^2\theta + \beta \cos^2\theta; \\ \nu' = \xi \sin\theta - \eta \cos\theta + \xi' \cos\theta + \eta' \sin\theta \\ = 2\,(\alpha - \beta) \cos\theta \sin\theta. \end{array}\right\} \ldots.(2.)$$

Those three equations are exactly analogous to the equations 3 and 4 of Article 112, from which they may be formed by substituting α for p_x, and β for p_y in both equations; and then, in the first place, α' for p_n, and θ for $\widehat{x\,n}$; in the second place, β' for p_n, and $(90° - \theta)$ for $\widehat{x\,n}$, and in the third place, ν' for p_t, and θ for $\widehat{x\,n}$.

This illustrates the general principle of analogy of stresses and strains stated in Article 251. That principle is further illustrated by the following geometrical construction of the preceding problem. In fig. 115, make $\overline{o\,a} = \alpha$, $\overline{o\,b} = \beta$, and draw the ellipse $b\,c\,a\,f$, and the circumscribing circle C a F. Let $\angle\, a\,o\,\mathrm{C} = \theta$, and let o F be perpendicular to o C, so that those lines represent the direction of the new rectangular axes, to which the strain composed of α and β is to be referred. Draw C c, F f, parallel to $o\,b$, cutting the ellipse in c and f, from which points respectively draw $c\,m \perp o\,\mathrm{C}$, and $f\,n \perp o\,\mathrm{F}$. Then

$$\overline{o\,m} = \alpha',\ \overline{o\,n} = \beta',\ 2\,\overline{c\,m} = 2\,\overline{f\,n} = \nu',$$

are the components of the strain, referred to the new axes; and the *ellipse of strain* $b\,c\,a\,f$ is analogous to the *ellipse of stress* of Article 112.

The results of the preceding investigation are applicable not only to an uniform state of strain, but to a state of strain varying from point to point of the body, provided the variation is continuous, so that it shall be possible, by diminishing the space under consideration, to make the strain within that space deviate from uniformity by less than any given deviation.

261. **Ellipsoid of Strain.**—A strain by which the size and figure of a body are altered in three dimensions may be represented in a manner analogous to that of the preceding Article, by conceiving a sphere of the radius unity to be transformed by the strain into an ellipsoid, and considering the displacement of various particles, from their original places in the sphere, to their new places in the ellipsoid. The three axes of the ellipsoid are the principal axes of strain, and their extensions or compressions, as compared with the coincident diameters of the sphere, are the three principal elementary strains which compose the entire strain. It is by this method, which it is unnecessary here to give in detail, that the general principles stated in Articles 249 and 251 are arrived at.

262. **Transverse Elasticity of an Isotropic Substance.**—Let the two principal elementary strains in one plane be of equal magnitude, but opposite kinds; that is, supposing the strain in fig. 114 along O A to be an extension, α, let the strain along O B be a compression, $\beta = -\alpha$. The ellipse will fall beyond the circle at A, and as much within it at B, and will cut it at an intermediate point near the middle of each quadrant.

Take a pair of new axes bisecting the right angles between the original axes; that is, let $\theta = 45°$; then the equations 2 of Article 260 give the following result :—

$$\alpha' = 0;\ \beta' = 0;\ \nu' = 2\,\alpha;\ldots\ldots\ldots\ldots\ldots\ldots(1.)$$

that is to say, *an extension, and an equal compression, along a pair of rectangular axes, are equivalent to a simple distortion relatively to a pair of axes making angles of* 45° *with the original axes; and the amount of the distortion is double that of either of the two direct strains which compose it;* a proposition which is otherwise evident, by considering that a distortion of a square is equivalent to an elongation of one diagonal, and a shortening of the other, in equal proportions.

The body being *isotropic*, or equally elastic in all directions, let A be its direct and B its lateral elasticity; then the pair of principal strains $\alpha, \beta = -\alpha$, will be accompanied by a pair of principal stresses along O A and O B respectively, given by the following equations :—

$$\text{along O A, } p_x = \mathrm{A}\,\alpha + \mathrm{B}\,\beta = (\mathrm{A} - \mathrm{B})\,\alpha;$$

$$\text{O B, } p_y = \mathrm{B}\,\alpha + \mathrm{A}\,\beta = (\mathrm{B} - \mathrm{A})\,\alpha = -\,p_x;\ldots\ldots(2.)$$

that is to say, there will be a *pull along* O A, *and an equal thrust along* O B.

It has already been proved, in Article 111, that such a pair of principal stresses, of equal intensities and opposite kinds, are equivalent to a pair of shearing stresses of the same intensity on a pair of planes making angles of 45° with the axes of principal stress; or taking p_t to represent the intensity of the shearing stress on each of a pair of planes normal to the new pair of axes,

$$p_t = p_x = (\mathrm{A} - \mathrm{B})\,\alpha;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.)$$

but if C be the co-efficient of transverse elasticity of the substance, we have also

$$p_t = \mathrm{C}\,\nu;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4.)$$

and consequently, for an isotropic substance,

$$\mathrm{C} = \frac{\mathrm{A} - \mathrm{B}}{2};\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5.)$$

or *the transverse elasticity is half the difference of the direct and lateral elasticities.*

This is the demonstration of a principle already stated in Article 256. The corresponding principle for pliabilities, viz. :—that the *transverse pliability is twice the sum of the direct and lateral extensibilities*, is demonstrated by a similar process, of which the steps may be briefly summed as follows :—

$$\alpha = \mathfrak{a}\, p_x - \mathfrak{b}\, p_y = (\mathfrak{a} + \mathfrak{b})\, p_x;$$
$$\beta = \mathfrak{a}\, p_y - \mathfrak{b}\, p_x = -(\mathfrak{a} + \mathfrak{b})\, p_x = -\alpha;$$
$$\therefore \nu' = 2\,\alpha = 2\,(\mathfrak{a} + \mathfrak{b})\, p_x = 2\,(\mathfrak{a} + \mathfrak{b})\, p_t = \mathfrak{c}\, p_t,$$
$$\therefore \mathfrak{c} = 2\,(\mathfrak{a} + \mathfrak{b}).\text{—Q. E. D.} \ldots\ldots\ldots\ldots\ldots\ldots (6.)$$

263. **Cubic Elasticity.**—If the three rectangular dimensions of a body or particle are changed in the respective proportions $1 + \alpha$, $1 + \beta$, $1 + \gamma$, its volume is altered in the proportion

$$(1 + \alpha)\,(1 + \beta)\,(1 + \gamma);$$

and when the elementary strains α, β, γ, are very small fractions this is sensibly equal to

$$1 + \alpha + \beta + \gamma.$$

Consequently, as in Article 249,

$$\alpha + \beta + \gamma$$

may be called the *cubic strain*, or *alteration of volume.*

In an isotropic substance, the three rectangular direct stresses which accompany those three strains are

$$\left.\begin{aligned} p_{xx} &= \mathrm{A}\,\alpha + \mathrm{B}\,(\beta + \gamma); \\ p_{yy} &= \mathrm{A}\,\beta + \mathrm{B}\,(\gamma + \alpha); \\ p_{zz} &= \mathrm{A}\,\gamma + \mathrm{B}\,(\alpha + \beta); \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots (1.)$$

The third part of the sum of those stresses, which may be called the *mean direct stress*, has the following value :—

$$\frac{p_{xx} + p_{yy} + p_{zz}}{3} = \frac{(\mathrm{A} + 2\,\mathrm{B})}{3} \cdot (\alpha + \beta + \gamma); \ldots\ldots\ldots (2.)$$

The co-efficient contained in this expression, being the ratio of the mean direct stress to the cubic strain, is the *cubic elasticity*, or *elasticity of volume*, already mentioned in Article 256, its reciprocal being the *cubic compressibility.*

264. **Fluid Elasticity.**—The distinction between solids and fluids is well illustrated by applying to fluids the equations of Articles 262 and 263. Fluids offer no resistance to distortion, that is, they have no transverse elasticity; therefore for them

$$\mathrm{C} = \frac{\mathrm{A} - \mathrm{B}}{2} = 0; \text{ or } \mathrm{A} = \mathrm{B}.$$

Introducing this into the equations 1 and 2 of Article 263, we find

$$p_{xx} = p_{yy} = p_{zz} = \mathrm{B}(\alpha + \beta + \gamma),$$

and the cubic elasticity

$$\frac{\mathrm{A} + 2\,\mathrm{B}}{3} = \mathrm{B}.$$

The equality of the pressures in all directions at a given point in a fluid has already been proved by another process in Article 110.

The equations of Article 256 show the *pliabilities* of a perfect fluid to be infinite, with the exception of the cubic compressibility, which is $\frac{1}{\mathrm{B}}$.

SECTION 3.—*On Resistance to Stretching and Tearing.*

265. **Stiffness and Strength of a Tie-Bar.**—If a cylindrical or prismatic bar, whose cross section is S (as in Article 97, fig. 46), be subjected to a pull whose resultant acts along the axis of figure of the bar, and whose amount is P, the intensity of the pull will be uniform on each cross section of the bar, and will have the value

$$p = \frac{\mathrm{P}}{\mathrm{S}} \quad \text{.................................(1.)}$$

This direct stress will produce a strain, whose principal element will be a longitudinal extension of each unit of length of the bar, of the value

$$\alpha = \mathfrak{a}\, p = \frac{p}{\mathrm{E}} \quad \text{...........................(2.)}$$

where $\mathfrak{a}$ denotes the *direct extensibility*, and E its reciprocal, the *modulus of elasticity*, or *co-efficient of resistance to stretching*, as explained in Articles 256 and 257.

Let x denote the length of the bar, or of any portion of it, in the free or unloaded state; that length, under the tension p, becomes $(1 + \alpha)\, x$.

The co-efficient

$$\mathrm{E} = \frac{p}{\alpha},$$

is nearly constant until p passes the limit of the *proof stress;* but after that limit has been passed, that co-efficient diminishes; that is to say, the extension α increases faster than the intensity of the stretching force p, until the bar is torn asunder.

The *ultimate strength* of the bar, or the total pull required to tear it instantly asunder—the *proof strength,* or the greatest pull of which it can safely bear the long-continued or repeated application—and the *working load*—are computed by means of the formula

$$p = f, \text{ or } \mathrm{P} = f\mathrm{S}, \text{ ....} (3.)$$

where f represents the *ultimate tenacity*, the *proof tenacity*, or the *working stress*, as the case may be.

The *toughness* of the bar, or the extension corresponding to the *proof load*, is given by the formula

$$\alpha = \frac{f}{\mathrm{E}}, \text{ ....} (4.)$$

where f is the *proof tenacity*.

266. The **Resilience, or spring** of the bar, or the work performed in stretching it to the limit of proof strain, is computed as follows:—x being the length, as before, the elongation of the bar under the proof load is

$$\alpha x = \frac{f x}{\mathrm{E}};$$

the force which acts through this space has for its least value 0, for its greatest value $\mathrm{P} = f\mathrm{S}$, and for its mean value $\frac{f\mathrm{S}}{2}$; so that the work performed in stretching the bar to the proof strain is

$$\frac{f\mathrm{S}}{2} \cdot \frac{f x}{\mathrm{E}} = \frac{f^2}{\mathrm{E}} \cdot \frac{\mathrm{S} x}{2} \text{ ....} (1.)$$

The co-efficient $\frac{f^2}{\mathrm{E}}$, by which one-half of the volume of the bar is multiplied in the above formula, is called the MODULUS OF RESILIENCE.

267. **Sudden Pull.**—A pull of $\frac{f\mathrm{S}}{2}$, or *one-half of the proof load*, being *suddenly* applied to the bar, will produce the *entire proof strain* of $\frac{f}{\mathrm{E}}$, which is produced by the *gradual* application of the proof load itself; for the work performed by the action of the constant force $\frac{f\mathrm{S}}{2}$ through a given space, is the same with the work performed by the action, through the same space, of a force increasing at an uniform rate from 0 up to $f\mathrm{S}$. Hence a bar, to resist with safety the sudden application of a given pull, requires to have twice the strength that is necessary to resist the gradual application and steady action of the same pull.

The principle here applied belongs to the subject of dynamics, and is stated by anticipation, on account of its importance as respects the strength of materials. It is the chief reason for making the factor of safety for a moving load considerably greater than for a steady load (see Article 247).

268. A **Table of the Resistance of Materials to Stretching and Tearing,** by a direct pull, in pounds per square inch, is given at the end of the volume.

The tenacity, or resistance to tearing, given in that table, is in each case the *ultimate tenacity*, being the quantity as to which experimental data are most abundant and precise. The proof tenacity and working tension, when required, are to be found by dividing the ultimate tenacity by the proper factors, according to Article 247.

The modulus of elasticity in each case is given from experiments made within the limits of proof strain.

Both co-efficients, for fibrous substances, have reference to the effects of tension acting *along the fibres*, or "grain." Both the tenacity and the elasticity of timber against forces acting across the grain are much smaller than against forces acting along the grain, and are also of uncertain amount, the results of experiments being few and contradictory.

269. **Additional Data.** — The following are a few experimental results in addition to those given in the table :—

Welded joint of a wrought iron retort.—Ultimate tenacity, by a single experiment, in lbs. per square inch,... 30750·
Iron wire-ropes.—Strength in lbs., for each lb. weight per fathom, ... Ultimate, 4480·
Proof, ... 2240·
Working load $\frac{1}{6}$ of ultimate, or $\frac{1}{3}$ of proof strength.
Hempen cables.—Ultimate strength = (girth in inches)^2 × 448 lb.
Leathern belts.—Working tension in lbs. per square inch, according to General Morin .... 285·

Chain cables, when the tendency of each link to collapse is resisted by means of a cross-bar, as shown in fig. 116, have a strength per square inch of cross section of the link equal to that of the iron of which they are made, when it is in the form of bars.

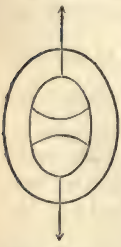

Fig. 116.

270. The **Strength of Rivetted Joints** of iron plates is given in the table, in *lbs. per square inch of section of the plate*, from the experiments of Mr. Fairbairn. The strength of a double-rivetted joint is seven-tenths of that of the iron plate, simply because of three-tenths of the breadth of the plate being punched out in each row of rivet-holes. The strength of a single-rivetted joint is diminished not merely by the removal of the iron at the

rivet-holes, but by the unequal distribution of the stress. Rivetted joints will be further considered in the sequel.

271. **Thin Hollow Cylinders; Boilers; Pipes.**—Let q denote the uniform intensity of the pressure exerted by a fluid which is confined within a hollow cylinder of the radius r, and of a thickness, t, which is small as compared with that radius.

The demonstration in Article 179 shows, that if we consider a *ring*, being a portion of the cylinder of the length *unity*, the tension on that ring will be

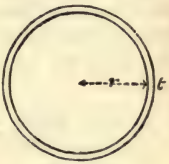

Fig. 117.

$$P = q r, \qquad (1.)$$

being the force per unit of length with which the internal pressure tends to split the cylinder from end to end.

The sectional area of the ring under consideration is t. Then assuming, what is very nearly correct, that the tension is uniformly distributed, the intensity of that tension is

$$p = \frac{q r}{t} \qquad (2.)$$

The ratio of thickness to radius, which a thin hollow cylinder requires, to fit it for a given intensity of *bursting pressure*, *proof pressure*, or *working pressure*, is given by the formula

$$\frac{t}{r} = \frac{q}{f}; \qquad (3.)$$

f being the *ultimate tenacity*, the *proof tension*, or the *working tension*, as the case may be.

It is considered prudent, in STEAM-BOILERS, to make the working tension only *one-eighth* of the ultimate tenacity. The joints of plate iron boilers are single-rivetted; but from the manner in which the plates break joint, analogous to the bond in masonry, the tenacity of such boilers is considered to approach more nearly to that of a double-rivetted joint than that of a single-rivetted joint. Mr. Fairbairn estimates it at 34,000 lbs. per square inch; so that the values of f for wrought iron boilers may be thus stated:—

Bursting tension, ...	34,000
Proof tension, ...	17,000
Working tension, ...	4,250

For CAST IRON WATER PIPES, the working tension may be made *one-sixth* of the bursting tension, which for cast iron, on an average, is 16,500 lbs. per square inch; that is to say, the values of f are

Bursting tension, ...	16,500
Proof tension (one-third),......	5,500
Working tension, ...	2,750

For steam-pipes, as for steam-boilers, the factor of safety should be *eight.*

272. **Thin Hollow Spheres.**—Let fig. 117 now be conceived to represent a diametral section of a thin hollow sphere, filled with a fluid which presses from within with the intensity q. The area of the fluid cut by the section is

$$\pi r^2;$$

hence the whole force to be resisted by the tenacity of the section of the spherical shell is

$$P = \pi q r^2. \quad \text{(1.)}$$

The area of the section of the spherical shell, supposing the thickness t to be small as compared with the radius r, is very nearly

$$S = 2 \pi r t; \quad \text{(2.)}$$

hence assuming, what is very nearly correct, that the tension is uniform, its intensity is

$$p = \frac{P}{S} = \frac{q r}{2 t}; \quad \text{(3.)}$$

or, *one-half* of the tension round a cylindrical shell having the same internal pressure, and the same proportion of thickness to radius; so that, in these circumstances, the sphere is twice as strong as the cylinder.

Equation 3 gives also the *longitudinal* tension in a thin hollow cylinder, which, being only one-half of the circumferential tension round the cylinder, does not require to be considered in practice.

The proper ratio of thickness to radius in a thin hollow sphere is given by the formula

$$\frac{t}{r} = \frac{q}{2 f}; \quad \text{(4.)}$$

f being the bursting, proof, or working tension, according as q is the bursting, proof, or working pressure.

273. **Thick Hollow Cylinder.**—The assumption that the circumferential tension, or *hoop-tension* as it may be called, in a hollow cylinder is uniformly distributed, is approximately true only when the thickness is small as compared with the radius; for if a ring of the cylinder be conceived to be divided into several concentric hoops, one within another, the tension of the innermost hoop balances part of the radial pressure of the confined fluid, so that a diminished radial pressure is transmitted to the second hoop, which has therefore a less tension than the first hoop, and so on.

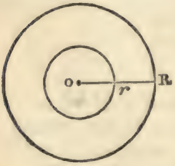

Fig. 118.

Equation 2 of Article 271 gives the *mean* hoop-tension in a thick as well as in a thin cylinder; but it is not the mean, but the *greatest* hoop-tension (that is, the tension round the inner surface of the cylinder), which is limited by the strength of the material. The object of the present investigation is to show what law the variation of hoop-tension follows, and thence, what relation the maximum tension bears to the fluid pressure.

To make the solution perfectly general, it will be supposed that the cylinder is pressed from without as well as from within. Let fig. 118 represent a cross section of the cylinder; let R denote its external and r its internal radius. Let q_0 denote the fluid pressure from within, and q_1 that from without; p_0 the hoop-tension at the inner surface of the cylinder, and p_1 the hoop-tension at the outer surface.

Consider, as before, a ring whose length, parallel to the axis of the cylinder, is unity. The radial section of that ring, from r to R in fig. 118, has to sustain the difference between the total pressures from within and without, in a direction perpendicular to the radius O r R, on a quadrant bounded by that radius. That difference is

$$q_0 r - q_1 \mathrm{R}.$$

Conceive the ring to be divided into an indefinite number of concentric hoops, each of the thickness $d\,r$, and exerting a tension of the intensity p; then the total hoop-tension will be

$$\int_r^{\mathrm{R}} p\,d\,r = q_0 r - q_1 \mathrm{R} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

From the symmetry of the ring and *f* the forces acting on it in all directions round the centre O, it is obvious that the axes of stress of any particle of metal must be respectively in the direction of a radius, and perpendicular to that direction. The principal stresses at any particle are a *radial pressure*, q (which for each particle at the inner surface is q_0, and for each particle at the outer surface, q_1) and a *hoop-tension* p.

As in the case of the ellipse of stress, Article 112, we may conceive this pair of principal stresses to be made up of two component pairs, viz.:—

A pair of equal stresses of the same kind, constituting a *fluid pressure* or *tension*, whose common intensity, stated so as to be a tension when positive, a pressure when negative, is

$$\frac{p-q}{2} = m;$$

and a pair of equal stresses of contrary kinds, whose common intensity is

$$\frac{p+q}{2}=n.$$

Thus we have $p=n+m$, $q=n-m$; and the problem is to be solved by first supposing m to act alone, then supposing n to act alone, and lastly combining their effects; observing, that the only solutions of equation 1 which are admissible, are those which are true for all values of R and r.

CASE 1. *Equal and similar stresses*, or $n=0$. In this case

$$p=-q=m,$$

showing, that instead of a radial pressure, there is a radial tension equal to the hoop-tension, and constituting along with it simply a fluid tension of the intensity m at each point. Equation 1 is fulfilled by making

$$p=-q=m=\text{constant},\ldots\ldots\ldots\ldots\ldots\ldots(2.)$$

which reduces both sides of equation 1 to

$$m\,(\mathrm{R}-r).$$

CASE 2. *Equal and contrary stresses*, or $m=0$. In this case

$$p=q=n,$$

and the solution of equation 1 is

$$p=q=n=\frac{a}{r'^2}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.)$$

a being an arbitrary constant, and r' any value of the radius, from r to R inclusive; for this reduces both sides of equation 1 to

$$a\left(\frac{1}{r}-\frac{1}{\mathrm{R}}\right).$$

CASE 3. *General solution.* By combining the two partial solutions of equations 2 and 3 together, we find

$$\left.\begin{array}{ll}\text{Radial pressure,} & q=n-m=\dfrac{a}{r'^2}-m;\\[2ex] \text{Hoop-tension,} & p=n+m=\dfrac{a}{r'^2}+m;\end{array}\right\}\ldots\ldots\ldots\ldots(4.)$$

To determine the constants a and m we have the equations

$$\frac{a}{r^2}-m=q_0;\ \frac{a}{\mathrm{R}^2}-m=q_1;$$

whence we obtain by elimination

$$\left.\begin{aligned} a &= \frac{(q_0 - q_1)\,R^2 r^2}{R^2 - r^2}; \\ m &= \frac{q_0 r^2 - q_1 R^2}{R^2 - r^2}; \end{aligned}\right\} \text{..................} (5.)$$

giving, finally, for the *maximum hoop-tension,*

$$p_0 = \frac{a}{r^2} + m = \frac{q_0 (R^2 + r^2) - 2\,q_1 R^2}{R^2 - r^2} \text{..........} (6.)$$

The mean hoop-tension is

$$\frac{q_0 r - q_1 R}{R - r}, \text{..........................} (7.)$$

which is exceeded by the maximum in the proportion

$$\frac{q_0 (R^2 + r^2) - 2\,q_1 R^2}{(q_0 r - q_1 R)(R + r)}, \text{.....................} (8.)$$

a proportion which tends towards equality, as R and r become more nearly equal.

A transposition of equation 6 gives the following value of the ratio of the external to the internal radius, required in order that p_0 may be $= f$, the bursting, proof, or working tension, as the case may be :—

$$\frac{R}{r} = \sqrt{\left\{ \frac{f + q_0}{f - q_0 + 2\,q_1} \right\}} \text{..................} (9.)$$

In most cases which occur in practice, the external fluid pressure q_1 is so small compared with the internal, that it may be neglected.

One important consequence of equation 9 is, that *if the internal pressure q_0 is equal to or greater than the sum* $f + 2\,q_1$ *of the coefficient of strength and twice the external pressure, no thickness, how great soever, will enable the cylinder to resist the pressure.*

The following is a geometrical representation of the foregoing solution. In fig. 119, let O represent the centre of the cylinder; O r its internal, and O R its external radius. To represent the value of $n = \frac{a}{r'^2}$, draw two ordinates r A, R B, at right angles to the direction of those radii, such that

$$\overline{r\,A} : \overline{R\,B} :: R^2 : r^2.$$

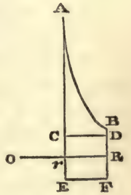

Fig. 119.

Then A and B will be points in a *hyperbola of the second order,* A B, which has the property that

$$\text{area } r\,A\,B\,R = r \times \overline{r\,A} - R \times \overline{R\,B};$$

so that it represents case 2.

Draw C D || O r R, cutting off from the ordinates the parts C A, D B, which bear to each other the proportions

$$\overline{C\,A} : \overline{D\,B} :: q_0 : q_1.$$

Then $\overline{r\,C} = \overline{R\,D}$ will represent m, the solution of case 1. Draw E F || O r R at the same distance $\overline{r\,E} = \overline{r\,C}$ on the opposite side. Then if any ordinate be drawn across the two straight lines E F and C D, and the curve A B, at a given distance r' from O, the segment of that ordinate between C D and A B will represent the radial pressure q, and the entire ordinate from E F to A B will represent the hoop-tension p, at that distance from O; and in particular $\overline{E\,A}$ will represent the maximum hoop-tension p_0.

The formulæ of this Article are the same with those given by M. Lamé in his *Traité de l'Elasticité;* but they are arrived at in a different manner.

274. **Cylinder of Strained Rings.**—To obviate, in whole or in part, the unequal distribution of the hoop-tension in thick hollow cylinders for withstanding great pressures, it has been proposed to construct such cylinders of concentric hoops or rings built together, the outer hoops being "*shrunk*" on to the inner hoops, in such a manner, that before any internal pressure is applied, the hoops within a certain distance of the centre may be in a state of circumferential compression, and those beyond that distance in a state of circumferential tension. If the stress thus produced by the mutual action of the concentric hoops could be adjusted with such accuracy, as to be at each point exactly equal and opposite to the difference between the actual hoop tension at the same point due to the internal pressure, as given by equations 4, 5, and 6, of Article 273, and the mean hoop-tension as given by equation 7, then upon applying the proper internal pressure, there would result simply an uniform tension equal to the mean, and the formulæ of Article 271 would become applicable to thick as well as to thin cylinders. Even although it may be impracticable to adjust the previous stress with the accuracy above described, any approach to its proper distribution must increase the strength of the cylinder. This method of construction has been carried into effect in Captain Blakely's gun, Mr. Mallet's mortar, and some other pieces of artillery.

The only equation which the stress of the concentric hoops will of itself fulfil is

$$\int_r^R p\,d\,r = 0.$$

275. **Thick Hollow Sphere.**—Let fig. 118 now represent a diametral section of a hollow sphere, the fluid pressures within and without being q_0 and q_1, as before. The pressure to be resisted at the section is

$$\pi (q_0 r^2 - q_1 \mathrm{R}^2);$$

and if the section of the metal be conceived to be divided into an indefinite number of concentric rings, the breadth of one of these rings being $d r$, its radius r', and the tension at it p, it appears that the total resistance of the section will be

$$2 \pi \int_{r'}^{\mathrm{R}} p \, r' \, d r;$$

and hence the equation to be fulfilled, for all values of q_0, q_1, r, and R, is

$$2 \int_{r}^{\mathrm{R}} p \, r' \, d r = q_0 r^2 - q_1 \mathrm{R}^2 \text{..................} (1.)$$

From symmetry it appears, that the axes of stress at any particle must be, one in the direction of a radius, with the pressure q along it, and the other two in any two directions perpendicular to the first and to each other, with equal tensions p along them. Two partial solutions are obtained in the following manner:—

Let

$$\frac{2p - q}{3} = m,$$

$$\frac{p + q}{3} = n;$$

so that

$$p = n + m; \; q = 2n - m.$$

CASE 1. $n = 0$, $p = -q = m$; being the case of a *fluid tension*, equal in all directions. In this case, equation 1 is solved by making

$$p = -q = m = \text{constant}, \text{..................} (2.)$$

which reduces both sides of that equation to

$$m (\mathrm{R}^2 - r^2)$$

CASE 2. $m = 0$, $p = \frac{q}{2} = n$; being the case of a pair of circumferential tensions, each equal to half of the radial pressure. In this case, equation 1 is solved by making

$$p = \frac{q}{2} = n = \frac{a}{r'^3}; \text{......................} (3.)$$

which reduces both sides of that equation to

$$2a \left(\frac{1}{r} - \frac{1}{\mathrm{R}}\right).$$

CASE 3. *General solution.*

$$\left.\begin{aligned} q &= 2n - m = \frac{2a}{r'^3} - m, \\ p &= n + m = \frac{a}{r'^3} + m, \end{aligned}\right\} \dots\dots\dots\dots (4.)$$

The constants a and m, deduced from the equations

$$q_0 = \frac{2a}{r^3} - m; \; q_1 = \frac{2a}{R^3} - m,$$

are found by elimination to have the following values :—

$$\left.\begin{aligned} a &= \frac{(q_0 - q_1) R^3 r^3}{2(R^3 - r^3)} \\ m &= \frac{q_0 r^3 - q_1 R^3}{R^3 - r^3} \end{aligned}\right\} \dots\dots\dots\dots (5.)$$

giving finally, for the maximum tension,

$$p_0 = \frac{a}{r^3} + m = \frac{q_0 (R^3 + 2r^3) - 3 q_1 R^3}{2(R^3 - r^3)} \dots\dots (6.)$$

A transformation of this equation gives the following value of ratio of the external to the internal radius of the sphere, required in order that p_0 may be $= f$, the bursting, proof, or working tension, as the case may be :—

$$\frac{R}{r} = \sqrt[3]{\left\{ \frac{2(f + q_0)}{2f - q_0 + 3q_1} \right\}} \dots\dots\dots\dots (7.)$$

This equation shows, that if

$$q_0 = \text{or} > 2f + 3q_1,$$

no thickness will be sufficient to enable the sphere to withstand the pressure.

The formulæ of this Article agree with those given by M. Lamé, though arrived at by a different process.

276. **Boiler Stays.**—The sides of locomotive fire-boxes, the ends of cylindrical boilers, and the sides of boilers of irregular figures like those of marine steam engines, are often made of flat plates, which are fitted to resist the pressure from within by being connected together across the water-space or steam-space between them by tie-bars, called stays when long, bolts when short. For example, fig. 120 represents part of the flat side of a locomotive fire-box, and shows the arrangement of the bolts by which it is tied to the flat plate at the other side of the water-space.

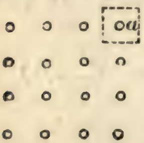

Fig. 120.

Each of these bolts or stays sustains the pressure of the steam against a certain area of the plate to which it is attached. Thus, in fig. 120, the bolt a resists the pressure of the steam on the square area which surrounds it, and whose side is equal to the distance from centre to centre of the bolts.

Let a be the sectional area of a stay; A, that of the portion of flat plate which it holds; q, the bursting, proof, or working pressure, and f the ultimate, proof, or working tension of the material of the stay. Then

$$f a = q \mathrm{A}.$$

The proper factor of safety is *eight*, as for other parts of boilers. Experience has shown, that the plate, if its material is as strong as that of the stay, should have its thickness equal to *half the diameter* of the stay. If the plate be of a weaker material than the stay, its thickness should be proportionally increased.

The flat ends of cylindrical boilers are sometimes stayed to the cylindrical sides by means of triangular plates of iron called "*gussets*." These plates are placed in planes radiating from the axis of the boiler, and have one edge fixed to the flat end, and the other to the cylindrical body. Each gusset sustains the pressure of the steam against a *sector* of the flat circular end. Considering that the resultant tension of a gusset must be concentrated near one edge, it appears advisable that its sectional area should be three or four times that of a stay-bar suited for sustaining the pressure on the same area.

The best experimental data respecting the strength of boilers are due to the researches of Mr. Fairbairn, especially those recorded in his work called *Useful Information for Engineers*.

277. **Suspension Rod of Uniform Strength.**—In fig. 121, let W be a weight hung from the lower end of a vertical rod B C, whose weight per unit of volume is w, and let it be required to find how the transverse section S of the rod must vary with the height x above B, in order that the tension may be everywhere of equal intensity f.

The total load at any point is, W from the weight hung at B, $w \int_0^x \mathrm{S}\, dx$ from the weight of the rod for a height x above B; and this must be equal to the pull $f\mathrm{S}$. Hence

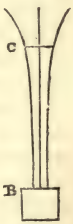

Fig. 121.

$$\mathrm{W} + w \int_0^x \mathrm{S}\, dx = f\mathrm{S}; \text{ ....(1.)}$$

which being solved, gives for the cross section of the rod,

$$\mathrm{S} = \frac{\mathrm{W}}{f} \cdot e^{\frac{wx}{f}}; \text{ ....(2.)}$$

and for its weight, for a height x above B,

$$fS - W = W\left(e^{\frac{wx}{f}} - 1\right) \text{ .... } (3.)$$

The most useful application of this is to the determination of the dimensions of the pump-rods of deep mines. They are not made with the section varying continuously, according to the formula 2, but in a series of divisions, each of uniform scantling; nevertheless that formula will serve to show approximately the law which the dimensions of those divisions should follow.

SECTION 4.—*On Resistance to Shearing.*

278. **Condition of Uniform Intensity.**—The present section refers to those cases only in which the shearing stress on a body is uniform in direction and in intensity. The effects of shearing stress varying in intensity will be considered under the head of Resistance to Bending, which is in general accompanied by such a stress; and the effects of shearing stress varying in direction as well as in intensity under the head of Resistance to Torsion.

It has been shown in Article 103 that shearing stresses can only exist in pairs, every shearing stress on a given plane being necessarily accompanied by a shearing stress of equal intensity on another plane. In Article 112, Problem II., it is shown that for any combination of stress parallel to a given plane, the planes relatively to which the shearing stress is greatest are at right angles to each other, and make angles of 45° with the axes of principal stress.

When equal forces are applied to the opposite sides of a wedge, bolt, rivet, or other body, in such a manner as to tend to shear it into two parts at a particular transverse plane of section, then at any given point in that transverse sectional plane the shearing stress is of equal intensity relatively to that plane itself, and to a longitudinal plane traversing the same point, perpendicular to the direction of the externally-applied shearing forces. If the wedge, bolt, or rivet is loose in its hole or socket at and near the plane of shearing, there can be no shearing stress on those free parts of its external surface which are at right angles to the direction of the external shearing force; and hence the intensity of the shearing stress at the plane of shearing, how great soever it may be in the internal parts of the body, must diminish to nothing at certain parts of the external edges of that sectional plane, and must be unequally distributed; so that the most intense shearing stress must be greater than the intensity of a stress of equal amount uniformly distributed.

To insure uniform distribution of the stress, it is necessary that the rivet or other fastening should fit so tight in its hole or socket, that the friction at its surface may be at least of equal intensity to the shearing stress. When this condition is fulfilled, the intensity of that stress is represented simply by $\frac{F}{S}$; F being the shearing force, and S the sectional area which resists it.

279. A **Table of the Resistance of Materials to Shearing and Distortion,** in lbs. avoirdupois per square inch, is given at the end of the volume. It is of small extent, because of the small number of substances whose resistances to shearing and distortion have been ascertained by satisfactory experiments. The resistance of timber to shearing is in each case that which acts between contiguous layers of fibres.

280. **Economy of Material in Bolts and Rivets.**—There are many structures, such as boilers, wrought iron bridges, and frames of timber or iron, in which the principal pieces, such as plates, links, or bars, being themselves subjected to a direct pull, are connected with each other at their joints by fastenings, such as rivets, bolts, pins, or keys, which are under the action of a shearing force. It is in every such case important, that the pieces connected and their fastenings should be of equal strength; for if the fastenings be the weaker, either the whole structure is insufficiently strong, or the material which gives the additional strength to the plates or bars is wasted: and if the fastenings be the stronger, the plates and bars are weakened more than is necessary by the holes or sockets; and as before, either the structure is too weak, or material is wasted.

Let f denote the resistance per square inch of the material of the principal pieces to tearing; S, the total sectional area, whether of one piece or of two or more parallel pieces, which must be torn asunder in order that the structure may be destroyed; f', the resistance per square inch of the material of the fastenings to shearing; S′, the total sectional area of fastenings at one joint, which must be sheared across in order that the structure may be destroyed; then, if the conditions of uniform distribution of stress are fulfilled, the principal pieces and their fastenings ought to be so proportioned, that

$$f\,S = f'\,S'; \text{ or } \frac{S'}{S} = \frac{f}{f'} \ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

For wrought iron rivetted plates, taking the value of f' from the table (as determined by the experiments of Mr. Doyne), we have

$$\frac{f}{f'} = 1 \text{ nearly, and } \therefore S' = S \ldots\ldots\ldots\ldots\ldots\ldots (2.)$$

For wrought iron bars connected by bolts or rivets, we have

$$\frac{f}{f'} = \frac{6}{5} \text{ nearly, and } \therefore S' = \frac{6}{5}\ S \ldots\ldots\ldots\ldots\ldots\ldots (3.)$$

Example I. *Plate-joint overlapped, single-rivetted.* Fig. 122. A, front view; B, side view. Let

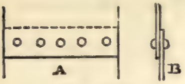

Fig. 122.

$t =$ thickness of plate.
$d =$ diameter of rivet.
$c =$ distance from centre to centre of rivets.

Then

$$1 = \frac{S'}{S} = \frac{\text{Sectional area of one rivet}}{\text{Sectional area of plate between two holes}}$$

$$= \frac{0{\cdot}7854\,d^2}{t\,(c-d)}; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4.)$$

so that, d and t being given, and c required, we have

$$c = \frac{0{\cdot}7854\,d^2}{t} + d \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5.)$$

d in practice is usually from $2t$ to $1\frac{1}{2}t$; and the overlap from c to $1\frac{1}{10}c$.

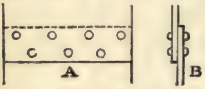

Fig. 123.

Example II. *Plate-joint overlapped, double-rivetted.* Fig. 123.

$$1 = \frac{S'}{S} = \frac{\text{Sectional area of two rivets}}{\text{Sectional area of plate between two holes in same line}}$$

$$= \frac{1{\cdot}5708\,d^2}{t\,(c-d)}; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.)$$

$$\therefore c = \frac{1{\cdot}5708\,d^2}{t} + d \ldots\ldots\ldots\ldots\ldots\ldots\ldots (7.)$$

Overlap in practice $= 1\frac{2}{3}c$ to $1\frac{3}{4}c$.

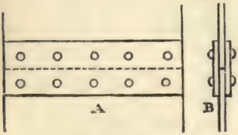

Fig. 124.

Example III. *Plate Butt-joint, with a pair of covering plates, single-rivetted.* Fig. 124. Here each rivet can give way only by being sheared across in two places at once; therefore

$$1 = \frac{S'}{S}$$

$$= \frac{2 \times \text{Sectional area of rivet}}{\text{Sectional area of plate between two holes}} = \frac{1{\cdot}5708\,d^2}{t\,(c-d)}; \ldots (8.)$$

$$\therefore c = \frac{1{\cdot}5708\,d^2}{t} + d \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9.)$$

Length of each covering plate $= 2 \times$ overlap $=$ from $2c$ to $2\frac{1}{5}c$.

Example IV. *Plate Butt-joint, with a pair of covering plates, double-rivetted.* Fig. 125.

$$1 = \frac{S'}{S} = \frac{4 \times \text{Sectional area of rivet}}{\text{Sectional area of plate between two holes in one row}}$$

$$= \frac{3{\cdot}1416\, d^2}{t\,(c-d)}; \ldots\ldots\ldots\ldots(10.)$$

$$\therefore c = \frac{3{\cdot}1416\, d^2}{t} + d \ldots\ldots(11.)$$

Length of each covering-plate $= 2 \times$ overlap

$= \text{from } 3\tfrac{1}{3} \text{ to } 3\tfrac{1}{2}\, c.$

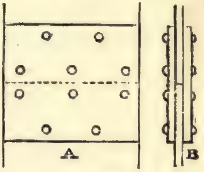

Fig. 125.

Note.—The *length of a rivet,* before being clenched, measuring from the head, is about $4\frac{1}{2}\, t$ for overlapped-joints, and $5\frac{1}{2}\, t$ for butt-joints with covering-plates.

Example V. *Suspension bridge chain-joint.* The chain of a suspension bridge consists of long and short links alternately. Each long link consists of one or more, say of n, parallel flat bars, of a shape resembling fig. 64, Article 138, placed side by side; each bar has a round eye at each end. Each short link consists of $n+1$ parallel flat bars, with round eyes at their ends, which are placed between and outside of the ends of the parallel bars of the long links; so that the end of each long bar is between the ends of a pair of short bars. The eyes of the long and short bars at each joint form one continuous cylindrical hole or socket, into which a bolt or pin is fitted, to connect the links together. To break the chain at a joint, by the giving way of the bolt, that bolt must be sheared across at $2\,n$ places at once. Hence, let S denote the total sectional area of the bars in a link, and d the diameter of the bolt; then $S' = 2\,n \times 0{\cdot}7854\, d^2 = 1{\cdot}5708\, n\, d^2$; and because S′ should be $= \frac{6}{5}$ S, we have

$$d = \sqrt{\frac{S}{1{\cdot}309\, n}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(12.)$$

281. **Fastenings of Timber Ties.**—In timber framing, a tie may be connected with the adjoining pieces of the frame either by having their ends abutting against notches cut in the tie (as shown at A, A, fig. 81, Article 161), or by means of bolts or pins. In either case, the tie may yield to the stress in two ways,—by being torn asunder at the place where its transverse section is least (that is, where it is notched or pierced, as the case may be),—or by having the part beyond the notch, or beyond the bolt-hole, sheared off or sheared

out, as the case may be. In order that the material may be economically used, equation 1 of Article 280 should be fulfilled, viz. :—

$$f\,\mathrm{S}=f'\,\mathrm{S}'; \text{ or } \frac{\mathrm{S}'}{\mathrm{S}}=\frac{f}{f'} \qquad (1.)$$

This condition serves to determine the distance of the notch, or of the bolt-hole, or of the nearest bolt-hole where there are more than one, from the end of the tie, in the following manner :—

Let h be the *effective* depth of the tie, left after deducting the depth of the notch, or the diameters of bolt-holes, and d the distance of the notch, or of the nearest bolt-hole, from the end of the tie; then for a notch

$$\frac{\mathrm{S}'}{\mathrm{S}}=\frac{d}{h} \therefore d=\frac{f}{f'}h; \qquad (2.)$$

and for bolt-holes, if n be their number,

$$\frac{\mathrm{S}'}{\mathrm{S}}=\frac{2\,n\,d}{h} \therefore d=\frac{f}{2\,n\,f'}h \qquad (3.)$$

In determining the number n, it is to be observed, that if *two or more bolts pierce the same layer of fibres*, the resistance to the shearing out of the part of that layer between the end of the tie and the most distant of the bolts is nearly the same as if that bolt existed alone; so that *the most distant only of such a set of bolts is to be reckoned* in using equation 3. In general, the piercing of the same layer of fibres by more than one bolt is unfavourable to economy.

SECTION 5.—*On Resistance to Direct Compression and Crushing.*

282. **Resistance to Compression,** when the limit of proof stress is not exceeded, is sensibly equal to the resistance to extension, and is expressed by the same "*modulus of elasticity*," already mentioned and explained in Articles 257, 265, 266, and 268. When that limit is exceeded, the irregular alterations undergone by the figure of the substance render the precise determination of the resistance to compression difficult, if not impossible.

283. **Modes of Crushing.—Splitting, Shearing, Bulging, Buckling, Cross-breaking.**—*Crushing*, or breaking by compression, is not a simple phenomenon like tearing asunder, but is more or less complex and varied, according to the texture of the substance. The modes in which it takes place may be classed as follows :—

I. *Crushing by splitting* (fig. 126) into a number of prismatic fragments, separated by smooth surfaces whose general direction is nearly parallel to the direction of the crushing force, is characteristic of hard homogeneous substances of a glassy texture, such as vitrified bricks.

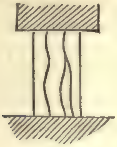

Fig. 126.

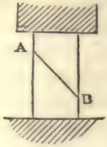

Fig. 127.

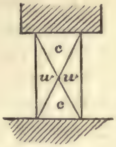

Fig. 128.

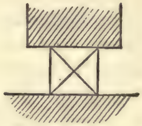

Fig. 129.

II. *Crushing by shearing or sliding* of portions of the block along oblique surfaces of separation is characteristic of substances of a granular texture, like cast iron, and most kinds of stone and brick. Sometimes the sliding takes place at a single plane surface, like A B in fig. 127; sometimes two cones or pyramids are formed, like *c, c,* in fig. 128, which are forced towards each other, and split or drive outwards a number of wedges surrounding them, like *w, w,* in the same figure. Sometimes the block splits into four wedges, as in fig. 129.

The surfaces of shearing make an angle with the direction of the crushing force, which Mr. Hodgkinson (who first fully investigated those phenomena) found to have values depending on the kind and quality of material. For different qualities of cast iron, for example, that angle ranges from 42° to 32°. The greatest intensity of shearing stress is on a plane making an angle of 45° with the direction of the crushing force; and the deviation of the plane of shearing from that angle shows that the resistance to shearing is not purely a cohesive force, independent of the normal pressure at the plane of shearing, but consists partly of a force analogous to friction, increasing with the intensity of the normal pressure.

Mr. Hodgkinson considers that in order to determine the true resistance of substances to direct crushing, experiments should be made on blocks in which the proportion of length to diameter is not less than that of 3 to 2, in order that the material may be free to divide itself by shearing. When a block which is shorter in proportion to its diameter is crushed, the friction of the flat surfaces between which it is crushed has a perceptible effect in *holding its parts together*, so as to resist their separation by shearing; and thus the apparent strength of the substance is increased beyond its real strength.

In all substances which are crushed by splitting and by shearing, the resistance to crushing considerably exceeds the tenacity, as an examination of the tables will show. The resistance of cast iron to crushing, for example, was found by Mr. Hodgkinson to be somewhat more than *six* times its tenacity.

III. *Crushing by bulging*, or lateral swelling and spreading of the block which is crushed, is characteristic of ductile and tough materials, such as wrought iron. Owing to the gradual manner in which materials of this nature give way to a crushing force, it is difficult to determine their resistance to that force exactly; that resistance is in general less, and sometimes considerably less, than the tenacity. In wrought iron, the resistance to the direct crushing of short blocks, as nearly as it can be ascertained, is from $\frac{2}{3}$ to $\frac{4}{5}$ of the tenacity.

IV. *Crushing by buckling or crippling* is characteristic of fibrous substances, under the action of a thrust along the fibres. It consists in a lateral bending and wrinkling of the fibres, sometimes accompanied by a splitting of them asunder. It takes place in timber, in plates of wrought iron, and in bars longer than those which give way by bulging. The resistance of fibrous substances to crushing is in general considerably less than their tenacity, especially where the lateral adhesion of the fibres to each other is weak compared with their tenacity. The resistance of most kinds of timber to crushing, when dry, is from $\frac{1}{2}$ to $\frac{2}{3}$ of the tenacity. Moisture in the timber weakens the lateral adhesion of the fibres, and reduces the resistance to crushing to about one-half of its amount in the dry state.

V. *Crushing by cross-breaking* is the mode of fracture of columns and struts in which the length greatly exceeds the diameter. Under the breaking load, they yield sideways, and are broken across like beams under a transverse load. This mode of crushing will be considered after the subject of resistance to bending.

284. A **Table of the Resistance of Materials to Crushing by a Direct Thrust,** in pounds avoirdupois per square inch, is given at the end of the volume. So far as that table relates to the strength of brick and stone, reference has already been made to it in Article 235. It is condensed from the experimental data given by various authorities, especially by Tredgold, Mr. Fairbairn, Mr. Hodgkinson, and Captain Fowke.

285. **Unequal Distribution of the Pressure** on a pillar arises from the line of action of the resultant of the load not coinciding with the axis of figure of the pillar, so that the *centre of pressure* of a cross section of the pillar does not coincide with its *centre of figure*, but deviates from it in a certain direction by a certain distance, which may be denoted by r_0.

In this case the strength of the pillar is diminished in the same ratio in which the mean intensity of the pressure is less than the maximum intensity; that is to say, in a ratio which may be denoted by

$$\frac{\text{mean intensity}}{\text{maximum intensity}} = \frac{p_0}{p_1}.$$

That ratio may be found with a precision sufficient for practical purposes, by considering the pressure at any cross section of the pillar as an *uniformly varying stress,* as defined in Article 94. Consequently the following is the process to be pursued :—

Find, by the methods of Article 95, the principal axes and moments of inertia of the cross section of the pillar; and thence determine the neutral axis conjugate to the direction of the deviation r_0. Let θ be the angle made by that axis with the direction of the deviation r_0; then the perpendicular distance of the centre of pressure from the neutral axis will be

$$x_0 = r_0 \sin \theta.$$

Find the moment of inertia of the cross section relatively to the neutral axis, and denote it by I; then from equations 1, 2, and 4 of Article 94, it appears that if x_1 be the *greatest perpendicular distance* of the edge of the cross section from the neutral axis in the same direction with x_0, the greatest intensity of pressure will be

$$\left.\begin{aligned} &p_1 = p_0 + a\,x_1; \\ \text{in which}\qquad &a = \frac{x_0\,\mathrm{P}}{\mathrm{I}} = x_0\,p_0 \cdot \frac{\mathrm{S}}{\mathrm{I}}; \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots (1.)$$

P being the total pressure, and S the area of the section of the pillar. Consequently the ratio required is

$$\frac{p_0}{p_1} = \frac{1}{1 + \frac{x_0\,x_1\,\mathrm{S}}{\mathrm{I}}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.)$$

Values of S, for certain symmetrical figures, and of I for the principal axes of these figures, have already been given in the table of Article 205, from which are computed the following values of the factor $\frac{x_1\,\mathrm{S}}{\mathrm{I}}$ in the denominator of the preceding formula :—

FIGURE OF CROSS SECTION.	$\frac{x_1\,\mathrm{S}}{\mathrm{I}}$.
I. Rectangle, $h\,b$; b, neutral axis, II. Square, h^2,........................	$\frac{6}{h}$.
III. Ellipse : neutral axis, b; other axis, h; IV. Circle : diameter, h,........................	$\frac{8}{h}$.

V. Hollow rectangle: outside dimensions, h, b; inside dimensions, h', b'; neutral axis, b,... $\left.\right\}\ \dfrac{6\,h\,(h\,b - h'b')}{h^3\,b - h'^3\,b'}$.

VI. Hollow square, $h^2 - h'^2$,........................ $\dfrac{6\,h}{h^2 + h'^2}$.

VII. Circular ring: diameter, outside, h; inside, h', $\dfrac{8\,h}{h^2 + h'^2}$.

286. **Limitations of the Preceding Formulæ.**—The formulæ of the preceding Article of this section have reference to direct crushing only, and are therefore limited in their application to those cases in which the pillars, blocks, or struts along which the pressure acts are not so long in proportion to their diameter as to have a sensible tendency to be crushed by bending. Those cases comprehend—

Stone and brick pillars, and blocks of ordinary proportions;

Pillars and struts of cast iron, in which the length is not more than five times the diameter, approximately;

Pillars and struts of wrought iron, in which the length is not more than ten times the diameter, approximately;

Pillars and struts of dry timber, in which the length is not more than about twenty times the diameter.

287. **Crushing and Collapsing of Tubes.**—When a hollow cylinder is exposed to a pressure from without, there is a circumferential thrust round it, whose greatest intensity takes place at the inner surface of the cylinder, and may be computed by suitably modifying the formulæ of Article 273. That is to say, let R and r denote respectively the outer and inner radii of the cylinder, q_1 the intensity of the radial pressure from without, q_0 that of the radial pressure from within, and let p_0 *now* denote, not a *tension*, but a *thrust*, viz., the maximum circumferential thrust which acts round the inner surface of the cylinder. Then reversing the signs of the second side of equation 6 of Article 273, we obtain

$$p_0 = \frac{2\,q_1\,\mathrm{R}^2 - q_0\,(\mathrm{R}^2 + r^2)}{\mathrm{R}^2 - r^2} \ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

When the pressure from within is null or insensible, this becomes

$$p_0 = \frac{2\,q_1\,\mathrm{R}^2}{\mathrm{R}^2 - r^2}; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.)$$

and *supposing the material to give way by direct crushing*, the proper ratio of the internal to the external radius is given by the equation

$$\frac{r}{\mathrm{R}} = \sqrt{1 - \frac{2 q_1}{f}}, \dots\dots\dots\dots\dots\dots\dots (3.)$$

q_1 being the working, proof, or crushing external pressure, and f the working, proof, or crushing thrust of the material, as the case may be.

This formula gives correct results for *thick hollow cylinders.* But where the thickness is small (as in the internal flues of boilers), the cylinder gives way, not by direct crushing, but by COLLAPSING, which, as it consists in an alteration of figure, is analogous to crushing by bending. According to Mr. Fairbairn's experiments, published in the *Philosophical Transactions* for 1858, the intensity of the pressure from without which makes a thin wrought iron tube collapse is inversely as the length, inversely as the radius, and directly as the power of the thickness whose index is 2·19. In most calculations for practical purposes, the *square* of the thickness may be used instead of that power. For plate iron flues, let l be the length, d the diameter, t the thickness, all in the same units of measure, and let q be the collapsing pressure in lbs. on the square inch; then

$$q = 9{,}672{,}000 \frac{t^2}{l\,d} \text{ nearly} \dots\dots\dots\dots\dots\dots (4.)$$

Mr. Fairbairn strengthens long flues by means of rings of T-iron; in which case l is the distance between two adjacent rings.

SECTION 6.—*On Resistance to Bending and Cross-Breaking.*

288. **Shearing Force and Bending Moment in General.**—It has already been shown, in Articles 141 and 142, how to determine the proportions between the resultant of the gross load of a beam and the two forces which support it,—whether those three forces are perpendicular or oblique to the beam,—and whether they are parallel or inclined to each other. In the present section those cases alone will be considered in which the loading and supporting forces are perpendicular to the beam, and parallel to each other, and in one plane; for such forces alone tend simply to bend the beam, and if sufficiently great, to break it across.

In Article 161 it has been shown how to determine the resistances exerted by the pieces of a frame which are cut by an ideal sectional plane, in terms of the forces and couples which act on one of the portions into which that plane of section divides the frame; and in Articles 162, 163, 164, and 165, that *method of sections,* as it is called, has been applied to the determination of the stresses acting along the bars of half-lattice or Warren girders and of lattice girders.

The method followed in determining the effect of a transverse load on a continuous beam is similar; except that the resistance at the plane section, which is to be determined, does not consist of a finite number of forces acting along the axes of certain bars, but of a distributed stress, acting with various intensities, and, it may be, in various directions, at different points of the section of the beam.

In what follows, the load of the beam will be conceived to consist of weights acting vertically downwards, and the supporting forces will also be conceived to be vertical. The longitudinal axis of the beam being perpendicular to the applied forces, will accordingly be horizontal. The conclusions arrived at will be applicable to cases in which the axis of the beam and the direction of the applied forces are inclined, so long as they are perpendicular to each other.

Let any point in the longitudinal axis of the beam be taken as the origin of co-ordinates; and at a given horizontal distance x from that origin, conceive a vertical section perpendicular to the longitudinal axis to divide the beam into two parts. To fix the ideas, let horizontal distances to the $\left\{\begin{matrix}\text{left}\\ \text{right}\end{matrix}\right\}$ be considered as $\left\{\begin{matrix}\text{positive}\\ \text{negative}\end{matrix}\right\}$; let vertical distances and forces in an $\left\{\begin{matrix}\text{upward}\\ \text{downward}\end{matrix}\right\}$ direction, be considered as $\left\{\begin{matrix}\text{positive}\\ \text{negative}\end{matrix}\right\}$; and let the moments of couples be $\left\{\begin{matrix}\text{positive}\\ \text{negative}\end{matrix}\right\}$ according as they are $\left\{\begin{matrix}\text{left-handed}\\ \text{right-handed}\end{matrix}\right\}$.

Let F denote the resultant of all the vertical forces, whether loading or supporting, which act on the part of the beam to the left of the vertical plane of section, and let x' be the horizontal distance of the line of action of that resultant from the origin.

If the beam is strong enough to sustain the forces applied to it, there will be a *shearing stress* whose amount is equal to F, distributed (in what manner will afterwards appear) over the given vertical section; and that shearing stress, or vertical resistance, will constitute, along with the applied force F, a couple whose moment is

$$\mathrm{M} = \mathrm{F}\,(x' - x) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

This is called the *bending moment* or *moment of flexure* of the beam at the vertical section in question; and it is resisted by the normal stress at that section, in a manner to be explained in the sequel.

If the bending moment is $\left\{\begin{matrix}\text{positive}\\ \text{negative}\end{matrix}\right\}$, it tends to make the originally straight longitudinal axis of the beam become concave $\left\{\begin{array}{l}\text{upwards}\\ \text{downwards}\end{array}\right\}$.

The determination of the magnitude and position of the resultant F consists simply in finding the resultant of a number of parallel forces in one plane, as explained in Article 44, the supporting forces having first been found by the principles of Articles 39 and 141. These processes are expressed by general formulæ as follows :—

CASE 1. *The load applied at detached points.*—Let W denote one of the weights of which the load consists ; x'' its horizontal distance from the origin ; then

$-\Sigma \cdot \mathrm{W}$ is the total load, made negative as acting downwards ; and

$-\Sigma \cdot x'' \mathrm{W}$ is its moment relatively to the origin.

Let x_1 and x_2 be the horizontal distance of the points of support from the origin, and let P_1, P_2, be the supporting forces ; then to determine those forces we have the conditions of equilibrium

$$\mathrm{P}_1 + \mathrm{P}_2 - \Sigma \cdot \mathrm{W} = 0 ;$$
$$x_1 \mathrm{P}_1 + x_2 \mathrm{P}_2 - \Sigma \cdot x'' \mathrm{W} = 0 ;$$

from which follow the equations

$$\left.\begin{aligned} \mathrm{P}_1 &= \frac{x_2 \Sigma \cdot \mathrm{W} - \Sigma \cdot x'' \mathrm{W}}{x_2 - x_1} ; \\ \mathrm{P}_2 &= \frac{x_1 \Sigma \cdot \mathrm{W} - \Sigma \cdot x'' \mathrm{W}}{x_1 - x_2}. \end{aligned}\right\} \text{................(2.)}$$

To show how the shearing force and moment of flexure at any cross section are found, let W be applied to the left of the origin, and let the plane of section, whose distance from the origin is x, lie between P_1 and P_2 ; then the force acting on the beam to the left of x will be

$$\left.\begin{array}{l} \mathrm{F} = \mathrm{P}_1 - \Sigma_x^{x_1} \cdot \mathrm{W} ; \\ \text{and the moment of flexure} \\ \mathrm{M} = (x_1 - x)\, \mathrm{P}_1 - \Sigma_x^{x_1} \cdot (x'' - x)\, \mathrm{W} ; \end{array}\right\} \text{..........(3.)}$$

the symbol $\Sigma_x^{x_1} \cdot$ denoting in each case, that the summation extends to that part of the beam only which lies between the given plane of vertical section and the point of support (if any) to the left of that plane.

CASE 2. *The load continuously distributed.*—On any indefinitely short division of the beam whose length is $d\,x$, and distance from the origin x'', let the intensity of the load per unit of length be w. Then in the equations 2 and 3, given above, it is only necessary to substitute $w\,d\,x$ for W, and the sign $\int$ for the sign Σ.

289. In **Beams Fixed at One End Only,** and loaded on the projecting portion, as in fig. 67 of Article 141, and figs. 133 to 136 of a subsequent Article, the shearing force and moment of flexure can be determined for any vertical section of the projecting part of the beam, without considering the supporting pressures.

Let the plane at which the beam is fixed be taken as the origin; let c be the length of the projecting part of the beam. The results in the cases most important in practice are given in the following table:—

Example.	Shearing Force F		Bending Moment M	
	Anywhere. F	Greatest. F_0	Anywhere. M	Greatest. M_0
I. Loaded at extreme end with W,......	$-W$	$-W$	$-(c-x)W$	$-cW$
II. Uniform load of intensity w,..........	$-w(c-x)$	$-wc$	$-\frac{w(c-x)^2}{2}$	$-\frac{wc^2}{2}$
III. Uniform load of intensity w, and additional load at extreme end W′,	$-W'-w(c-x)$	$-W'-wc$	$-W'(c-x)-\frac{w(c-x)^2}{2}$	$-W'c-\frac{wc^2}{2}$

290. In **Beams Supported at Both Ends,** and loaded on the intermediate portion, like those represented in fig. 66 of Article 141, and in figs. 138 and 140 of a subsequent Article, it is most convenient to take the *middle of the beam* as the origin of co-ordinates. Then let c denote the *half-span* of the beam, so that $2\,c$ is the *span*, or distance between the points of support; the positions of those points will be expressed by

$$x_1 = c;\ x_2 = -c;\ x_1 - x_2 = 2\,c; \ldots\ldots\ldots\ldots\ldots (1.)$$

which substitutions convert equation 2 of Article 288 into the following:—

$$\left.\begin{aligned} P_1 &= \frac{\Sigma \cdot W}{2} + \frac{\Sigma \cdot x'' W}{2\,c}; \\ P_2 &= \frac{\Sigma \cdot W}{2} - \frac{\Sigma \cdot x'' W}{2\,c}. \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (2.)$$

If the load is symmetrically distributed,

$$\Sigma \cdot x'' \, \mathrm{W} = 0,$$

and

$$\mathrm{P}_1 = \mathrm{P}_2 = \frac{\Sigma \cdot \mathrm{W}}{2} = \Sigma_0^c \cdot \mathrm{W} \ldots\ldots\ldots\ldots\ldots (2 \text{ A.})$$

The equations 3 of Article 288 also become

$$\left.\begin{array}{l} \mathrm{F} = \mathrm{P}_1 - \Sigma_x^c \cdot \mathrm{W}\,; \\ \mathrm{M} = (c-x)\,\mathrm{P}_1 - \Sigma_x^c \cdot (x''-x)\,\mathrm{W}\,; \end{array}\right\} \ldots\ldots\ldots\ldots\ldots (3.)$$

and for a symmetrically distributed load,

$$\mathrm{F} = \Sigma_0^x \cdot \mathrm{W}\,;\; \mathrm{M} = (c-x)\,\Sigma_0^c \cdot \mathrm{W} - \Sigma_x^c \cdot (x''-x)\,\mathrm{W} \ldots . (3 \text{ A.})$$

The results in the cases most important in practice are given in the following table:—

Example.	Shearing Force F		Bending Moment M	
	Anywhere. F	Greatest. F_1 or F_2	Anywhere. M	Greatest. M_0 or M''.
IV. Single load, W, in middle—				
Left of O,.......	$\frac{W}{2}$	$\frac{W}{2}$	$\frac{(c-x)W}{2}$	$\frac{cW}{2} = M_0$
Right of O,.....	$-\frac{W}{2}$	$-\frac{W}{2}$		
V. Single load, W, applied at x''—				
Left of x'', ...	$\frac{(c+x'')W}{2c}$	$\frac{(c+x'')W}{2c}$	$\frac{(c+x'')(c-x)W}{2c}$	$\frac{(c^2-x''^2)W}{2c}$ $= M''$ at x''
Right of x'',.....	$\frac{-(c-x'')W}{2c}$	$\frac{-(c-x'')W}{2c}$	$\frac{(c-x'')(c+x)W}{2c}$	
VI. Uniform load of intensity, w, ...	wx	wc	$\frac{w(c^2-x^2)}{2}$	$\frac{wc^2}{2} = M_0$

291. **Moments of Flexure in Terms of Load and Length.**—For practical purposes, it is often convenient to express the greatest bending moment of a beam in terms of the *total load*, W, and *unsupported length*, *l*, of a beam, by means of a formula of this kind,

$$\mathrm{M}_0 = m\,\mathrm{W}\,l, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

where m is a numerical factor. For beams fixed at one end, $l = c$;

for beams supported at both ends, $l = 2c =$ the span; for an uniform load, $W = wl$. Hence, comparing equation 1 with Examples I., II., IV., V., and VI. of Articles 289 and 290, we find the following values of the factor m:—

	m
I. Beam fixed at one end, loaded at the other,……	$1.$
II. Beam fixed at one end, loaded uniformly,……..	$\frac{1}{2}.$
IV. Beam supported at both ends, loaded in the middle,………………………………………	$\frac{1}{4}.$
V. Beam supported at both ends, loaded at x'' from the middle,………………………………	$\frac{1}{4}\left(1-\frac{4x''^2}{l^2}\right).$
VI. Beam supported at both ends, uniformly loaded,	$\frac{1}{8}.$

292. **Uniform Moment of Flexure.**—If a pair of equal and opposite couples, acting in the same longitudinal plane, be applied at or near the ends of a beam, the part of the beam intermediate between the portions to which the couples are applied is under the influence of an *uniform moment of flexure,* and of *no shearing force.*

An illustration of this is the condition of that part of the axle of a railway carriage which lies between the pair of wheels, if the bearings are outside of the wheels, or between the bearings if the bearings are inside of the wheels. Let W be the weight which rests on one pair of wheels; then $\frac{W}{2}$ is the weight resting on each wheel, and on each bearing. Let l be the distance from the centre of each wheel to the middle of the adjoining bearing. Then a pair of equal and opposite couples, each of the moment,

$$M = \frac{Wl}{2},$$

are applied to the two ends of the axle; and this is the uniform moment of flexure of the portion of the axle lying between the portions acted upon by the forces which constitute the couples; and the shearing force on the same portion is null.

293. **Resistance of Flexure** means, the moment of the resistance which a beam opposes to being bent or broken across; and if the beam is strong enough, that moment, at each cross section of the beam, is equal and opposite to the moment of the bending forces at the same cross section.

Let fig. 130 represent a side view of part of a beam which is of uniform cross section, and which is subjected to an uniform moment of flexure; and let fig. 130* represent the cross section of the same beam. It is self-evident that the curvature produced in the part of the beam in question must be uniform; that is to say, that any longitudinal line in the beam, such as its upper edge A A′, or its lower edge B B′, which in the free condition of the beam is straight, must be bent into an arc of a circle; and that any surface originally plane and longitudinal, and perpendicular to the plane in which the curvature takes place, such as the upper surface A A′, or the lower surface B B′, must be bent into a cylindrical form; and the cylindrical surfaces so produced will have a common axis. Any two transverse sectional planes, such as A B and A′ B′, which in the free state of the beam are parallel to each other, will have, in the curved state of the beam, positions radiating from the axis of curvature.

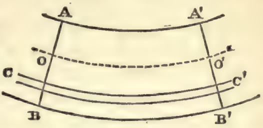

Fig. 130.

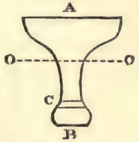

Fig. 130*.

Therefore, if the portion of the beam between the transverse planes A B, A′ B′, be conceived to be divided into layers, such as C C′, originally plane, parallel, and of equal length, these layers, in the bent condition of the beam, must have lengths proportional to their distances from the axis of curvature. The layers near the concave side of the beam, A A′, are shortened by the bending, and the layers near the convex side, B B′, lengthened; and there must be some intermediate layer which is neither lengthened nor shortened, but preserves its free length. Let O O′ be the surface originally plane, now curved, at which that layer is situated; this is called the *neutral surface* of the beam, and the line O O, fig. 130*, in which it intersects a given cross section, is called the *neutral axis* of that section.

The *direct strains*, or proportionate elongations and compressions, of the layers of the beam are proportional to their distances below and above the neutral surface; and hence, within the limits of proof stress, the *direct stresses*, or tensions and pressures, at the different points of the cross section A B, fig. 130*, *have intensities sensibly proportional to their distances from the neutral axis* O O.

Therefore the *direct stress* at each section, such as A B, whose moment constitutes the *resistance to bending*, is an *uniformly-varying stress*, as defined in Article 91; and in order that the *longitudinal resultant* of that stress may be null, the neutral axis (as shown in that Article) must traverse the *centre of gravity of the cross section* A B.

The *moment of a bending stress* has already been given in Article 92, equations 3 and 4; and the methods of determining the integrals I and K, which occur in those equations, have been explained and illustrated in Article 95.

To apply the equations of those Articles to the present purpose, let p be the intensity of the direct stress at a layer of the beam whose distance from the neutral axis is y: height above the neutral axis being considered as positive, and depth below it as negative. Then because a moment of flexure tending to make the beam concave upwards has been treated as positive, it is convenient, in order to avoid the unnecessary use of negative signs, to consider the constant ratio $\frac{p}{y}$ as positive when it is such as to give resistance to an upward moment of flexure; that is, when p is a thrust for positive values of y, and a pull for negative values; consequently, p is to be considered as $\left\{\begin{matrix}\text{positive}\\\text{negative}\end{matrix}\right\}$ according as it is a $\left\{\begin{matrix}\text{thrust.}\\\text{pull.}\end{matrix}\right\}$

This being understood, we have, for the moment of the resistance opposed by the beam to bending,

$$M = \frac{p}{y} \cdot \sqrt{(I^2 + K^2)}; \dots\dots\dots\dots\dots\dots (1.)$$

and for the angle made by the neutral axis with the direction of the axes of the bending couples,

$$\mu = -\operatorname{arc} \cdot \tan \frac{K}{I}; \dots\dots\dots\dots\dots\dots (2.)$$

I and K being found by the methods of Article 95.

In some cases, a more convenient form of equation 2 is that which gives θ, the angle made by the neutral axis with its *conjugate axis*, in which the plane of the bending forces cuts the plane of section A B, viz.:—

$$\operatorname{cotan} \cdot \theta = \frac{K}{I} \dots\dots\dots\dots\dots\dots (3.)$$

In almost every case which occurs in practice, the plane of the bending forces cuts each cross section of the beam in one or other of its *principal axes*, for which $K = 0$, $\mu = 0$, $\theta = 90°$; and then equation 1 becomes

$$M = \frac{p\,I}{y} \dots\dots\dots\dots\dots\dots (4.)$$

In beams whose transverse sections and moments of flexure are not uniform, no error appreciable in practice is produced by applying equation 4 to each cross section, and to the moment of flexure which acts upon it, as if the given section and moment belonged to an uniform beam with an uniform moment of flexure.

294. The **Transverse Strength** of a beam, ultimate, proof, or working, as the case may be, is the load required to break it across, or to produce the proof stress or the working stress, as the case may be. It is found by equating the greatest moment of flexure, expressed in terms of the load and length, as in Article 291, to the moment of resistance at the cross section where that moment of flexure acts: such moment of resistance being found from the equations of Article 293, by putting for p the ultimate, proof, or working direct stress of the material, as the case may be, and for y the distance from the neutral axis to the point in the given cross section where the limiting stress p is first attained. That point will be at the $\left\{\begin{matrix}\text{concave}\\ \text{convex}\end{matrix}\right\}$ side of the beam, according as the material gives way most readily to $\left\{\begin{matrix}\text{pressure.}\\ \text{tension.}\end{matrix}\right\}$

In fig. 131, A represents a beam of a granular material, like cast iron, giving way by the crushing of the concave side, out of which a sort of wedge is forced. B represents a beam giving way by the tearing asunder of the convex side.

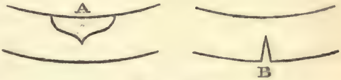

Fig. 131.

In a beam symmetrical above and below, or otherwise of such a form that the neutral axis is at the middle of the depth of the cross section, if h is that depth,

$$y = \pm \frac{h}{2},$$

and the limiting value of p is the resistance to pressure or to tension, whichever is least.

For other forms of section, let

$$\begin{aligned} y &= y_a \text{ for the concave side; and} \\ &= y_b \text{ for the convex side;} \end{aligned}$$

and let the limiting stresses be

$$\begin{aligned} p &= f_a \text{ for pressure; and} \\ &= f_b \text{ for tension;} \end{aligned}$$

then the beam will give way by $\left\{\begin{matrix}\text{crushing}\\ \text{tearing}\end{matrix}\right\}$ according as $\frac{y_a}{y_b}$ is

$$\left\{\begin{matrix}\text{greater}\\ \text{less}\end{matrix}\right\} \text{ than } \qquad \frac{f_a}{f_b} \ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

This point having been determined, the equation from which the strength of the beam may be found is

$$M_0 = m W l = \frac{f I}{y} \text{ ....(2.)}$$

When the *breaking* load is in question, the co-efficient f is what is called the *modulus of rupture* of the material. It does not always agree with the resistance of the same material to direct crushing or direct tearing, but has a special value, which can be found by experiments on cross-breaking only. One of the causes of this phenomenon is probably the fact, already stated in Article 257, that the resistance of a material to a direct stress is increased by preventing or diminishing the alteration of its transverse dimensions; and another cause may be the fact, that the strength of masses of metal, especially when cast, is greater in the external layer, or *skin*, than in the interior of the mass. When a bar is directly torn asunder, the strength indicated is that of the weakest part of the mass, which is in the centre; when it is broken across, the strength indicated is that either of the skin, which is the strongest part, or of some part near the skin (See the Article 296).

When the *proof load* or *working load* is in question, the co-efficient f is the modulus of rupture divided by a suitable *factor of safety*, as to which see Article 247.

295. **Transverse Strength in Terms of Breadth and Depth.**—From the principles explained in Article 95, it is obvious that the moments of inertia, I, of similar sections are to each other as the breadths, and as the cubes of the depths. If, therefore, b be the breadth, and h the depth, of the rectangle circumscribing the cross section of a given beam at the point where the moment of flexure is greatest, we may put

$$I = n' b h^3 \text{ ....(1.)}$$

n' being a numerical factor depending on the form of the section. It is also evident, that for similar figures, the values of y are as the depths; so that we may put

$$y = m' h \text{ ....(2.)}$$

m' being another numerical factor depending on the form of section. If the section is symmetrical above and below, $m' = \frac{1}{2}$. Thus it appears, that the *resistances of flexure of similar cross sections are as their breadths and as the squares of their depths*, and that equation 2 of Article 294, which expresses equality between the greatest moment of flexure, as stated in terms of the load and length, and the resistance of the cross section where that moment acts, is equivalent to the following:—

$$M_0 = m W l = n f b h^2 \text{ ....(3.)}$$

where $n = \frac{n'}{m'}$ is a numerical factor depending on the form of cross section of the beam, and m is the numerical factor depending on the mode of distribution of the loading and supporting forces, of which examples have been given in Article 291.

The following table gives examples of the values of the three factors, n', m', n, for some of the more usual forms of cross section:

FORM OF CROSS SECTIONS.	$n' = \frac{\mathrm{I}}{b h^3}$.	$m' = \frac{y}{h}$.	$n = \frac{\mathrm{I}}{y b h^2}$.
I. Rectangle $b\,h$, ... (including square)	$\frac{1}{12}$	$\frac{1}{2}$	$\frac{1}{6}$
II. Ellipse— Vertical axis h, ... Horizontal axis b, ... (including circle)	$\frac{\pi}{64} = \frac{1}{20 \cdot 4}$ $= 0 \cdot 0491$	$\frac{1}{2}$	$\frac{\pi}{32} = \frac{1}{10 \cdot 2}$ $= 0 \cdot 0982$
III. Hollow rectangle, $b\,h$ —$b'\,h'$; also I-formed section, where b' is the sum of the breadths of the lateral hollows,...	$\frac{1}{12}\left(1 - \frac{b'\,h'^3}{b\,h^3}\right)$	$\frac{1}{2}$	$\frac{1}{6}\left(1 - \frac{b'\,h'^3}{b\,h^3}\right)$
IV. Hollow square— $h^2 - h'^2$..................	$\frac{1}{12}\left(1 - \frac{h'^4}{h^4}\right)$	$\frac{1}{2}$	$\frac{1}{6}\left(1 - \frac{h'^4}{h^4}\right)$
V. Hollow ellipse,.........	$\frac{1}{20 \cdot 4}\left(1 - \frac{b'\,h'^3}{b\,h^3}\right)$	$\frac{1}{2}$	$\frac{1}{10 \cdot 2}\left(1 - \frac{b'\,h'^3}{b\,h^3}\right)$
VI. Hollow circle, ...	$\frac{1}{20 \cdot 4}\left(1 - \frac{h'^4}{h^4}\right)$	$\frac{1}{2}$	$\frac{1}{10 \cdot 2}\left(1 - \frac{h'^4}{h^4}\right)$

In using the equation 3 for any of the purposes to which it may be applied—such as computing the strength of a beam of which the dimensions and figure are given, or fixing the transverse dimensions of a beam of which the strength, length, and figure are given —care is to be taken to use the *same unit of measure* throughout the calculation; that is to say, when the transverse dimensions, as is usually the case, are stated in inches, and the co-efficient of strength f in pounds on the square inch, the length l should be stated in inches also. This caution is necessary on account of that diversity of units which is characteristic of British measures.

296. **A Table of the Resistance of Materials to Breaking Across** is given at the end of the volume. It gives values of the modulus of rupture, being that for which the co-efficient f stands in Article

294, equation 2, and in Article 295, equation 3, when $m\,W\,l$ is the breaking moment. It will be observed, that this modulus is, for most materials, intermediate between the tenacity and the resistance to direct crushing.

297. **Cast Iron Beams.**—The values of the modulus of rupture for cast iron require special remark. It had for some time been known, that while the direct tenacity of cast iron (as determined by Mr. Hodgkinson) is on an average 16,500 lbs. per square inch, the modulus of rupture of rectangular cast iron beams is on an average about 40,000 lbs. per square inch, or two and a-half times as great. This was supposed to be accounted for by the assumption, that the stress on a cross section of a cast iron beam is not an uniformly varying stress, and that the neutral axis does not traverse the centre of gravity of the section. But in 1855, Mr. William Henry Barlow, by experiments of which an account is published in the *Philosophical Transactions* for that year, showed,—in the first place, that the stress is an uniformly varying stress, and that the neutral axis, in symmetrical sections at all events, traverses the centre of gravity of the section,—and in the second place, that the modulus of rupture has various values, ranging from the mere direct tenacity of the iron up to about two and a-third times that tenacity, according to the figure of the cross section of the beam.

The beams on which the experiments of Mr. Barlow, now referred to, were made, were in some cases of a solid rectangular section, and in other cases of an open-work rectangular section, consisting of equal rectangular upper and lower horizontal bars, with alternate open spaces and vertical connecting bars between. As far as those experiments went, they were in accordance with the following empirical formula:—

$$f = f_0 + f' \cdot \frac{H}{h}, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

where f is the modulus of rupture of the beam in question; f_0, the direct tenacity of the iron of which it is made; f', a co-efficient determined empirically; and $\frac{H}{h}$, the ratio which the *depth of solid metal* H in the cross section of the beam bears to the *total depth of section h.* The following were the values of the constants for the cast iron experimented on:—

$$\text{Direct tenacity,}\quad \left.\begin{aligned} f_0 &= 18{,}750 \text{ lbs. per square inch;} \\ f' &= 23{,}000 \text{ lbs. per square inch;} \\ &= 1\tfrac{1}{4} f_0 \text{ nearly.} \end{aligned}\right\} \ldots\ldots\ldots (2.)$$

Mr. Barlow has since made further experiments on cast iron beams of various forms of section, and also experiments on wrought iron beams, showing, though not so conclusively, variations in the modulus of rupture of wrought iron analogous to those which have been proved to exist in the case of cast iron; but as those further experiments, though communicated to the Royal Society, have not yet been published in detail, it would be premature to make remarks on them here.

Mr. Barlow has proposed a theory of those phenomena, to the effect that the curvature of the layers of the beam produces a peculiar kind of resistance to bending, distinct from that which arises from the direct elasticity; and he adduces in support of that theory the fact that the additional strength represented by the second term of equation 1 increases with the ultimate curvature of the beam; that is, its curvature just before breaking. Another conceivable theory has already been mentioned in Article 294, viz., that the strength of a metal bar, and in particular of a cast iron bar, is greatest at the *skin*, and diminished towards the interior; that the tenacity found by directly tearing a bar asunder, f_0, is the tenacity of the interior; that the modulus of rupture of a solid rectangular beam, $f_0 + f'$, is the tenacity of the skin, and that the modulus of rupture of an open-work beam is the tenacity at a distance from the skin depending on the form of section. But until conclusive experimental data shall have been obtained, all theories on the subject must be considered as provisional only.

298. The **Section of Equal Strength for Cast Iron Beams** was first proposed by Mr. Hodgkinson, in consequence of his discovery of the fact, that the resistance of cast iron to direct crushing is more than six times its resistance to tearing. It consists, as in fig. 132, of a lower flange B, an upper flange A, and a vertical web connecting them. The sectional area of the lower flange, which is subjected to tension, is nearly six times that of the upper flange, which is subjected to thrust. In order that the beam, when cast, may not be liable to crack from unequal cooling, the vertical web has a thickness at its lower side equal to that of the lower flange, and at its upper side equal to that of the upper flange.

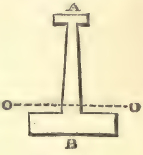

Fig. 132.

The tendency of beams of this class to break by tearing of the lower flange is slightly greater than the tendency to break by crushing of the upper flange; and their modulus of rupture is equal, or nearly equal, to the direct tenacity of the iron of which they are made, being, on an average of different kinds of iron, 16,500 lbs. per square inch.

Let the areas and depths of the parts of which the section in fig. 132 consists be denoted as follows :—

	Areas.	Depths.
Upper flange,........................	A_1,	h_1.
Lower flange,..............	A_2,	h_2.
Vertical web,........................	A_3,	h_3.
Totals,...	$A_1 + A_2 + A_3 = A$,	$h_1 + h_2 + h_3 = h$.

No appreciable error will arise from treating the section of the vertical web as rectangular instead of trapezoidal. The height of the neutral axis above the lower side of this section is

$$y_b = \frac{h}{2} - \frac{(h_3 + h_1)\,A_2 - (h_2 + h_3)\,A_1 - (h_2 - h_1)\,A_3}{2A} \text{.........(1.)}$$

Then by applying the formula of Article 95, Example VI., to this case, the moment of inertia of the section is found to be as follows:—

$$I = \frac{A_1 h_1^2 + A_2 h_2^2 + A_3 h_3^2}{12} + \frac{1}{4A}\left\{ A_1 A_2 (h_1 + h_2 + 2 h_3)^2 \right.$$
$$\left. + A_1 A_3 (h_1 + h_3)^2 + A_2 A_3 (h_2 + h_3)^2 \right\};\text{.........(2.)}$$

and the strength of the beam is expressed by the equation

$$M_0 = m W l = \frac{f_b I}{y_b} \text{........................(3.)}$$

It is seldom necessary, however, to use the formulæ 1 and 2 in all their complexity; the following approximate formula being usually sufficiently near the truth for practical purposes, and its error being on the safe side. Let h' be the depth from the middle of the upper flange to the middle of the lower flange; then

$$M_0 = m W l = f_b h' A_2 \text{......................(4.)}$$

299. **Beams of Uniform Strength** are those in which the dimensions of the cross section are varied in such a manner, that its ultimate or proof resistance bears at each point of the beam the same proportion to the moment of flexure. That resistance, for figures of the same kind, being proportional to the breadth and to the square of the depth, can be varied either by varying the breadth, the depth, or both. The

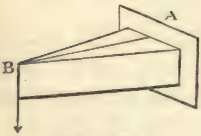

Fig. 133.

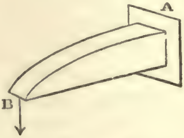

Fig. 134.

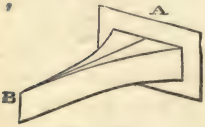

Fig. 135.

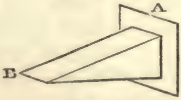

Fig. 136.

law of variation depends upon the mode of variation of the moment of flexure of the beam from point to point, and this depends on the distribution of the load and of the supporting forces, in a way which has been exemplified in Articles 289 and 290. When the depth of the beam is made uniform, and the breadth varied, the vertical longitudinal section is rectangular, and the plan is of a figure depending on the mode of variation of the breadth. When the breadth of the beam is made uniform, and the depth varied, the plan is rectangular, and the vertical longitudinal section is of a figure depending on the mode of variation of the depth. The following table gives examples of the results of those principles :—

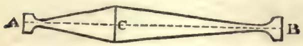

Fig. 137.

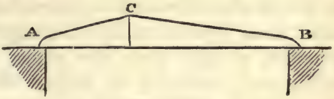

Fig. 138.

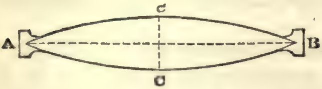

Fig. 139.

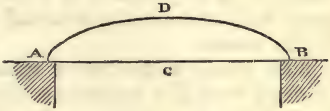

Fig. 140.

Mode of Loading and Supporting.	$b\,h^2$, proportional to	Depth h constant; Figure of Plan.	Breadth b constant; Figure of Vertical Longitudinal Section.
I. (Figs. 133, 134). Fixed at A, loaded at B, ...	Distance from B.	Triangle, apex at B, fig. 133.	Parabola, vertex at B, fig. 134.
II. (Figs. 135, 136). Fixed at A, uniformly loaded,...	Square of distance from B.	Pair of parabolas, vertices touching each other at B, fig. 135.	Triangle, apex at B, fig. 136.
III. (Figs. 137, 138). Supported at A and B, loaded at C, ...	Distance from adjacent point of support.	Pair of triangles, common base at C, apices at A and B, fig. 137.	Pair of parabolas, vertices at A and B, meeting at C, fig. 138.
IV. (Figs. 139, 140). Supported at A and B, uniformly loaded, ...	Product of distances from points of support.	Pair of parabolas, vertices at C, C, in middle of beam; common base A B, fig. 139.	Ellipse A D B, fig. 140.

The formulæ and figures for a *constant depth* are applicable to the breadths of the flanges of the ⊥-shaped girders described in Article 298. In applying the principles of this Article, it is to be borne in mind, that the *shearing force* has not yet been taken into account; and that, consequently, the figures described in the above table require, at and near the places where they taper to edges, some additional material to enable them to withstand that force. In figs. 137 and 139, such additional material is shown, disposed in the form of projections or palms at the points of support, which serve both to resist the shearing force, and to give lateral steadiness to the beams.

300. **Proof Deflection of Beams.**—Reverting to fig. 130, it is evident that if α represents the proportionate elongation of the layer C C′, whose distance from the neutral surface O O′ is y, and if r be the radius of curvature of the neutral surface, we must have

$$1 : 1 + \alpha : : r : r + y;$$

and consequently, the radius of curvature is

$$r = \frac{y}{\alpha};$$

and the *curvature*, which is the reciprocal of the radius of curvature, is expressed by the equation

$$\frac{1}{r} = \frac{\alpha}{y}.$$

Let p be the direct stress at the layer C C′, and E the *modulus of elasticity* of the material; then $\alpha = \frac{p}{\mathrm{E}}$, and consequently, the curvature has the following values :—

$$\frac{1}{r} = \frac{p}{\mathrm{E}\,y} = \frac{\mathrm{M}}{\mathrm{E\,I}}, \text{......................} (1.)$$

the second value being deduced from the first by means of equation 4 of Article 293.

When the quantity $\frac{p}{y} = \frac{\mathrm{M}}{\mathrm{I}}$ varies for different points of the beam, the curvature varies also.

Suppose now that the beam is under its *proof load*, and let M_0 denote the greatest moment of flexure arising from that load, I_0 the moment of inertia of the cross section at which that moment acts, and y_0 the distance from the neutral axis of that section to the layer where the limiting intensity f of the stress is attained. Then the curvature will be,

$$\left.\begin{array}{ll}\text{at the section of greatest stress,} & \dfrac{1}{r_0} = \dfrac{f}{\mathrm{E}\, y_0} = \dfrac{\mathrm{M}_0}{\mathrm{E\, I}_0}; \\ \text{at any other section,} & \dfrac{1}{r} = \dfrac{\mathrm{M}}{\mathrm{E\, I}} = \dfrac{f}{\mathrm{E}\, y_0} \cdot \dfrac{\mathrm{M\, I}_0}{\mathrm{I\, M}_0}. \end{array}\right\} \dots\dots\dots(2.)$$

The exact integration of this equation for slender springs, in certain cases, will be considered in a subsequent Article. For beams it is integrated approximately in the following manner:—

Let the middle of the neutral axis of the section of greatest stress be taken as the origin of co-ordinates, and represented by A in figs. 141 and 142. For a beam supported at both ends and symmetrically loaded, A is in the middle of the beam (fig. 141). For a beam fixed at one end and projecting, A is at the fixed end (fig. 142). Let the beam be so fixed or supported that at this point its neutral surface shall be horizontal, and let a horizontal tangent, A X C, to that surface at that point be taken as the axis of abscissæ. Let $\overline{\mathrm{A\,C}}$, the horizontal distance from the origin to one end of the beam, be denoted by c, which, as in Articles 289 and 290, is the length of the projecting portion of a beam fixed at one end, and the half-span of a beam supported at both ends and symmetrically loaded. Let $\overline{\mathrm{A\,X}}$, the abscissa of any other point in the beam $= x$. Let A B D be the curved form assumed by the neutral surface when the beam is bent, which form, in a beam supported at both ends, is concave upwards, as in fig. 141, and in a beam fixed at one end concave downwards, as in fig. 142. Let $\overline{\mathrm{X\,B}} = v$ be the ordinate of any point B in the curve A B D; being the difference of level between that point and the origin A. Let $\overline{\mathrm{C\,D}} = v_1$ be the greatest ordinate: this is what is termed *the deflection.*

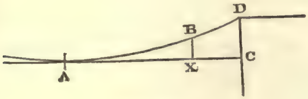

Fig. 141.

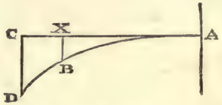

Fig. 142.

The *inclination* of the beam at any point B, is expressed by the equation

$$i = \text{arc tan}\, \frac{d\,v}{d\,x};$$

and the *curvature*, being the rate of variation of the inclination in a given length of the curve, is expressed by

$$\frac{1}{r} = \frac{d\,i}{d\,s} = \frac{d\,i}{d\,x\sqrt{1+\frac{d\,v^2}{d\,x^2}}}.$$

But in cases which occur in practice, the curvature of the beam is so slight, that the arc i is sensibly equal to its tangent, the *slope* $\frac{d\,v}{d\,x}$; and the elementary arc $d\,s$ is sensibly equal to its horizontal projection $d\,x$; so that the following equations may be used without sensible error :—

$$\left.\begin{array}{ll} \text{Slope,} & i = \frac{d\,v}{d\,x}; \\ \text{Curvature,} & \frac{1}{r} = \frac{d\,i}{d\,x} = \frac{d^2\,v}{d\,x^2}. \end{array}\right\} \dots\dots\dots\dots(3.)$$

Therefore, when the curvature at each point is given by equation 2, the slope and the ordinate are to be found by two successive integrations, as shown by the following equations :—

$$\left.\begin{array}{ll} \text{Slope,} & i = \int_0^x \frac{d\,x}{r} = \frac{f}{\mathrm{E}\,y_0} \cdot \int_0^x \frac{\mathrm{M\,I_0}}{\mathrm{I\,M_0}} \cdot d\,x; \\ \text{Ordinate,} & v = \int_0^x i\,d\,x = \frac{f}{\mathrm{E}\,y_0} \cdot \int_0^x \int_0^x \frac{\mathrm{M\,I_0}}{\mathrm{I\,M_0}}\,d\,x^2. \end{array}\right\} \dots\dots(4.)$$

The *greatest slope* i_1—that is, the slope at D—and the *deflection* or greatest ordinate v_1, are found by performing the *complete* integrations between the limits $x = 0$ and $x = c$.

[Readers who are not familiar with the integral calculus are referred to Article 81 for explanations of the nature of the process of integration.]

In both the integrals of the formulæ 4, the quantity $\frac{\mathrm{M\,I_0}}{\mathrm{I\,M_0}}$ is a *numerical ratio* depending on the mode of distribution of the loading and supporting forces, and the mode of variation of the section of the beam. Hence it is evident that we must have the complete integrals

$$\int_0^c \frac{\mathrm{M\,I_0}}{\mathrm{I\,M_0}} \cdot d\,x = m''\,c;\ \int_0^c \int_0^x \frac{\mathrm{M\,I_0}}{\mathrm{I\,M_0}} \cdot d\,x^2 = n''\,c^2; \dots\dots(5.)$$

where m'' and n'' are two *numerical factors* depending on the distribution of the forces and the figure of the beam; so that the greatest slope and the deflection are given by the equations

$$i_1 = \frac{m'' f c}{\mathrm{E}\, y_0};\ v_1 = \frac{n'' f c^2}{\mathrm{E}\, y_0} \ldots\ldots\ldots\ldots\ldots\ldots (6.)$$

For beams of similar figures, and similarly loaded and supported, y_0 is as the depth, and c as the length; hence, for such beams, the *greatest slope under the proof load is directly as the length, and inversely as the depth; and the proof deflection is directly as the square of the length, and inversely as the depth.*

The following table gives the values of the factors m'' and n'' for some of the more ordinary cases of beams of *uniform section,* in which the ratio $\frac{\mathrm{M\,I}_0}{\mathrm{I\,M}_0}$, being simply equal to $\frac{\mathrm{M}}{\mathrm{M}_0}$, depends on the distribution of the load alone, and may be found by the aid of the tables of Articles 289 and 290.

	$\frac{\mathrm{M}}{\mathrm{M}_0}$	m''	n''
I. Constant moment of flexure,	1	1	$\frac{1}{2}$
FIXED AT ONE END.			
II. Loaded at extreme end, ...	$1-\frac{x}{c}$	$\frac{1}{2}$	$\frac{1}{3}$
III. Uniformly loaded,............	$\left(1-\frac{x}{c}\right)^2$	$\frac{1}{3}$	$\frac{1}{4}$
SUPPORTED AT BOTH ENDS.			
IV. Loaded in the middle, ...	$1-\frac{x}{c}$	$\frac{1}{2}$	$\frac{1}{3}$
V. Uniformly loaded,............	$1-\frac{x^2}{c^2}$	$\frac{2}{3}$	$\frac{5}{12}$

For a beam of *uniform strength and uniform depth,* the quantity $\frac{\mathrm{M}}{\mathrm{I}}$ is constant; hence in every such beam, in what manner soever it may be supported and loaded, the curvature is uniform, as in the case of Example I. of the above table. For a beam of *uniform strength and uniform breadth,* the quantity $\frac{\mathrm{M}\,h}{\mathrm{I}}$ is constant; and therefore in such beams,

$$\frac{\mathrm{M\,I}_0}{\mathrm{I\,M}_0} = \frac{h_0}{h}; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7.)$$

h_0 being the depth at the section of greatest bending moment, and h the depth at any other section. The following table shows some of the consequences of these principles :—

	$\frac{\mathrm{M}\,\mathrm{I}_0}{\mathrm{I}\,\mathrm{M}_0}$	m''	n''
VI. Uniform strength and uniform depth,....	1	1	$\frac{1}{2}$
VII. Uniform strength, uniform breadth; fixed at one end, loaded at the other,................	$\sqrt{\frac{c}{c-x}}$	2	$\frac{2}{3}$
VIII. Uniform strength, uniform breadth; supported at both ends, loaded in the middle,..	$\sqrt{\frac{c}{c-x}}$	2	$\frac{2}{3}$
IX. Uniform strength, uniform breadth; fixed at one end, uniformly loaded, ...	$\frac{c}{c-x}$	Infinite.	1
X. Uniform strength, uniform breadth; supported at both ends, uniformly loaded,......	$\frac{c}{\sqrt{c^2-x^2}}$	$\frac{\pi}{2}=1{\cdot}5708$	$\frac{\pi}{2}-1=0{\cdot}5708$

It is to be borne in mind, that the values of m'' and n'' for beams of uniform strength, as given in the above table, are somewhat less than those which occur in practice, because, in computing the table, no account has been taken of the additional material which is placed at the ends of such beams, in order to give sufficient resistance to shearing.

The error thus arising applies chiefly to m'', the factor for the maximum slope. For the factor for the deflection, n'', the error is inconsiderable, as experiment has shown.

301. **Deflection found by Graphic Construction.**—The great length of the radii of curvature, which are the reciprocals of the curvatures given by equation 2 of Article 300, and the smallness of the ordinates of the curve of the neutral surface, in all cases which occur in practice, render it neither practicable nor useful to draw the figure of that curve in its natural proportions. But the following process, invented, so far as I am aware, by Mr. C. H. Wild, enables a diagram to be drawn, which represents, with a near approach to

accuracy, that curve, *with its vertical dimensions exaggerated*, so as to show conspicuously the slopes and ordinates

Compute, by equation 2 of Article 300, the radii of curvature for a series of equi-distant points in the beam. Diminish all those radii in any proportion which may be convenient, and draw a curve composed of small circular arcs with the diminished radii. Then in the same ratio that the radii, as compared with the horizontal scale of the drawing, are diminished, will the vertical scale of the drawing, according to which the ordinates are shown, be exaggerated.

302. **The Proportion of the Greatest Depth of a Beam to the Span** is so regulated, that its greatest deflection shall not exceed a certain proportion of the span which experience has shown to be consistent with convenience. That proportion, from various examples, appears to be—

$$\text{For the working load, } \frac{v_1}{2\,c} = \text{from } \frac{1}{600} \text{ to } \frac{1}{1200}.$$

$$\text{For the proof load, … } \frac{v_1}{2\,c} = \text{from } \frac{1}{200} \text{ to } \frac{1}{600}.$$

The determination of the proportion, $\frac{h_0}{2\,c}$, of the greatest depth of the beam to the span, so as to give the required stiffness, is effected by the aid of equation 6 of Article 300, from which we obtain

$$\frac{v_1}{2\,c} = \frac{n'' f c}{2\,\mathrm{E}\,y_0}.$$

Now $y_0 = m' h_0$, m' being a numerical factor, which for symmetrical sections is $\frac{1}{2}$; and consequently the required ratio is given by the equation

$$\frac{h_0}{2\,c} = \frac{y_0}{2\,m' c} = \frac{n'' f c}{2\,m'\,\mathrm{E}\,v_1} = \frac{n'' f}{4\,m'\,\mathrm{E}} \cdot \frac{2\,c}{v_1}, \text{ ....(1.)}$$

an expression consisting of three factors: a factor, $\frac{n''}{4\,m'}$, depending on the distribution of the load and the figure of the beam; a factor, $\frac{2\,c}{v_1}$, being the prescribed ratio of the span to the deflection; and a factor, $\frac{f}{\mathrm{E}}$, being the *proof* strain, or the *working* strain, of the material, as the case may be.

To illustrate this, let the beam be under its *working load*, uniformly distributed, and let it be of uniform section, alike above and below. Then $n'' = \frac{5}{12}$, $m' = \frac{1}{2}$. Let $\frac{2c}{v_1} = 1000$ be the prescribed ratio of the span to the working deflection. Let the material be wrought iron, for which $\frac{1}{3000}$ is a safe value for the working strain $\frac{f}{\mathrm{E}}$. Then

$$\frac{h_0}{2c} = \frac{5}{24} \cdot \frac{1000}{3000} = \frac{5}{72} = \frac{1}{14 \cdot 4};$$

which is very nearly the average proportion of depth to span adopted for wrought iron girders in practice.

303. The **Slope and Deflection of a Beam under any Load** are given by the following formulæ :—

$$\left.\begin{aligned} i' &= \int \frac{dx}{r} = \frac{1}{\mathrm{E}} \int \frac{\mathrm{M}}{\mathrm{I}} dx, \\ v' &= \int\int \frac{dx^2}{r} = \frac{1}{\mathrm{E}} \int\int \frac{\mathrm{M}}{\mathrm{I}} dx^2 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

To integrate these equations, it is only necessary to substitute for the constant factor $\frac{f}{y_0}$, in the equations 4, 5, 6, Article 300, its equivalent $\frac{\mathrm{M}'_0}{\mathrm{I}_0}$, M'_0 being now not the *proof* moment of flexure, but the actual moment of flexure at the point where the beam is horizontal; that is to say,

$$\text{Greatest slope } i'_1 = \frac{m'' \mathrm{M}'_0 c}{\mathrm{E}\,\mathrm{I}_0}; \text{ deflection } v'_1 = \frac{n'' \mathrm{M}'_0 c^2}{\mathrm{E}\,\mathrm{I}_0} \ldots (2.)$$

m'' and n'' being factors depending on the distribution of the load, and having the values given in the table of Article 300. Now the value of the moment of flexure is given in terms of the load and length by equation 1 of Article 291, and the ensuing table, viz., $\mathrm{M}_0 = m \mathrm{W} l$; and the value of I_0, in terms of the dimensions of the rectangle circumscribing the cross section, is given by equation 1 of Article 295, and the ensuing table, viz., $\mathrm{I}_0 = n' b h^3$; hence the above equations 2 become

$$i'_1 = \frac{m'' m \mathrm{W} l c}{n' \mathrm{E} b h^3}; \; v'_1 = \frac{n'' m \mathrm{W} l c^2}{n' \mathrm{E} b h^3} \ldots\ldots\ldots\ldots\ldots (3.)$$

Moreover, $l = c$, or $= 2c$, according as the beam is fixed at one end only, or supported at both; so that if m''', n''', be a pair of numerical factors, whose values are, for beams fixed at one end only,

$$m''' = m''m\,;\ n''' = n''m\,;$$

and for beams supported at both ends,

$$m''' = 2\,m''m\,;\ n''' = 2\,n''m\,;$$

the equations 3 become

$$i'_1 = \frac{m'''\,\mathrm{W}\,c^2}{n'\,\mathrm{E}\,b\,h^3};\ v'_1 = \frac{n'''\,\mathrm{W}\,c^3}{n'\,\mathrm{E}\,b\,h^3} \ldots\ldots\ldots\ldots\ldots\ldots (4.)$$

Whence it appears, that the deflections of similar beams under equal loads are as the *cubes of their lengths,* and *inversely as their breadths and the cubes of their depths.*

The values of $n' = \frac{\mathrm{I}_0}{b\,h^3}$, for the ordinary forms of cross section, are given in the table of Article 295. The following table gives the values of m''' and n''' for different modes of loading and supporting, for beams of uniform cross section, and for beams of uniform strength:—

A. UNIFORM CROSS SECTION.	m''' Factor for Slope.	n''' Factor for Deflection.
I. Fixed at one end, loaded at the other,	$\frac{1}{2}$	$\frac{1}{3}$.
II. Fixed at one end, loaded uniformly,	$\frac{1}{6}$	$\frac{1}{8}$.
III. Supported at both ends, loaded in the middle,	$\frac{1}{4}$	$\frac{1}{6}$.
IV. Supported at both ends, uniformly loaded,	$\frac{1}{6}$	$\frac{5}{48}$.
B. UNIFORM STRENGTH AND UNIFORM DEPTH.		
V. Fixed at one end, loaded at the other,	1	$\frac{1}{2}$.
VI. Fixed at one end, loaded uniformly,	$\frac{1}{2}$	$\frac{1}{4}$.
VII. Supported at both ends, loaded in the middle,	$\frac{1}{2}$	$\frac{1}{4}$.
VIII. Supported at both ends, loaded uniformly,	$\frac{1}{4}$	$\frac{1}{8}$.

C. UNIFORM STRENGTH AND UNIFORM BREADTH.	m''' Factor for Slope.	n''' Factor for Deflection.
IX. Fixed at one end, loaded at the other,…….	2 ……	$\frac{2}{3}$.
X. Fixed at one end, uniformly loaded,…….	Infinite. ……	$\frac{1}{2}$.
XI. Supported at both ends, loaded in the middle,	1 ……	$\frac{1}{3}$.
XII. Supported at both ends, uniformly loaded,	0·3927 …	0·1427.

304. **Deflection with Uniform Moment.**—In Article 292 the case has already been described, in which a beam or bar of uniform section has a pair of equal and opposite couples in the same plane applied to its ends, and the same case is the first given in the table of Article 300. In this case, M and I are constants, $m'' = 1$, and $n'' = \frac{1}{2}$; and accordingly, if c be the length of the part of the beam under consideration, and i'_1 the slope, and v'_1 the deflection, of one end relatively to a tangent at the other,

$$i'_1 = \frac{\mathrm{M}\,c}{\mathrm{E\,I}}; \quad v'_1 = \frac{\mathrm{M}\,c^2}{2\,\mathrm{E\,I}}.$$

305. The **Resilience or Spring of a Beam** is the *work performed* in bending it to the proof deflection. This, if the load is concentrated at or near one point, is the product of half the proof load into the proof deflection; that is to say,

$$\frac{\mathrm{W}\,v_1}{2} \text{..............................} (1.)$$

If the load is distributed, the length of the beam is to be divided into a number of small elements, and half the proof load on each element multiplied by the distance through which that element is moved during the proof deflection of the beam. Let u be that distance; then for beams fixed at one end,

$$\left.\begin{array}{r} u = v; \\ \text{and for beams supported at both ends,} \\ u = v_1 - v. \end{array}\right\} \text{.......................} (2.)$$

Let $d\,x$ be the length of an element of the beam; w the intensity of the load on it, per unit of length; then the resilience is

$$\frac{1}{2}\int u w \cdot d x \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.)$$

The cases in which the determination of resilience is most useful in practice are those in which the load is applied at one point.

Let the beam be fixed at one end and loaded at the other, c being the length of its projecting part. Then by Article 295, equation 3 (observing that $m = 1$, $l = c$),

$$\mathrm{W} = \frac{n f b h^2}{c},$$

(n being given by the table of Article 295), and by Article 300, equation 6,

$$v_1 = \frac{n'' f c^2}{\mathrm{E}\, y_0} = \frac{n'' f c^2}{m' \,\mathrm{E}\, h},$$

(n'' being given by the table of Article 300, and m' by that of Article 295). Consequently,

$$\textit{Resilience} = \frac{\mathrm{W}\, v_1}{2} = \frac{n\, n''}{2\, m'} \cdot \frac{f^2}{\mathrm{E}} \cdot c\, b\, h \ldots\ldots\ldots\ldots (4.)$$

It will be observed that this expression consists of three factors, viz.:—

(1.) The volume of the prism circumscribed about the beam, $c\, b\, h$.

(2.) A *Modulus of Resilience*, $\frac{f^2}{\mathrm{E}}$, of the kind already mentioned in Article 266.

(3.) A numerical factor, $\frac{n\, n''}{2\, m'}$; in which n and m' (Article 295) depend on the form of cross section of the beam, and n'' (Article 300) on the form of longitudinal section and of plan. The following are values of this compound factor for a *rectangular cross section*, for which $n = \frac{1}{6}$, $m' = \frac{1}{2}$, and therefore $\frac{n\, n''}{2\, m'} = \frac{n''}{6}$:—

	$\frac{n''}{6}$.
I. Uniform breadth and depth,..	$\frac{1}{18}$.
II. Uniform strength, uniform depth,................................	$\frac{1}{12}$.
III. Uniform strength, uniform breadth,............................	$\frac{1}{9}$.

If a beam be supported at both ends and loaded in the middle, its length being $l = 2c$, its proof deflection is the same with that of a beam of the same transverse dimensions and of the length c, fixed at one end and loaded at the other; and its proof load is double of that of the latter beam; therefore its resilience is double of that of the latter beam. Consequently, for rectangular beams of the half-span c, supported at both ends and loaded in the middle, we have the following values for the numerical factor of the resilience :—

	$\frac{n''}{6}$.
IV. Uniform breadth and depth,......................................	$\frac{1}{9}$.
V. Uniform strength, uniform depth,................................	$\frac{1}{6}$.
VI. Uniform strength, uniform breadth,............................	$\frac{2}{9}$.

306. **A Suddenly-Applied Transverse Load,** like the suddenly-applied pull of Article 267, produces at first double the maximum stress, and double the strain, which the application of a load gradually increasing from nothing to the amount of the given load would produce. It is unnecessary to demonstrate this in detail, the reasoning being the same with that employed in Article 267.

The contingency of the sudden application of a moving load is provided for by the factor of safety, which expresses the ratio of the proof load to the working load (Article 247).

The action of the rolling load to which a railway bridge is subjected is intermediate between that of an absolutely sudden load and a perfectly gradual load. It has been investigated mathematically by Mr. Stokes, and experimentally by Captain Galton, and the results are given in the Report of the Commissioners on the Application of Iron to Railway Structures. The practical conclusion to be drawn from them is, that a moving load requires a larger factor of safety than a steady load.

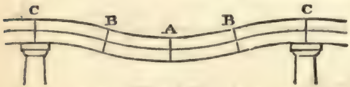

Fig. 143.

307. **Beam Fixed at Both Ends.**—A beam is *fixed*, as well as supported, at both ends, when a pair of equal and opposite couples are made to act on the vertical sectional planes at its points of support, of magnitude sufficient to maintain its longitudinal axis horizontal there, and so to diminish the deflection, slope, and curvature of its middle por-

tion. This is generally accomplished by making the beam form part of one continuous girder with several points of support, or by making it project on either side beyond its points of support, and so fastening or loading the projecting portions, that their loads, or the resistance of their fastenings, shall give the required pair of couples.

In fig. 143, let C B A B C represent a beam supported at the points C, C, loaded along its intervening portion, and so fixed or loaded beyond these points that at them its longitudinal axis is horizontal, instead of having the slope i_1, which it would have if the beam were simply *supported* at C, C, and not fixed. At each of the vertical sections above the points of support, C, C, there is an *uniformly-varying horizontal stress*, being a pull above and a thrust below the neutral axis; and the moment of that pair of stresses is that of the pair of equal and opposite couples which maintain the beam horizontal at the points of support. It is required to find,—in the first place, that resisting moment at the vertical planes of support (from which the stress on the material there may at once be found); and secondly, the effect of that moment on the curvature, slope, deflection, and strength of the beam.

The general method of solution of this question is as follows:—Compute, by equation 3 of Article 303, i'_1, the slope which the neutral surface of the beam would have at the points C, C, if it were simply supported there, and not fixed. Then, by Article 304, find the *uniform* moment of flexure, which, if it acted on the beam in such a manner as to make it become convex upwards, would produce a slope at the points C, C, *equal and contrary* to i'_1. This will be the required moment of resistance at the vertical sections C, C, from which the greatest stress on the material at those sections can be found by equation 4 of Article 293. It will afterwards appear that this is the greatest stress on the beam; so that by putting it instead of $M_0 = m\,W\,l$ in equations 2 of Article 294, and 3 of Article 295, the conditions of strength of the beam are determined. Denote this moment by $-M_1$, the negative sign denoting that it tends to produce convexity upwards, while the load on the beam tends to produce convexity downwards.

Let M be what the moment of flexure at any point of the beam *would be*, if it were simply supported at C, C. Then the actual moment of flexure is

$$M - M_1,$$

and by substituting this for M in the equations of Articles 300 and 303, the curvature, slope, and deflection, with the proof load, or with any load, are found.

Where M is the greater, as at A, the beam is convex downwards. Where M_1 is the greater, as at C, the beam is convex upwards. There are a pair of points, B, B, at which $M = M_1$, so that the moment of flexure, and consequently the curvature, vanish, and the beam is subjected to a shearing force alone; these are called the *points of contrary flexure;* and they divide the middle part of the beam, which is convex downwards, from the two endmost parts, which are convex upwards.

In expressing the solution of this problem by formulæ, four cases will be taken into consideration, viz.:—

1. The case of an uniform beam, with a symmetrical load in general.
2. Beam of uniform section, loaded in the middle.
3. Beam of uniform section, loaded uniformly.
4. Beam of uniform strength and uniform depth, uniformly loaded.

CASE 1. *Symmetrical load on a beam of uniform section.* By Article 303, equation 3, observing that $l = 2c$, we have

$$i'_1 = \frac{2m''m}{n'} \cdot \frac{Wc^2}{Ebh^3};$$

and by Article 304,

$$M_1 = \frac{EIi'_1}{c} = \frac{n'Ebh^3 i'_1}{c};$$

consequently,

$$M_1 = 2m''mWc = m'' \cdot mWl = m'' \cdot M_0, \dots\dots\dots (1.)$$

M_0 being what the moment of flexure at A *would have been,* had the beam been simply supported.

The values of m'' are given in Article 300.

Let M'_0 be the actual moment of flexure at A. Then

$$M'_0 = (1 - m'')\, M_0 \dots\dots\dots\dots\dots\dots\dots\dots (2.)$$

The greatest moment of flexure must be either at A or C, or at both, if the moments at these sections be equal and opposite. But for beams of uniform section, m'' is never less than $\frac{1}{2}$; therefore the greatest moment of flexure is at C, or both at C and A, and never at A alone.

The *strength* of the beam is expressed by the following formula, obtained by putting M_1 instead of mWl, in equation 3 of Article 295:—

$$M_1 = m''mWl = nfbh^2;\ W = \frac{nfbh^2}{m''ml} \dots\dots\dots\dots (3.)$$

f being the limit of proof or working stress, as the case may be, and n a factor suitable to the form of section of the beam, as given by the table of Article 295.

Hence it appears, that *by fixing the ends of an uniform beam, so that they shall be horizontal, its strength is increased in the ratio* $1 : m''$.

The *deflection* is found, by subtracting that due to the uniform moment M_1 from that which the load would produce if the beam were simply supported at C and C. The former of these quantities, according to Article 304, is

$$\frac{M_1 c^2}{2 E I} = \frac{m'' M_0 c^2}{2 E I};$$

and the latter, according to Article 303, equation 2, is

$$\frac{n'' M_0 c^2}{E I} = \frac{n'' M_1 c^2}{m'' E I};$$

so that the deflection, their difference, is

$$v_1 = \left(\frac{n''}{m''} - \frac{1}{2}\right) \cdot \frac{M_1 c^2}{E I} = \left(n'' - \frac{m''}{2}\right) \cdot \frac{M_0 c^2}{E I} \text{......(4.)}$$

From the last of those expressions, it appears that by fixing the ends horizontal, an uniform beam is made stiffer under a given load in the ratio

$$n'' : \left(n'' - \frac{m''}{2}\right).$$

If, in the first expression for the deflection, it be considered that M_1 is the moment of resistance corresponding to the proof or limiting stress at the section C, we may make

$$\frac{M_1}{I} = \frac{f}{y_0};$$

so as to obtain the following expression for the deflection under the proof load :—

$$v_1 = \left(\frac{n''}{m''} - \frac{1}{2}\right) \frac{f c^2}{E y_0} \text{......................(5.)}$$

being less than the proof deflection of a beam simply supported, as given by equation 6, Article 300, in the ratio

$$\left(\frac{n''}{m''} - \frac{1}{2}\right) : n''.$$

The points of contrary flexure are to be found in each particular case by solving the equation

$$M - M_1 = 0 \text{..........................(6.)}$$

Case 2. *Uniform section, loaded in the middle.*

$$\left.\begin{aligned}
&m = \frac{1}{4};\ m'' = \frac{1}{2};\ n'' = \frac{1}{3};\\
&M'_0 = M_1 = \frac{1}{2} M_0 = \frac{1}{8} W\, l = \frac{1}{4} W\, c = n f b h^2;\\
&v_1 = \frac{1}{6} \cdot \frac{f c^2}{E y_0}.
\end{aligned}\right\} \ldots(7.)$$

The points of contrary flexure are midway between A and C.

Case 3.—*Uniform section, uniformly loaded.*

$$\left.\begin{gathered}
W = 2\, c\, w\\
m = \frac{1}{8};\ m'' = \frac{2}{3};\ n'' = \frac{5}{12};\\
M_1 = \frac{2}{3} M_0 = \frac{1}{12} W\, l = \frac{1}{6} W\, c = n f b h^2;\\
M'_0 = \frac{1}{2} M_1 = \frac{1}{3} M_0;\\
v_1 = \frac{1}{8} \cdot \frac{f c^2}{E y_0}.
\end{gathered}\right\} \ldots\ldots(8.)$$

The points of contrary flexure are thus found. By the table of Article 300, case 5,

$$M = \left(1 - \frac{x^2}{c^2}\right) M_0 = \frac{3}{2}\left(1 - \frac{x^2}{c^2}\right) M_1;$$

so that in order to have $M = M_1$, we must make

$$1 - \frac{x^2}{c^2} = \frac{2}{3};\ \text{or } x = \frac{c}{\sqrt{3}} = 0{\cdot}577\, c;\ldots\ldots\ldots\ldots(9.)$$

which equation gives the distance of each of the points of contrary flexure B, from A, the middle of the beam.

Case 4. *Uniform strength, uniform depth, uniform load.* In this case the uniformity of strength is attained by making the breadth at each point proportional to the moment of flexure, as shown in the plan, fig. 144, preserving, at the points of contrary flexure B, B, a sufficient thickness only to resist the shearing force.

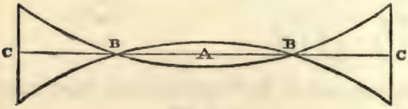

Fig. 144.

As shown in Article 300, case 6, the curvature of the beam is uniform in amount, changing in direction only at the points of contrary flexure. Therefore, in fig. 143, C B and B A, at each side of the beam, are two arcs of circles of equal radii, horizontal at A and C, and touching each other at B; therefore those arcs are of equal length; therefore each point of contrary flexure B is midway between the middle of the beam A and the point of support C.

It is evident also, that the proof deflection of the beam must be double of that of an uniformly curved beam of half the span, supported at the ends without being fixed; that is to say, one-half of that of an uniformly curved beam of the same span, supported but not fixed; or symbolically

$$v_1 = \frac{1}{4} \cdot \frac{f c^2}{\mathrm{E}\, y_0} \dots\dots\dots\dots (10.)$$

The actual moment of flexure at A must be the same as in an uniformly loaded beam, with the same intensity of load $w = \frac{\mathrm{W}}{2\,c}$, supported, but not fixed at B, B; that is to say,

$$\mathrm{M}'_0 = \frac{\mathrm{W}\,c}{16} = \frac{\mathrm{W}\,l}{32} = \frac{\mathrm{M}_0}{4} \dots\dots\dots\dots (11.)$$

and therefore, the moment of flexure at C is

$$n\,f\,b_1\,h^2 = \mathrm{M}_1 = \mathrm{M}_0 - \mathrm{M}'_0 = \frac{3\,\mathrm{M}_0}{4} = \frac{3\,\mathrm{W}\,c}{16} = \frac{3\,\mathrm{W}\,l}{32};\ (12.)$$

b_1 being the breadth of the beam at C, which is three times the breadth b_0 at A.

To find the breadth at any other point, it is to be observed, that the moment of flexure at the distance x from A is

$$\mathrm{M} - \mathrm{M}_1 = \frac{w\,(c^2 - x^2)}{2} - \frac{3\,w\,c^2}{8} = \left(\frac{1}{3} - \frac{4\,x^2}{3\,c^2}\right) \mathrm{M}_1;\dots(13.)$$

and that consequently the breadth b, which is proportional to the moment of flexure, is given by the equation

$$b = \frac{1}{3}\left(1 - \frac{4\,x^2}{c^2}\right) b_1 = \left(1 - \frac{4\,x^2}{c^2}\right) b_0 \dots\dots\dots (14.)$$

In using this equation, the positive or negative sign of the result merely indicates the direction of the curvature.

According to equation 14, the figure of the beam in plan (fig. 144) consists of two parabolas, having their vertices at A, and intersecting each other in the points of contrary flexure, B, B, for which $x = \pm \frac{c}{2}$.

The breadth which must be left at B, to resist shearing, will appear from the next Article.

308. **A Beam Fixed at One End and Supported at Both** is sensibly in the same condition with the part C B A B of the beam in fig. 143, extending from one of the fixed points C to the *farther* point of contrary flexure, which now represents a point *supported, but not fixed.* Hence if a continuous girder be supported on a series of piers, the span of each of the endmost bays should be to the span of each intermediate bay, in the ratio $c + x_0 : 2\,c$, where x_0 is the distance A B from the lowest point to a point of contrary flexure.*

309. **Shearing Stress in Beams.**—It has already been shown, in Article 288, how to find the amount F of the shearing force at a given vertical cross section of a beam; and examples of that force in particular cases have been given in Articles 289 and 290. The object of the present Article is to show the manner in which the stress which resists that force is distributed.

In Article 104 it has been shown, that the intensities of the tangential stresses at a given point, on a pair of planes at right angles to each other and to the plane parallel to which the stresses act, are necessarily equal. Hence, in order to determine the intensity of the vertical shearing stress at a given point in a vertical section of a beam, such as the point E in the vertical section G E B of the beam represented in fig. 145, it is sufficient to find the equal intensity of the horizontal shearing stress at the same point E in the horizontal plane E F. The existence of that horizontal shearing stress is familiarly known by the fact, that if a beam, instead of being one continuous mass, be divided into separate horizontal layers, those layers will slide on each other like the layers of a coach spring. The intensity of that stress is found as follows :—

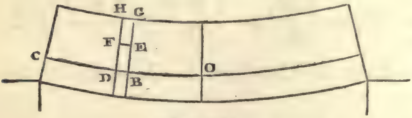

Fig. 145.

Let H F D be another vertical section near to G E B. If the moment of flexure at H F D differs from that at G E B, there must be a corresponding difference in the amount of the direct stress on two corresponding parts of the planes of section, such as G E and H F. (In the case shown in the figure, that direct stress is a thrust, and is greatest at G E). That difference constitutes a horizontal force acting on the solid H F E G; and in order to maintain the equilibrium of that solid, the amount of shearing stress on the plane F E must be equal and opposite to that horizontal force. That amount being divided by the area of the plane F E, gives the intensity of the shearing stress.—Q. E. I.

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* See Article 308A, p. 641.

From the foregoing solution it is obvious, that the shearing stress is *nothing* at the upper and lower surfaces of the beam; because the entire direct stress on each cross section is nothing. This might also be proved by reasoning like that of Article 278. It is also obvious that the shearing stress in the vertical layer between the two planes of section is greatest at D B, where they cut the neutral surface O C, at which the direct horizontal stress changes from thrust to pull; for at that surface the horizontal force to be balanced by the shearing stress reaches its maximum.

To express this solution symbolically in the case of a beam of uniform cross section; let $\overline{\mathrm{O\,B}} = x$, $\overline{\mathrm{O\,C}} = c$, $\overline{\mathrm{B\,E}} = y$, $\overline{\mathrm{B\,G}} = y_1$, $\overline{\mathrm{B\,D}} = \overline{\mathrm{E\,F}}$ (sensibly) $= d\,x$; let the breadth of the beam at any point E be denoted by z, and at the neutral surface by z_0.

Let p be the intensity of the direct horizontal stress at E, q that of the shearing stress at E, and q_0 that of the maximum shearing stress at B. Then by equation 4 of Article 293,

$$p = \frac{\mathrm{M}}{\mathrm{I}} y,$$

and the amount of the direct stress on the sectional plane between G and E is

$$\frac{\mathrm{M}}{\mathrm{I}} \int_y^{y_1} y\, z \cdot d\, y.$$

The horizontal force by which the solid H F E G is pressed from O towards C, is the excess of the value of the above quantity for G E above its value for H F; which excess arises from the excess of the moment of flexure M at G E B above the moment of flexure at H F D, farther from the middle of the beam by the distance $d\,x$. That difference of the moments of flexure is obviously equal to

$$\mathrm{F}\, d\, x.$$

F being the *amount* of the shearing force at the vertical layer in question; consequently, the horizontal force, which the shearing stress on the plane F E is to balance, is

$$\frac{\mathrm{F}\, d\, x}{\mathrm{I}} \int_y^{y_1} y\, z \cdot d\, y.$$

Dividing this by the area of the plane F E, which is $z\, d\, x$, the required intensity of the shearing stress is found to be

$$q = \frac{\mathrm{F}}{\mathrm{I}\, z} \int_{y}^{y_1} y\, z \cdot d\, y; \ldots\ldots\ldots\ldots\ldots\ldots(1.)$$

and the maximum value of that intensity, for the given vertical layer, which acts at D B in the neutral surface, is

$$q_0 = \frac{\mathrm{F}}{\mathrm{I}\, z_0} \int_{0}^{y_1} y\, z \cdot d\, y \ldots\ldots\ldots\ldots\ldots\ldots(2.)$$

The same results are in every case obtained, whether the upper or the lower surface of the beam be taken as the limit of integration indicated by y_1; the complete integral $\int y\, z \cdot d\, y$, for the whole cross section of the beam, being $= 0$, because of y being measured from the neutral axis, which traverses the centre of gravity of that section.

Let $\mathrm{S} = \int z\, d\, y$ be the area of the cross section of the beam. Then the *mean* intensity of the shearing stress is

$$\frac{\mathrm{F}}{\mathrm{S}},$$

and the *maximum* intensity exceeds the mean in the following ratio :—

$$\frac{q_0\, \mathrm{S}}{\mathrm{F}} = \frac{\mathrm{S}}{\mathrm{I}\, z_0} \int_{0}^{y_1} y\, z \cdot d\, y; \ldots\ldots\ldots\ldots\ldots\ldots(3.)$$

a ratio depending wholly on the figure of the cross section of the beam. The following table gives some of its values :—

FIGURE OF CROSS SECTION.	$\frac{q_0\, \mathrm{S}}{\mathrm{F}}$.
I. Rectangle, $z_0 = b$,........................	$\frac{3}{2}$.
II. Ellipse,..	$\frac{4}{3}$.
III. Hollow Rectangle— $\mathrm{S} = b\, h - b'\, h'$; $z_0 = b - b'$. This includes I-shaped sections,........................	$\frac{3}{2} \cdot \frac{(b\, h - b'\, h') \cdot (b\, h^2 - b'\, h'^2)}{(b - b') \cdot (b\, h^3 - b'\, h'^3)}$.
IV. Hollow square, $h^2 - h'^2$,....................	$\frac{3}{2}\left(1 + \frac{h\, h'}{h^2 + h'^2}\right)$.

V. VI. Hollow ellipse and hollow circle; the numerical factor $\frac{4}{3}$; the symbolical factor, the same as for the hollow rectangle and hollow square respectively.

For beams of variable cross section, the preceding results, though not absolutely correct, are near enough to the truth for practical purposes.

When a beam consists of strong upper and lower flanges or horizontal bars, connected by a thin vertical web or webs, like the wrought iron plate girders to be treated of in a subsequent section, the shearing force is to be treated as if it were entirely borne by the vertical web or webs, and uniformly distributed.

310. **Lines of Principal Stress in Beams.**—Let p be the intensity of the direct horizontal stress, and q that of the shearing stress, at any point, such as E, fig. 145, in a beam. Then the axes of principal stress at that point, and the intensities of the pair of principal stresses, may be found by Article 112, Problem IV., case 4. In the equations 21, 22, 23, which solve that problem, for p_n, the normal component of the stress on a vertical plane, is to be put p; for p'_n, the normal component of the stress on a horizontal plane, is to be put 0; and for p_t, the common tangential component, is to be put q. x and y having already been taken to denote the horizontal and vertical co-ordinates of the point E, p_1 and p_2 may be taken to represent the greatest and least principal stresses instead of p_x and p_y, and i_1 the angle which the axis of greatest stress makes with the horizon, instead of $\widehat{x\,n}$.

Then equation 21 of Article 112 becomes

$$\frac{p_1 + p_2}{2} = \frac{p}{2};$$

equation 22 becomes

$$\frac{p_1 - p_2}{2} = \sqrt{\left\{\frac{p^2}{4} + q^2\right\}};$$

from which we have

$$\left.\begin{aligned} p_1 &= \sqrt{\left\{\frac{p^2}{4} + q^2\right\}} + \frac{p}{2}; \\ -p_2 &= \sqrt{\left\{\frac{p^2}{4} + q^2\right\}} - \frac{p}{2} \end{aligned}\right\} \text{............(1.)}$$

These equations show, that the greatest principal stress is of the same kind with the direct horizontal stress, and the least principal stress of the contrary kind. Further, equation 23 becomes

$$\tan 2\,i_1 = \frac{2\,q}{p} \text{........................(2.)}$$

or in another form

$$\tan i_1 = \sqrt{1 + \frac{p^2}{4\,q^2}} - \frac{p}{2\,q} = -\frac{p_2}{q} \text{...............(3.)}$$

If i_2 be the angle which the axis of least stress makes with the horizon, then, because $i_1 - i_2 = 90°$, we have

$$-\tan i_2 = \frac{1}{\tan i_1} = \sqrt{1 + \frac{p^2}{4\,q^2}} + \frac{p}{2\,q} = \frac{p_1}{q} \ldots\ldots(4.)$$

Equations 3 and 4 show that the axes of greatest and least stress are inclined opposite ways to the horizon (as indeed they must be, being perpendicular to each other), the inclination of the axis of least stress being the steeper.

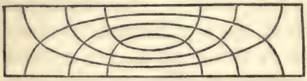

Fig. 146.

If those inclinations be computed for a number of different points in the vertical section of a beam, and the directions of the axes of stress at those points laid down on a drawing, a network of lines, consisting of two series of lines intersecting each other at right angles, as in fig. 146, may be drawn, so that each line shall touch the axes of stress traversing a series of points, and so that the tangents to the pair of lines which cross at any given point shall be the axes of stress at that point. These lines may be called the *lines of principal stress.* For a beam supported at the ends, the lines convex upwards are *lines of thrust,* and those convex downwards *lines of tension.* They all intersect the neutral surface at angles of 45°. The stress along each of those lines is greatest where it is horizontal, and gradually diminishes to nothing at the two ends of the line, where it meets the surface of the beam in a vertical direction.

311. **Direct Vertical Stress.**—It is to be observed, that no account has yet been taken of the *direct vertical stress* upon such planes as F E (fig. 145) in a loaded beam, that stress having been treated in the last Article as if it were null. The reasons for this are—first, That the direct vertical stress is in most practical cases of small intensity compared with the other elements of stress; secondly, That the mode of its distribution can be modified in an indefinite variety of ways by the modes of placing the load on or attaching it to the beam, so that formulæ applicable to one of those modes would not be applicable to another—(in fact, by a certain mode of loading, it can even be reduced to nothing); and thirdly, That its introduction would complicate the formulæ without adding materially to their accuracy.

312. **Small Effect of Shearing Stress upon Deflection.**—A shearing stress of the intensity q produces a distortion represented by $\frac{q}{C}$, C being the transverse elasticity, as already explained in Article 262. The *slope* of any given originally horizontal layer of the

beam at a given point will be increased by this distortion to the extent denoted by

$$i'' = \frac{q}{C} = \frac{F}{C I z}\int_y^{y_1} yz \cdot dy; \text{..............} (1.)$$

which additional slope is to be added to the slope due to the *bending stress*, in order to find the total slope. The curvature of the layer will also be increased by the amount

$$\frac{d i''}{dx} = \frac{dF}{dx} \cdot \frac{1}{C I z}\int_y^{y_1} yz \cdot dy, \text{...............} (2.)$$

for uniform beams, and to nearly the same amount for other beams; and there will be an additional deflection of the layer under consideration, of the amount

$$v''_1 = \int_0^c i'' dx \text{........................} (3.)$$

Observing that $\int_0^c F\, dx = M_0$, the above equation becomes, *for uniform beams*,

$$v''_1 = \frac{M_0}{C I z}\int_y^{y_1} yz \cdot dy \text{...............} (4.)$$

Supposing the beam to be under the proof load, we may put for $\frac{M_0}{I}$ its value $\frac{f}{y_1}$, making the equation

$$v''_1 = \frac{f}{C y_1 z} \cdot \int_y^{y_1} yz dy \text{..................} (5.)$$

The *greatest* value of this is that for the neutral surface, for which the limits of integration are 0 and y_1. To compare this additional deflection due to distortion with that due to flexure proper, let us take the case of a rectangular beam, in which $y_1 = \frac{h}{2}$, $z = b$, $\int_0^{y_1} yz\, dy = \frac{b h^2}{8}$. Then

$$v''_1 = \frac{fh}{4C} \text{..........................} (6.)$$

For the same beam, according to equation 6 of Article 300, we have the proof deflection due to flexure proper,

$$v_1 = \frac{n'' f c^2}{E y_0} = \frac{5}{6} \cdot \frac{f c^2}{E h};$$

so that the ratio of those two parts of the deflection is

$$\frac{v''_1}{v_1} = \frac{3}{10} \cdot \frac{\mathrm{E}}{\mathrm{C}} \cdot \frac{h^2}{c^2} \text{..........................} (7.)$$

For wrought iron (for example) $\frac{\mathrm{E}}{\mathrm{C}} =$ about 3. Suppose $\frac{h}{c} = \frac{1}{7}$, which is an ordinary proportion in practice; then $\frac{v''_1}{v_1} = \frac{9}{490}$ nearly, a quantity practically inappreciable.

It appears, then, that the distortion produced by the shearing stress in beams, even at the neutral surface, where it is greatest, produces a deflection which is very small compared with that due to the bending action of the load; and that the alteration of the external figure of the beam must be smaller still; from which it may be concluded, that in ordinary practical cases there is no occasion to compute the additional deflection due to the shearing stress.

313. **Partially-Loaded Beam.**—In designing beams for the support of roads and railways, or for any other situation in which one part of a beam may be loaded and another unloaded, it is necessary to consider whether a partial load may or may not produce, at any point of the beam, a more intense stress than an uniform load over the whole beam.

The case of this kind, which is most important in practice, is that in which a beam supported at both ends is uniformly loaded throughout a certain portion of its length and unloaded throughout the remainder; and its solution depends on two theorems.

THEOREM I. *For a given intensity of load per unit of length, an uniform load over the whole beam produces a greater moment of flexure at each cross section than any partial load.*

Let the two ends of the beam be called C and D, and any intermediate cross section E. Then for an *uniform* load, the moment of flexure at E is an upward moment, being equal to the upward moment of the supporting force at either of the ends relatively to E, *minus* the downward moment of the uniform load between that end and E. A *partial* load is produced by removing the uniform load from part of the beam, situated either between E and C, between E and D, or at both sides of E. First, let the load be removed from any part of the beam between E and C. Then the downward moment, relatively to E, of the load between E and D is unaltered; and the upward moment, relatively to E, of the supporting force at D is diminished, in consequence of the diminution of that force; therefore the moment of flexure is diminished. A similar demonstration applies to the case in which the load is removed from a part of the beam between E and D; and the combined effect of those two operations takes place when the load is removed from portions of the beam lying at both sides of E; so that *the removal* *of the load from any portion of the beam diminishes the moment of flexure at each point.*—Q. E. D.

Hence it follows, that *if a beam be strong enough to bear an uniform load of a given intensity, it will bear any partial load of the same intensity.*

THEOREM II. *For a given intensity of load per unit of length, the greatest shearing force at any given cross section of a beam takes place when the longer of the two parts into which that section divides the beam is loaded and the shorter unloaded.*

Let the ends of the beam, as before, be called C and D, and the given cross section E; and let C E be the longer part, and E D the shorter part of the beam. In the first place, let C E be loaded and E D unloaded. Then the shearing force at E is equal to the supporting force at D, and consists in a tendency of E D to slide upwards relatively to C E. The load may be altered, either by putting weight between D and E, or by removing weight between C and E. If any weight be put between D and E, a force equal to *part* of that weight is added to the supporting force at D, and therefore to the shearing force at E; but a force equal to the *whole* of that weight is taken away from that shearing force; therefore the shearing force at E is diminished by the alteration of the load. If weight be removed from the load between C and E, the shearing force at E is diminished also, because of the diminution of the supporting force at D. Therefore *any alteration from that distribution of the load in which the longer segment* C E *is loaded, and the shorter segment* E D *unloaded, diminishes the shearing force at* E. —Q. E. D.

In designing beams where the shearing force is borne by a thin vertical web, or by lattice work (as in plate, lattice, and other compound girders, to be considered more fully in a subsequent section), it is necessary to attend to this Theorem, and to provide strength, at each cross section, sufficient to bear the shearing force which may arise from the longer segment of the beam being loaded and the shorter unloaded.

To find a formula for computing that force, let c be the half-span of the beam, x the distance of the given cross section, E, from the middle of the beam, and w the uniform load per unit of length on the loaded part of the beam C E. The length of that part is

$$\overline{\mathrm{CE}} = c + x;$$

and the amount of the load upon it,

$$w(c + x).$$

The centre of gravity of that load lies at a distance from the end, C, of the beam which is represented by

$$\frac{c+x}{2};$$

and therefore the upward supporting force at the other end of the beam, D, which is also the shearing force at E, is given by the equation

$$\mathrm{F}' = w(c+x)\cdot\frac{c+x}{2} \div 2c = \frac{w(c+x)^2}{4c} \text{.........(1.)}$$

It has already been shown, in Article 290, that the shearing force at a given cross section with an uniform load is $\mathrm{F} = wx$; hence the *excess* of the greatest shearing force at a given cross section with a partial load, above the shearing force at the same cross section with an uniform load of the same intensity, is

$$\mathrm{F}' - \mathrm{F} = \frac{w(c-x)^2}{4c} \text{.......................(2.)}$$

At the ends of the beam this excess vanishes. At the middle, it consists of the whole shearing force $\mathrm{F}' = \frac{1}{4}wc$, or one quarter of the shearing force at the ends; that is, one-eighth of the amount of an uniform load.

314. **Allowance for Weight of Beam.**—When a beam is of great span, its own weight may bear a proportion to the load which it has to carry, sufficiently great to require to be taken into account in determining the dimensions of the beam. Before the weight of the beam can be known, however, its dimensions must have been determined, so that to allow for that weight, an indirect process must be employed.

As already explained in Article 302, the *depth* of a beam is determined by the deflection which it is desired to allow; and the *breadth* remains to be fixed by conditions of strength, the strength being simply proportional to the breadth.

Let b' denote the breadth as computed by considering the *external load alone*, W'. Compute the weight of the beam from that *provisional* breadth, and let it be denoted by B'. Then $\frac{\mathrm{B}'}{\mathrm{W}'}$ is the proportion which the weight of the beam must bear to the *entire* or *gross* load which it is calculated to support; and $\frac{\mathrm{W}'}{\mathrm{W}'-\mathrm{B}'}$ is the proportion in which the *gross* load exceeds the external load. Consequently, if for the *provisional* breadth b' there be substituted the *exact* breadth,

$$b = \frac{b'\,\mathrm{W}'}{\mathrm{W}'-\mathrm{B}'}, \text{..........................(1.)}$$

the beam will now be strong enough to bear both the proposed external load W′, and its own weight, which will now be

$$B = \frac{B' W'}{W' - B'};\text{.........................}(2.)$$

and the true gross load will be

$$W = \frac{W'^2}{W' - B'}\text{.........................}(3.)$$

In the preceding formulæ, both the external load and the weight of the beam are treated as if uniformly distributed—a supposition which is sometimes exact, and always sufficiently near the truth for the purposes of the present Article.

315. **Limiting Length of Beam.**—The gross load of beams of similar figures and proportions, varying as the breadth and square of the depth directly, and inversely as the length, is proportional to the square of a given linear dimension. The weights of such beams are proportional to the cubes of corresponding linear dimensions. Hence the weight increases at a faster rate than the gross load; and for each particular figure of a beam of a given material and proportion of its dimensions, there must be a certain size at which the beam will bear its own weight only, without any additional load.

To reduce this to calculation, let the gross working uniformly-distributed load of a beam of a given figure, as in Article 295, be expressed as follows:—

$$W = \frac{8 n f b h^2}{l};\text{.........................}(1.)$$

l, b, and h being the length, breadth, and depth of the beam, f the limit of working stress, and n a factor depending on the form of cross section. The weight of the beam will be expressed by

$$B = k w' l b h;\text{.........................}(2.)$$

w' being the weight of an unit of volume of the material, and k a factor depending on the figure of the beam. Then the ratio of the weight of the beam to the gross load is

$$\frac{B}{W} = \frac{k w' l^2}{8 n f h};\text{.........................}(3.)$$

which increases in the simple ratio of the length, if the proportion $\frac{l}{h}$ is fixed. When this is the case, the length L of a beam, whose weight (treated as uniformly distributed) is its working load, is given by the condition $\frac{B}{W} = 1$; that is,

$$L = \frac{8nfh}{kw'l} = \frac{Wl}{B} \text{.........................} (4.)$$

This *limiting length* having once been determined for a given class of beams, may be used to compute the ratios of the gross load, weight of the beam, and external load to each other, for a beam of the given class, and of any smaller length, l, according to the following proportional equation:—

$$L : l : L - l :: W : B : W - B \text{................} (5.)$$

To illustrate this by a numerical example, let the beams in question be plain rectangular cast iron beams, so that $n = \frac{1}{6}$, $k = 1$, $w' = 0{\cdot}257$ lb. per cubic inch; let 40,000 lbs. per square inch be taken as the modulus of rupture, and 4 as the factor of safety, so that $f = 10{,}000$ lbs. per square inch; and let $\frac{h}{l} = \frac{1}{15}$. Then

$$L = 3{,}459 \text{ inches} = 288 \text{ feet, nearly.}$$

316. **A Sloping Beam,** like that represented in fig. 68, Article 142, is to be treated like a horizontal beam, so far as the bending stress produced by that component of the load which is normal to the beam, is concerned. The component of the load which acts along the beam, is to be considered as producing a direct thrust along the beam, which is to be combined with the stress due to the bending component of the load.

317. **An Originally Curved Beam,** at any given cross section made at right angles to its neutral surface, so far as the bending stress is concerned, is in the same condition with an originally straight beam at a similar and equal cross section to which the same moment of flexure is applied. Beams are sometimes made with a slight convexity upwards, called a *camber*, equal and opposite to the curvature which the intended working load would produce in an originally straight beam. The effect of this is to make the beam become straight under the working load, instead of curved, and to diminish the additional stress due to rapid motion of the load, which additional stress arises partly from the curvature of the beam.

318. The **Expansion and Contraction of Long Beams,** which

arise from the changes of atmospheric temperature, are usually provided for by supporting one end of each beam on rollers of steel or hardened cast iron. The following table shows the proportion in which the length of a bar of certain materials is increased by an elevation of temperature from the melting point of ice (32° Fahr., or 0° Centigrade) to the boiling point of water under the mean atmospheric pressure (212° Fahr., or 100° Centigrade); that is, by an elevation of 180° Fahr., or 100° Centigrade :—

METALS.

Brass,	·00216
Bronze,	·00181
Copper,	·00184
Gold,	·0015
Cast iron,	·00111
Wrought iron and steel,	·00114 to ·00125
Lead,	·0029
Platinum,	·0009
Silver,	·002
Tin,	·002 to ·0025
Zinc,	·00294

EARTHY MATERIALS.

(The expansibilities of stone from the experiments of Mr. Adie.)

Brick, common,	·00355
„ fire,	·0005
Cement,	·0014
Glass, average of different kinds,	·0009
Granite,	·0008 to ·0009
Marble,	·00065 to ·0011
Sandstone,	·0009 to ·0012
Slate,	·00104

TIMBER.

(Expansion along the grain, when dry, according to Mr. Joule, *Proceed. Roy. Soc.*, Nov. 5, 1857.)

Baywood,	·000461 to ·000566
Deal,	·000428 to ·000438

Mr. Joule found that moisture diminishes, annuls, and even reverses, the expansibility of timber by heat, and that tension increases it.

319. The **Elastic Curve,** in the widest sense of the term, is the figure assumed by the longitudinal axis of an originally straight

bar under any system of bending forces. All the examples of the curvature, slope, and deflection of beams in Article 300 and the subsequent Articles, are cases in which the elastic curve has been determined with a degree of approximation sufficiently close under the circumstances; that is, when the deflection is a very small fraction of the length. The present Article relates to the figure of the elastic curve for a *slender flat spring* of *uniform section,* when acted upon either by a pair of equal and opposite couples, or by a pair of equal and opposite forces.

The general equation of Article 300 applies to this case, viz.:—

$$\frac{1}{r} = \frac{\mathrm{M}}{\mathrm{E\,I}};\qquad(1.)$$

I being the uniform moment of inertia of the section of the spring, E the modulus of elasticity, M the moment of flexure at a given point, and r the radius of curvature at that point.

When a spring is under the action of *a pair of equal and opposite couples* applied to its two ends, then, as in Article 304, M is constant, r is constant, and the elastic curve is a circular arc of the radius r.

When a spring is under the action of *a pair of equal and opposite forces,* let A and B denote the two points to which those forces are applied, and A B their common line of action. The figures from 146 *a* to 146 *f,* inclusive, represent various forms which the spring may assume, viz.:—

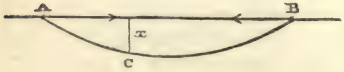

Fig. 146 *a.*

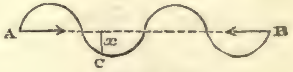

Fig. 146 *b.*

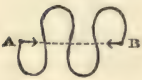

Fig. 146 *c.*

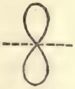

Fig. 146 *d.*

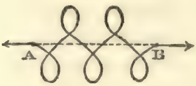

Fig. 146 *e.*

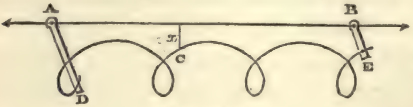

Fig. 146 *f.*

I. When the forces are directed towards each other—

a. A simple arc, like a bow, meeting A B at the points A and B only.

b, c. An undulating figure, crossing A B at any number of intermediate points.

d. The points A and B coinciding, which may give, with an endless spring, a figure of 8.

II. When the forces are directed from each other—

e. One or more loops, with the ends and intermediate portions meeting or crossing A B.

f. The forces acting from each other at the points A, B, in two rigid levers A D, B E, to which the spring is fixed at D and E : the spring forming one or more looped coils, lying altogether at one side of the line of action A B.

Let P be the common magnitude of the equal and opposite forces applied at A and B, and x the perpendicular distance of any point C in the elastic curve from the line of action A B. Then the moment of flexure at that point is obviously

$$M = x P; \qquad (2.)$$

and consequently the radius of curvature at that point is given by the equation

$$r = \frac{EI}{M} = \frac{EI}{xP}; \qquad (3.)$$

that is to say, *the radius of curvature is inversely proportional to the perpendicular distance from the line of action of the forces.* At each of the points in figs. 146 *a, b, c, d,* and *e,* where the curve meets or crosses A B, the radius of curvature is infinite; that is, there is a point of contrary flexure.

The above geometrical property is common to all the varieties of curves formed by an uniform spring bent by a pair of forces, and is sufficient to enable any one of them to be drawn approximately, by means of a series of short circular arcs. It is sufficient, also, to establish all their other geometrical properties, such as the relations between their rectangular co-ordinates, and the lengths of their arcs. These are expressed by means of elliptic functions; and it is unnecessary to give them in detail in this treatise, except in one case, which will be mentioned in the next Article, 319 A.

There is one important proposition, however, which it is here necessary to prove; and that is the following

THEOREM. *That a spring of a given length and section, to the ends of whose neutral surface a pair of forces are applied, will not be bent if those forces are less than a certain finite magnitude.* Let A and B in fig. 146 *a* be the two ends of the spring, to which two equal and opposite forces of the magnitude P are applied, directed towards each other; the spring forming a single arc A C B, of the length l. x being, as before, the ordinate of any point C, let y be the distance of that ordinate from A.

The smaller the force P, the more nearly will the arc A C B approach to the straight line A B; and in order to find the smallest value of P which is compatible with any bending of the spring, that force must be computed on the supposition that the ordinate x at each point is insensibly small compared with the length of the spring, and consequently, that the length of the arc A C does not sensibly differ from that of its abscissa y. This being the case, the curvature at any point C is to be taken as sensibly given by the following equation:—

$$\frac{1}{r} = -\frac{d^2 x}{d y^2};$$

which value being inserted in equation 3, gives

$$-\frac{d^2 x}{d y^2} = \frac{\mathrm{P}}{\mathrm{E\,I}} \cdot x \text{.......................} (4.)$$

The integral of this equation is

$$\left. \begin{aligned} x &= a \cdot \sin \cdot \frac{y}{c}, \\ \text{where } c &= \sqrt{\frac{\mathrm{E\,I}}{\mathrm{P}}}. \end{aligned} \right\} \text{..................} (5.)$$

In order that x may be $= 0$ at the points A and B, it is necessary that when $y = l$, $\frac{y}{c}$ should be $= n\,\pi$, n being any whole number; and consequently that

$$c = \frac{l}{n\,\pi} \text{..............................} (6.)$$

Now of all the possible values of n, that which gives the least value of P is $n = 1$; whence we find

$$c = \sqrt{\frac{\mathrm{E\,I}}{\mathrm{P}}} = \frac{l}{\pi}; \text{ and } \mathrm{P} = \frac{\pi^2 \mathrm{E\,I}}{l^2}; \text{..........} (7.)$$

and this *finite quantity* is the *smallest force which will bend the given spring* in the manner proposed.—Q. E. D.

This investigation proves the Theorem in question, and gives the least bending force; but as it leaves the constant a indeterminate, it does not give the figure assumed by the spring, which cannot be found exactly except by the use of elliptic functions.

319 A. The **Hydrostatic Arch,** described in Article 183, is of the same figure with the coiled and looped elastic curve represented in fig. 146 *f*; for its radius of curvature at any point is inversely proportional to the perpendicular distance of that point from a given straight line. In order to transform all the equations given in that Article for the hydrostatic arch into the corresponding equations for the coiled and looped elastic curve of fig. 146 *f*, it is only necessary to put for the constant product of the ordinate and radius of curvature the following value :—

$$x r = \frac{\mathrm{E\,I}}{\mathrm{P}}.$$

An instrument consisting of an uniform spring attached to a pair of levers, might be used for tracing the figures of hydrostatic arches on paper.

This property of the coiled and looped elastic curve is analogous to that discovered by James Bernouilli in the simple bow of fig. 146 *a*, viz., that it is the figure assumed by the vertical longitudinal section of an indefinitely broad sheet, containing a liquid mass whose upper horizontal surface is represented by A B.

SECTION 7.—*On Resistance to Twisting and Wrenching.*

320. The **Twisting Moment,** or moment of torsion, applied to a bar, is the moment of a pair of equal and opposite couples applied to two cross sections of the bar, in planes perpendicular to the axis of the bar, and tending to make the portion of the bar between those cross sections rotate in opposite directions about that axis. In the following Articles, twisting moments are supposed to be expressed in *inch-pounds*.

321. **Strength of a Cylindrical Axle.**—A cylindrical axle, A B, fig. 147, being subjected to the twisting moment of a pair of equal and opposite couples applied to the cross sections A and B, it is required to find the condition of stress and strain at any intermediate cross section such as S, and also the angular displacement of any cross section relatively to any other.

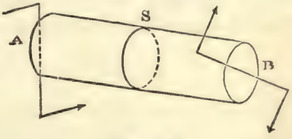

Fig. 147.

From the uniformity of the figure of the bar, and the uniformity of the twisting moment, it is evident that the condition of stress and strain of all cross sections is the same; also, because of the circular figure of each cross section, the condition of stress and strain of all particles at the same distance from the axis of the cylinder must be alike.

Suppose a circular layer to be included between the cross section S, and another cross section at the distance dx from it. The twisting moment causes one of those cross sections to rotate relatively to the other, about the axis of the cylinder, through an angle which may be denoted by di. Then if there be two points at the same distance r from the axis of the cylinder, one in the one cross section, and the other in the other, which points were originally opposite to each other, in a line parallel to the axis, the twisting moment shifts one of those points laterally, relatively to the other, through the distance $r\,di$. Consequently the part of the layer which lies between those points is in a condition of *distortion*, in a plane perpendicular to the radius r; and the distortion is expressed by the ratio

$$\nu = r \cdot \frac{di}{dx} \text{..................................(1.)}$$

which varies *proportionally to the distance from the axis.* There is therefore a *shearing stress* at each point of the cross section C, whose direction is perpendicular to the radius drawn from the axis to that point, and whose intensity is *proportional to that radius,* being represented by

$$q = \mathrm{C}\,\nu = \mathrm{C}\,r \cdot \frac{di}{dx} \text{..........................(2.)}$$

The STRENGTH of the axle is determined in the following manner:—Let f be the limit of the shearing stress to which the material is to be exposed, being the *ultimate* resistance to wrenching if it is to be broken, the *proof* resistance if it is to be tested, and the *working* resistance if the working moment of torsion is to be determined. Let r_1 be the external radius of the axle. Then f is the value of q at the distance r_1 from the axis; and at any other distance r, the intensity of the shearing stress is

$$q = \frac{fr}{r_1} \text{..................................(3.)}$$

Conceive the cross section S to be divided into narrow concentric rings, each of the breadth dr. Let r be the *mean radius* of one of these rings. Then its area is $2\,\pi\,r\,dr$; the intensity of the shearing stress on it is that given by equation 3, and the leverage of that stress relatively to the axis of the cylinder is r; consequently, the moment of the shearing stress of the ring in question, being the product of those three quantities, is

$$\frac{2\pi f}{r_1} \cdot r^3 d r;$$

which being integrated for all the rings from the centre to the circumference of the cross section S, gives for the moment of torsion, and of resistance to torsion,

$$\mathrm{M} = \frac{2\pi f}{r_1} \cdot \int_0^{r_1} r^3 d r = \frac{\pi f r_1^3}{2} \text{ ....(4.)}$$

$$\left(\frac{\pi}{2} = 1 \cdot 5708\right).$$

If the axle is *hollow*, r_0 being the radius of the hollow, the integral is to be taken from $r = r_0$ to $r = r_1$; and the moment of torsion becomes

$$\mathrm{M} = \frac{2\pi f}{r_1} \cdot \int_{r_0}^{r_1} r^3 d r = \frac{\pi f (r_1^4 - r_0^4)}{2 r_1} \text{ ....(5.)}$$

It is in general more convenient to express the strength of an axle in terms of the diameter than in terms of the radius. Let h_1 be the external diameter of the axle, and h_0 its internal diameter, if hollow; then

$$\left.\begin{array}{ll} \text{For a } \textit{solid axle,} & \mathrm{M} = \dfrac{\pi f h_1^3}{16} = \dfrac{f h_1^3}{5 \cdot 1}; \\ \text{For a } \textit{hollow axle,} & \mathrm{M} = \dfrac{\pi f (h_1^4 - h_0^4)}{16 h_1} = \dfrac{f (h_1^4 - h_0^4)}{5 \cdot 1 h_1} \end{array}\right\} \text{ .... (6.)}$$

If these formulæ be compared with those applicable to solid and hollow cylindrical beams in Article 295, it will be seen that they differ only in the numerical factor, which, for the moment of flexure, is $\frac{\pi}{32} = \frac{1}{10 \cdot 2}$, and for the moment of torsion, $\frac{\pi}{16} = \frac{1}{5 \cdot 1}$. Hence we have this useful principle, that for *equal values of the limiting stress* f, *the resistance of a cylinder, solid or hollow, to wrenching, is double of its resistance to breaking across.*

Values of the co-efficient of ultimate resistance to shearing for cast and wrought iron, are given in a table which has already been referred to. The co-efficient for cast iron is somewhat doubtful, because the experiments give varying results. That given in the table, viz., 27,700, is adopted on the authority of Mr. Hodgkinson's work *On Cast Iron*, as the mean of the experiments considered by him the most trustworthy; but some experiments give a value as low as 24,000, and others a value as high as 30,000.

With respect to the *working* values of the limiting stress f, the following are those adopted by Tredgold in his practical rules :—

For cast iron,.................7,650 lbs. per square inch.
For wrought iron,..........8,570 ,, ,,

This amounts to allowing a factor of safety of about 4 for cast iron and 6 for wrought. Practical experience of the strength of wrought iron axles confirms the co-efficient given above for wrought iron very closely, it having been found that such axles bear a working stress of 9,000 lbs. per square inch for any length of time, if well manufactured of good material. The co-efficient for cast iron appears to leave too small a factor of safety for any motion except one that is very smooth and steady, and it may be considered that 5,000 lbs. per square inch is a safer co-efficient for general use. Hence we may put, as the limit of working stress in shafts,

For cast iron,...........$f = 5{,}000$ lbs. per square inch.
For wrought iron,.....$f = 9{,}000$,, ,,

322. **Angle of Torsion of a Cylindrical Axle.**—Suppose a pair of diameters, originally parallel, to be drawn across the two circular ends, A and B, of a cylindrical axle, solid or hollow; it is proposed to find the angle which the directions of those lines make with each other when the axle is twisted, either by the working moment of torsion, or by any other moment.

This question is solved by means of equation 2 of Article 321, which gives for the *angle of torsion per unit of length*,

$$\frac{d\,i}{d\,x} = \frac{q}{\mathrm{C}\,r}.$$

The condition of the axle being uniform at all points of its length, the above quantity is constant; and if x be the length of the axle, and i the angle of torsion sought, expressed in length of arc to radius 1, we have $\frac{i}{x} = \frac{d\,i}{d\,x}$, and therefore,

$$i = \frac{x\,q}{\mathrm{C}\,r} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.)$$

I. Let the moment of torsion be the *working moment*, for which

$$\frac{q}{r} = \frac{f}{r_1}.$$

Then the angle of torsion is

$$i = \frac{f x}{C r_1} = \frac{2 f x}{C h_1} \dots\dots (2.)$$

and is the same whether the axle is solid or hollow.

A value of C, the co-efficient of transverse elasticity for cast iron, is given in the table; but it is uncertain, as experiments are discordant. For wrought iron, that constant has been found with more precision, its mean value being about 9,000,000 lbs. per square inch. Hence, for the *working torsion* of wrought iron shafts, we may make

$$\frac{f}{C} = \frac{1}{1{,}000} \dots\dots (3.)$$

II. Let the moment of torsion have any amount M consistent with safety. Then for $\frac{q}{r}$, we have to put the equal ratio deduced from the equations 4 and 5 of Article 321, by substituting q for f in the numerators and r for r_1 in the denominators; that is to say,

$$\left.\begin{array}{ll} \textit{For solid axles,} & \frac{q}{r} = \frac{2 M}{\pi r_1^4}; \text{ and} \\ & i = \frac{q x}{C r} = \frac{2 M x}{\pi C r_1^4} = \frac{32 M x}{\pi C h_1^4} = 10{\cdot}2 \frac{M x}{C h_1^4} \\ \textit{For hollow axles,} & \frac{q}{r} = \frac{2 M}{\pi (r_1^4 - r_0^4)}; \text{ and} \\ & i = \frac{q x}{C r} = \frac{2 M x}{\pi C (r_1^4 - r_0^4)} = \frac{32 M x}{\pi C (h_1^4 - h_0^4)} = 10{\cdot}2 \frac{M x}{C (h_1^4 - h_0^4)} \end{array}\right\} (4.)$$

323. The **Resilience of a Cylindrical Axle** is the product of one-half of the greatest moment of torsion into the corresponding angle of torsion; and it is given by the following equation:—

$$\left.\begin{array}{l} \frac{M i}{2} = \frac{f^2 h_1^2 x}{5{\cdot}1 C} \text{ for a solid shaft; or} \\ \frac{M i}{2} = \frac{f^2 (h_1^4 - h_0^4) x}{5{\cdot}1 C h_1^2} \text{ for a hollow shaft.} \end{array}\right\} \dots\dots (1.)$$

324. **Axles not Circular in Section.**—When the cross section of a shaft is not circular, it is certain that the ratio $\frac{q}{r}$ of the shearing stress at a given point to the distance of that point from the axis of the shaft, is not a constant quantity at different points of the cross section, and that in many cases it is not even approximately constant; so that formulæ founded on the assumption of its being constant are erroneous. The mathematical investigations of M. de St. Venant have shown how the intensity of the shearing stress is distributed in certain cases.

The most important case in practice to which M. de St. Venant's method has been applied is that of a square shaft; and it appears that its moment of torsion is given by the formula

$$\mathrm{M} = 0{\cdot}281\, f h^3 \text{ nearly.}$$

325. **Bending and Twisting combined; Crank and Axle.**—A shaft is often acted upon by a bending load and a pair of twisting couples at the same time. In that case, the greatest direct stress due to the bending load, and the greatest shearing stress due to the moment of torsion, are to be combined in the manner already illustrated for beams, in Article 310.

That is to say, let p be the greatest stress due to bending, and q that due to twisting; let p_1 be the intensity of the greatest resultant stress, and i the angle which its direction makes with the axis of the shaft. Then

$$\left.\begin{aligned} p_1 &= \sqrt{\left\{\frac{p^2}{4} + q^2\right\}} + \frac{p}{2}\,; \\ \tan 2i &= \frac{2q}{p}\,; \end{aligned}\right\} \text{.................(1.)}$$

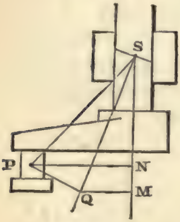

Fig. 148.

One of the most important examples of this is illustrated in fig. 148, which represents a shaft having a crank at one end. At the centre of the crank-pin, P, is applied the pressure of the connecting rod; and at the bearing, S, acts the equal and opposite resistance of that bearing. Representing the common magnitude of those forces by P, they form a couple whose moment is

$$\mathrm{M} = \mathrm{P} \cdot \overline{\mathrm{S\,P}}.$$

Draw P N perpendicular to S N, the axis of the shaft; and let the angle P S N $= j$. Then the couple M may be resolved into

A bending couple $P \cdot \overline{NS} = M \cos j$; and

A twisting couple $P \cdot \overline{NP} = M \sin j$.

Equal and opposite couples act on the farther end of the shaft. Let h be its diameter.

By the formulæ of Article 295, the greatest stress produced at S by the bending couple is

$$p = \frac{10 \cdot 2\, M \cos j}{h^3}; \qquad (2.)$$

and that produced by the twisting couple, according to Article 321, is

$$q = \frac{5 \cdot 1\ M \sin j}{h^3} = \frac{p \tan j}{2}; \qquad (3.)$$

consequently, by the equations 1 of this Article, the resultant greatest stress at S, and its inclination to the axis of the shaft, are

$$\left.\begin{array}{c} p_1 = \frac{p}{2}(\sec j + 1) = \frac{5 \cdot 1\ M}{h^3}(1 + \cos j); \\ i = \frac{j}{2}; \end{array}\right\} \qquad (4.)$$

and by making $p_1 = f$, the proper diameter can be determined.

These results may be represented graphically as follows:—Draw S Q bisecting the angle N S P, and P Q perpendicular to S Q. S Q will be the direction of the resultant greatest stress at S, and the intensity of that stress will be the same as if it were caused by the bending action of a force equal to P and applied at Q, on an oblique section of the shaft perpendicular to S Q; and also the same as the greatest intensity of the stress which would be produced at S by the direct bending action of a force equal to P applied at M in the axis of the shaft, with the leverage

$$\overline{SM} = \overline{SP}\,\frac{1 + \cos j}{2} = \frac{\overline{SP} + \overline{SN}}{2} \qquad (5.)$$

326. The **Teeth of Wheels** are made sufficiently strong, to provide against an action analogous to combined twisting and bending, which may arise from the whole force transmitted by a pair of wheels happening to act on one corner of one tooth, such as C or D, fig. 149.

In fig. 150, let the shaded part represent a portion of a cross section of the rim of the wheel A of fig. 149, and let E H K P be the face of a tooth, on one corner of which, P, acts the force represented by that letter. Conceive any sectional plane E F to intersect the tooth from the side E P to the crest P K, and let P G be perpendicular to that plane. Let h be the thickness of the tooth, and let $\overline{\mathrm{EF}} = b$, $\overline{\mathrm{PG}} = l$.

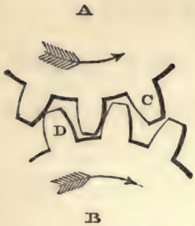

Fig. 149.

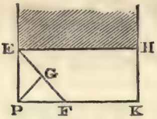

Fig. 150.

Then the moment of flexure at the section E F is $\mathrm{P}\,l$, and the greatest stress produced by that moment of flexure at that section is

$$p = \frac{6\,\mathrm{P}\,l}{b\,h^2},$$

which is a maximum when $\angle \mathrm{PEF} = 45^\circ$, and $b = 2\,l$, having then the value,

$$f = \frac{3\,\mathrm{P}}{h^2}.$$

Consequently, the proper thickness for the tooth is given by the equation

$$h = \sqrt{\frac{3\,\mathrm{P}}{f}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

This formula is Tredgold's; according to whom the proper value for the greatest working stress f is 4,500 lbs. per square inch, when the teeth are of cast iron.

SECTION 8.—*On Crushing by Bending.*

327. **Introductory Remarks.**—Pillars and struts whose lengths exceed their diameters in considerable proportions (as is almost always the case with those of timber and metal), give way not by direct crushing, but by bending sideways and breaking across, being crushed at one side, as at A, fig. 151, and torn asunder at the other, as at B.

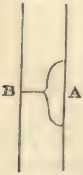

Fig. 151.

There does not yet exist any complete theory of this phenomenon. The formulæ which have been provisionally adopted are founded on a mode of investigation partly theoretical and partly empirical. Those which will first be explained are of a form proposed by Tredgold on theoretical grounds. Having fallen for a time into disuse, they were revived by Mr. Lewis Gordon, who determined the values of the constants contained in them by a comparison of them with Mr. Hodgkinson's experiments. Then will be given Mr. Hodgkinson's own empirical formulæ for the ultimate strength of cast iron pillars.

328. **Strength of Iron Pillars and Struts.**—Let P be the load which acts on a long pillar or strut, and S its sectional area. Then one part of the intensity of the greatest stress on the material is simply the intensity due to the uniform distribution of the load over the section, and may be represented thus:—

$$p' = \frac{\mathrm{P}}{\mathrm{S}}.$$

Another part of the greatest stress is that which arises from the lateral bending, which will take place in that direction in which the pillar is most flexible; that is, in the direction of its least diameter, if the diameters are unequal. Let h be that diameter, and b the diameter perpendicular to it; let l be the length of the pillar, and let v be the greatest deflection of the axis of the pillar from its original straight position. Then, as in the case of a spring, Article 319, the greatest moment of flexure is P v; and the greatest stress produced by that moment (which will be denoted by p'') is directly as the moment, and inversely as the breadth and square of the thickness of the pillar (Article 295); that is,

$$p'' \propto \frac{\mathrm{P}\, v}{b\, h^2}.$$

But the greatest deflection consistent with safety is directly as the square of the length, and inversely as the thickness (Article 300); that is,

$$v \propto \frac{l^2}{h};$$

also, the product $b\,h^2$ is proportional to the sectional area S and to the thickness h. Consequently we have the proportional equation

$$p'' \propto \frac{\mathrm{P}\, l^2}{\mathrm{S}\, h^2} \propto p' \cdot \frac{l^2}{h^2};$$

that is, *the additional stress due to bending is to the stress due to direct pressure, in a ratio which increases as the square of the proportion in which the length of the pillar exceeds the least diameter.*

The whole intensity of the greatest stress on the material of the pillar, being made equal to a co-efficient of strength f, is expressed by the following equation:—

$$f = p' + p'' = \frac{\mathrm{P}}{\mathrm{S}}\left(1 + a \cdot \frac{l^2}{h^2}\right); \ldots\ldots\ldots\ldots\ldots (1.)$$

in which a is a constant co-efficient, to be determined by experiment. Hence the following is the strength of a long pillar :—

$$P = \frac{fS}{1 + a \cdot \frac{l^2}{h^2}} \dots\dots\dots\dots (2.)$$

The following are the values of f and a for the *ultimate strength*, as computed by Mr. Gordon from Mr. Hodgkinson's experiments on pillars FIXED AT THE ENDS, by having flat capitals and bases, as in fig. 152 :—

	f, lbs. per inch.	a.
Wrought iron, solid rectangular section,	36,000	$\frac{1}{3,000}$.
Cast iron, hollow cylinder,...............	80,000	$\frac{1}{800}$.
„ solid „	80,000	$\frac{1}{400}$.

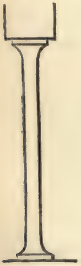
Fig. 152.

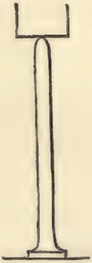
Fig. 153.

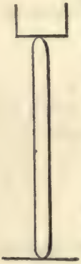
Fig. 154.

A pillar ROUNDED AT BOTH ENDS, as in fig. 154, is as flexible as a pillar of the same diameter, fixed at both ends, and of double the length; and its strength might therefore be expected to be the same; a conclusion verified by the experiments of Mr. Hodgkinson. Hence, for such pillars,

$$P = \frac{fS}{1 + 4a\frac{l^2}{h^2}} \dots\dots\dots (3.)$$

Mr. Hodgkinson found the strength of a pillar, *fixed at one end and rounded at the other* (fig. 153), to be a mean between the strengths of two pillars of the same length and diameter, one fixed at both ends, and the other rounded at both ends.

Taking the proof load as one-half of the breaking load for wrought iron, and one-third for cast iron, and the working load as from one-fourth to one-sixth of the breaking load for both materials, the following are the values to be assigned to the limit of stress f under different circumstances :—

LOAD—Breaking.		Proof.	Working.
Wrought iron, ...	36,000	18,000	6,000 to 9,000
Cast iron, ...	80,000	26,700	13,300 to 20,000

In using the formulæ 2 and 3, the ratio $\frac{l}{h}$ is generally fixed beforehand, to a degree of approximation sufficient for the purposes of the calculation.

329. **Connecting Rods** of engines are to be considered as in the condition of struts rounded at both ends; **Piston Rods,** as in the condition of struts fixed at one end and rounded at the other.

330. **Comparison of Cast and Wrought Iron.**—When the ultimate strength per square inch of section of pillars is computed by means of equation 2 of Article 328, it appears that for the smaller proportions of length to diameter, cast iron is the stronger material; but that its strength diminishes as the proportion of length to diameter increases, faster than that of wrought iron; so that for the proportion

$$l : h :: \sqrt{695} : 1 :: 26\tfrac{1}{2} : 1 \text{ nearly,}$$

those materials in the shape of solid pillars, rectangular for wrought iron, cylindrical for cast, are equally strong, and beyond that proportion wrought iron is the stronger. This result was pointed out by Mr. Gordon. The following table illustrates it:—

$\frac{l}{h}$ …………………………		10	20	26·4	30	40
Breaking load, lbs. per square inch, $= \frac{P}{S}$, …..	Wrought, solid rectangular,	34,840	31,765	29,230	27,700	23,480
	Cast, solid cylindrical,	64,000	40,000	29,230	24,620	16,000

331. **Mr. Hodgkinson's Formulæ for the Ultimate Strength of Cast Iron Pillars,** as deduced by that author from his own experiments, are as follows:—

I. When the length is not less than thirty times the diameter.

For solid cylindrical pillars, h being the diameter, *in inches,* and L the length *in feet,*

$$P = A \frac{h^{3\cdot6}}{L^{1\cdot7}} \ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

For hollow cylindrical pillars, h_1 being the external, and h_0 the internal diameter, *in inches,* and L the length *in feet,*

$$P = A \cdot \frac{h_1^{3\cdot6} - h_0^{3\cdot6}}{L^{1\cdot7}} \ldots\ldots\ldots\ldots\ldots\ldots (2.)$$

The values of the co-efficient A are as follows:—

		Tons.
(1.)	For solid pillars with rounded ends,...............	14·9
(2.)	„ „ flat ends,.....................	44·16
(3.)	For hollow pillars with rounded ends,............	13·0
(4.)	„ „ flat ends,..................	44·3

II. When the length is less than thirty times the diameter.

Let b denote the breaking load of the pillar, as computed by the preceding formulæ. Let c denote the crushing load of a short block of the same sectional area S, as computed by the formula

$$c = 49 \text{ tons} \times \text{S in square inches} \ldots\ldots\ldots\ldots (3.)$$

Then the correct crushing load of the pillar is

$$\mathrm{P} = \frac{b\,c}{b + \frac{3\,c}{4}} \ldots\ldots\ldots\ldots\ldots\ldots (4.)$$

332. In **Wrought Iron Framework,** the bars which act as struts, in order that they may have sufficient stiffness, are made of various figures in cross section, of which some examples are given in figs. 155 (angle iron), 156 (channel iron), 157 (a cross-shaped section, used in half-lattice girders), and 158 (T-iron). In some large lattice girders, the struts are composed of a pair of parallel T-iron bars, such as fig. 158, with their middle ribs turned towards each other, and connected together by a lattice work of small diagonal bars. In applying to wrought-iron struts the formulæ of Article 328, pages 361, 362, for $\frac{l^2}{h^2}$ there is to be substituted $\frac{l^2 \mathrm{S}}{12 \mathrm{J}}$; J being the *least moment of inertia* of the section (Article 95, pages 77-82).

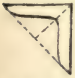
Fig. 155.

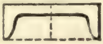
Fig. 156.

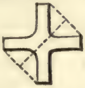
Fig. 157.

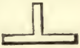
Fig. 158.

333. **Wrought Iron Cells** are rectangular tubes (generally square) composed of four plate iron sides, rivetted to angle iron bars at the corners, as shown in the section, fig. 159. This mode of construction was designed by Mr. Fairbairn, to resist a thrust along the axis of the tube. The *ultimate resistance* of a single square cell to crushing by the buckling or bending of its sides, when the thickness of the plates is *not less than one-thirtieth of the diameter of the cell*, as determined by Mr. Fairbairn and Mr. Hodgkinson, is

27,000 lbs. per square inch section of iron;

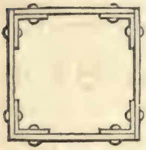
Fig. 159.

but when a number of cells exist side by side in one girder, their stiffness is increased, and their ultimate resistance to a thrust may be taken at

33,000 to 36,000 lbs. per square inch section of iron.

The latter co-efficients apply also to cylindrical cells.

334. The **Sides of Plate Iron Girders** are subjected to a diagonal thrust arising from the shearing stress, and are usually stiffened by means of T-iron ribs, in the manner shown in fig. 160. The entire depth across the ribs may be taken to represent h in the formulæ of Article 328.

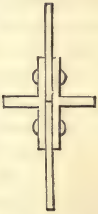
Fig. 160.

335. **Timber Posts and Struts.**—The following formula is given on the authority of Mr. Hodgkinson's experiments, for the *ultimate* resistance of posts of *oak* and *red pine* to crushing by bending :—

$$P = A \frac{h^2}{l^2} S; \ldots\ldots\ldots\ldots\ldots\ldots(1.)$$

S being the sectional area in square inches, $h : l$ the ratio of the least diameter to the length, and $A = 3,000,000$ lbs. per square inch.

The *factor of safety* for the working load of timber being 10, A is to be made $= 300,000$ only, if P is the working load.

For square posts and struts, the formula becomes

$$P = A \frac{h^4}{l^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.)$$

If the strength of a timber post be computed both by this formula and by the formula for direct crushing, viz. :—

$$P = f S, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.)$$

the *lesser* value should be adopted as the true strength.

The above formulae are for posts and struts fixed at both ends. For those which are freely jointed at both ends, the strength is reduced to one fourth.

Weisbach applies to timber posts and struts a formula identical with equation 2 of page 362, with the following values of the constants:—

$$f = 7,200 \text{ lbs. on the square inch.}$$

$$a = \frac{1}{250}$$

The resistance of timber to crushing, while green, is about one-half of its resistance after having been dried.

SECTION 9.—*On Compound Girders, Frames, and Bridges.*

336. **Compound Girders in General.**—A compound girder is a structure which, as a whole, acts as a beam, resisting bending and breaking by a transverse load; but whose parts are subjected to a variety of stresses of different kinds, requiring to be separately considered; such as the Warren girder of Articles 162 and 163, and the Lattice girder of Articles 164 and 165.

In Part II., Chapter II., Section 1, it has already been shown how to determine the total stresses which act on the several pieces of a frame; in section 6 of the present chapter, it has been shown how the stress is distributed in a continuous beam; and in that and other sections, the resistance of materials to the various kinds of stress has been considered. The principal object of the present section is to indicate, by referring back to previous Articles, where the data and formulæ for determining the strength of the different parts of certain compound structures are to be found.

A girder consists of three principal parts: a *lower rib*, to resist tension; an *upper rib*, to resist thrust; and a *vertical web* or *frame*, to resist shearing force.

337. **Plate Iron Girders** are treated of in this section rather than in section 6, because the slender proportions of the parts subjected to a thrust sometimes render it necessary to compute their strength according to the laws of resistance to crushing by bending, explained in Article 328. Some of the forms of cross section employed in such beams are shown in figs. 161, 162, 163, 164, and 165. Fig. 161 is a plain I-shaped beam, rolled in one piece. In fig. 162, the upper and lower ribs consist each of a flat bar or narrow plate rivetted to a pair of angle irons, the two pairs of angle irons being rivetted to the upper and lower edges of the vertical web. In fig. 163 the construction is the same, except that the vertical web is double: this is the "*box-beam*," long employed in the platforms of blast furnaces, and first used in a railway bridge by Andrew Thomson about 1832, on the Pollok and Govan Railway. In fig. 164, the upper and lower ribs are each *built* of several layers of narrow plates or flat bars, rivetted to each other and to a pair of angle

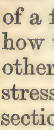

Fig.161.

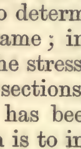

Fig.162.

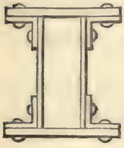

Fig. 163.

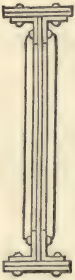

Fig. 164.

irons; the upper and lower pairs of angle irons are rivetted to the upper and lower edges of the vertical web, and the plates of the vertical web are connected and stiffened at each of their vertical joints by a pair of T-irons, in the manner of which a horizontal section has been already given in fig. 160, Article 334. The object of building the larger sizes of horizontal ribs in layers, instead of making them in one piece, is to make them of those sizes of iron which can easily be rolled of good quality, and which are usually found in the market. Beams resembling fig. 164 are sometimes made with a double vertical web, for the sake of lateral stiffness.

Fig. 165 represents the general form of the cross section of *great tubular* or *cellular girders*, characterized by Mr. Stephenson's principle, of carrying the railway through the interior of the beam, and by Mr. Fairbairn's principle, of giving stiffness by means of cells, already described in Article 333. The joints of the cells are connected and stiffened by covering plates outside as well as angle irons inside; and the plates of the two sides, which form a double vertical web, are stiffened and connected by T-irons, like those of fig. 164.

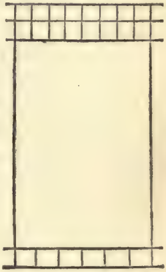
Fig. 165.

Smaller cellular girders are sometimes used, in which the top alone consists of one or two lines of cells, the girder in other respects being similar to fig. 164, with either a single or a double vertical web.

In all plate iron girders, the joints exposed to tension should have covering plates, double rivetted if the stress is great enough to require it, which is almost always the case in the lower rib (see Article 280). The joints exposed to thrust should be exactly plane, exactly perpendicular to the direction of the thrust, accurately fitted, and perfectly close, that the surfaces may abut equally over their whole extent. Should open or irregular abutting joints be discovered after the girder has been put together, they should be filed out, and a flat plate of steel driven tight into each opening. The plates or bars of which built ribs are composed should break joint in a manner similar to the bond of brickwork.

In plate iron girders generally, it is sufficiently accurate for practical purposes to consider the whole bending moment M at any vertical section as borne by the upper and lower ribs, and the whole shearing stress F by the vertical web; and also to consider the resistance of each of the horizontal ribs as concentrated at the centre of gravity of its section. Let h be the vertical depth between the centres of gravity of the sections of the upper and lower ribs; then the common value of the thrust along the compressed rib, and the tension along the stretched rib, is

$$P = \frac{M}{h} \dots\dots\dots\dots (1.)$$

Let S_1 be the sectional area of the compressed rib, f_1 its resistance to crushing per square inch, S_2 the sectional area of the stretched rib, f_2 its resistance to tearing per square inch; then

$$S_1 = \frac{P}{f_1} = \frac{M}{f_1 h}; \; S_2 = \frac{P}{f_2} = \frac{M}{f_2 h} \dots\dots\dots\dots (2.)$$

The values of the tenacity f_2 have already been considered in section 3. For plate beams with double-rivetted covering plates, its ultimate value may be taken at about 45,000 lbs. per square inch of section of rib. The ultimate resistance to crushing, f_1, may be taken at its full value of 36,000 lbs. per square inch in great tubular girders; but when the compressed rib is narrow as compared with its length, the tendency to lateral bending may be allowed for by means of the following empirical formula, of the kind already explained in section 8, Article 328:—

$$f_1 = \frac{f}{1 + a \cdot \frac{l'^2}{h'^2}}; \dots\dots\dots\dots (3.)$$

where $f = 36{,}000$, $a = \frac{1}{5{,}000}$, $h' =$ the breadth of the compressed rib, and $l' =$ the span of the girder, if it is not laterally stiffened by framing. In cases in which parallel beams are stiffened by horizontal diagonal braces, l' may be taken to denote the distance along the rib between a pair of the points to which braces are attached.

Let t be the thickness of the vertical web if single, or the sum of the thicknesses if double. Then its sectional area is $h\,t$ nearly; consequently, if f_3 be its resistance per unit of section to the shearing force,

$$ht = \frac{F}{f_3}; \text{ and } t = \frac{F}{f_3 h}; \dots\dots\dots\dots (4.)$$

and as the shearing stress is equivalent to a pull and a thrust in directions perpendicular to each other, and at angles of 45° to the horizon, f_3 should be the resistance of the vertical web to crushing, as determined by equation 2 of Article 328, page 362, in which, for $\frac{l}{h}$ is to be substituted $\frac{h}{h''}$, h being the depth of the web, as before, and h'' the width across the flanges of the stiffening ribs.

The shearing force F at each cross section is to be computed as for a *partial load,* extending over the greater of the two segments into which the section divides the beam, as explained in Article 313. The weight of the beam itself may be allowed for, either by the method of Article 314, or by the approximate method of Article 315.

Owing probably to the yielding of the joints, it is found that in computing the deflection of plate girders, when first loaded (Articles 300 to 303), a smaller modulus of elasticity ought to be taken than for continuous iron bars. Its value in lbs. per square inch is about two-thirds of the value for a continuous bar, so that the deflection is about one-half greater. But the part of that deflection due to the yielding of the joints is permanent; so that after the joints have "come to their bearing" the modulus of elasticity becomes the same as for a continuous bar.

338. For **Half-Lattice Beams and Lattice Beams,** the methods of determining the total stresses have been fully considered in Articles 162, 163, 164, and 165; and it has only to be added here, that the shearing force should be computed for a partial load, as in Article 313. The ultimate tenacity of the ties may be taken at $f_2 =$ from 50,000 to 60,000 lbs. per square inch. The resistance of the struts is to be computed as in Article 328. The figure of the strut diagonals has been considered in Article 332. The compressed rib may be a T-bar in small beams, and in larger beams a built rib or a cell. The remarks made in the last Article on abutting joints and on deflection are equally applicable in the present case. In designing those joints which are connected by means of bolts, rivets, or keys, the principles of Article 280 should be observed.

339. **A Bowstring Girder** consists of an arched rib resisting thrust; a horizontal tie resisting tension, and holding together the ends of the arched rib; a series of vertical suspending bars, by

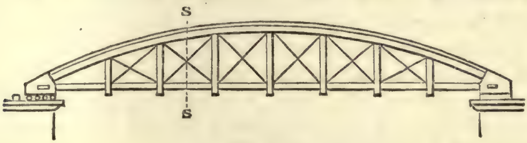

Fig. 166.

which the platform is hung from the arched rib, and a series of diagonal braces between the suspending bars. Such girders are executed in timber and in iron; sometimes the arched rib is made of cast iron, as being stronger against crushing than wrought iron, and the remainder of the structure of wrought iron.

The arched rib may be treated as uniformly loaded. According to Article 178, its condition is like that of an uniformly-loaded chain inverted, and its proper form a *parabola;* and the thrust along it at each point is to be found by the formulæ of Article 169. The tension along the horizontal tie is equal to the uniform horizontal component of the thrust along the arched rib.

The tension on each vertical suspending bar is the weight of those portions of the platform and of the tie rod which hang from it. To give lateral stability to the girder, the suspending bars are usually made of considerable breadth, and of a form of horizontal section resembling figs. 160 and 161, and are firmly bolted to the cross beams of timber or of wrought iron which carry the roadway.

When the beam is uniformly loaded, the arched rib is equilibrated, and *there is no stress on the diagonals.* The strength of the two diagonals which cross each other at a given plane of section S S′, is to be adapted to sustain the *excess of the greater shearing force due to a partial load above that due to an uniform load*, as given by the formulæ of Article 313.

340. **Stiffened Suspension Bridges.**—The suspension bridge is that which requires the least quantity of material to support a given load. But when it consists, as in Article 169, solely of cables or chains, suspending rods, and platform, it alters its figure with every alteration of the distribution of the load; so that a moving load causes it to oscillate in a manner which, if the load is heavy and the speed great, or even if the application of a small load takes place by repeated shocks, may endanger the bridge. To diminish this evil, it has long been the practice partially to stiffen suspension bridges by means of framework at the sides resembling a lattice girder.

It was formerly supposed that, to make a suspension bridge as stiff as a girder bridge, we should use lattice girders sufficiently strong to bear the load of themselves, and that, such being the case, there would be no use for the suspending chains. But Mr. P. W. Barlow, having made some experiments upon models, finds that very light girders, in comparison with what were supposed to be necessary, are sufficient to stiffen a suspension bridge. If mathematicians had directed their attention to the subject, they might have anticipated this result.

The present is believed to be the first investigation of its theory which has appeared in print.

The weight of the chain itself, being always distributed in the same manner, resists alteration of the figure of the bridge. By leaving it out of account, therefore, an error will be made on the safe side as to the stiffness of the bridge, and the calculation will be simplified.

Let fig. 167 represent one side of a suspension bridge, in which a girder is used to stiffen the bridge. In order that it may do so effectually, any partial or concentrated load on the platform must, by means of the girder, be transmitted to the chain in such a manner as to be uniformly distributed on the chain.

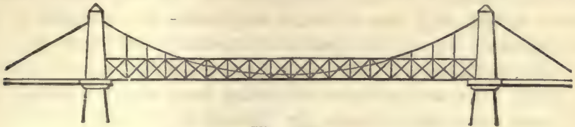

Fig. 167.

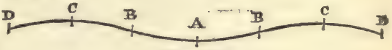

Fig. 168.

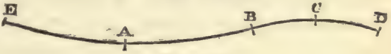

Fig. 169.

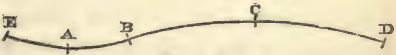

Fig. 170.

The girder must have its ends so fixed to the piers as to be incapable of rising or falling. Then the forces which act upon it may be thus classed :—*downward,* the load as applied; *downward or upward,* the resistances of the fastenings of the ends to their vertical displacement; *upward,* the uniformly distributed tension, acting through the suspension rods, between the girder and the chain.

The girder will be supposed to be of uniform section throughout its length.

Two cases will be considered :—first, that in which a given load is concentrated in the middle of the girder; and secondly, that in which a given portion of the length of that girder is uniformly loaded, and the remainder unloaded, like the partially loaded beam of Article 313. The second case is the most important in practice.

In each case, the *half-span* of the bridge will be denoted by c, and the horizontal distance of any point from the middle of the bridge by x.

Case I. *A single load* W, *applied at the centre of the girder,* tends to depress the chain in the middle, and consequently to raise it at the sides, and along with it to raise the beam near the ends; but the beam being, by its attachment to the piers, prevented from rising at the ends, takes a form like that represented by fig. 168: depressed in the middle at A, and concave upwards; elevated, and convex upwards at C, C; having points of contrary flexure at B, B; and again depressed at D, D, the points of attachment to the piers. Now this curved figure is the effect of three downward forces, applied at D, A, D, respectively, and of an uniformly distributed upward force, acting on the whole length of the girder. Each half of the girder, therefore, is in the condition of the beam described in Article 308, *inverted;* that is to say, the half-girder from A to D, if inverted, becomes a beam *supported* at D, *supported and fixed horizontal* at A, and loaded uniformly between A and D; and hence (referring to the formulæ of Article 307, case 3, and of Article 308) we have the following proportions amongst the lengths of the parts into which the half-girder is divided by the highest point C, and the point of contrary flexure B,

$$\overline{BC} = \overline{CD} = \frac{\overline{AC}}{\sqrt{3}} = 0{\cdot}577 \times \overline{AC}; \text{............} (1.)$$

and consequently, making $\overline{AC}$, the distance between the lowest and highest points, $= c'$, we have

$$\frac{c'}{c} = \frac{\overline{AC}}{\overline{AD}} = \frac{1}{1{\cdot}577} = 0{\cdot}634 \text{..................} (2.)$$

In order to determine the greatest moment of flexure, and the deflection, of the stiffening girder, $\overline{AC} = c'$ is to be taken as the half-span of a girder like that considered in Article 307, case 3, fixed at both ends, and loaded with an uniform load of the intensity

$$w = \frac{W}{2\,c'} = \frac{W}{1{\cdot}268\,c} \text{......................} (3.)$$

The greatest moment of flexure, as thus determined by the formulæ of Article 307, case 3, is at the point A, and has the following value:—

$$M_1 = \frac{w\,c'^2}{3} = \frac{c'\,W}{6} = 0{\cdot}1057\,c\,W; \text{............} (4.)$$

and to that moment of flexure must the strength of the stiffening girder be adapted.

The proof deflection may be measured in two ways: either between the highest and lowest points, C and A, or between the ends and the lowest point, D and A. The first may be called v_C, and the second v_D. Now by Article 307, case 3, we have

$$v_C = \frac{1}{8} \cdot \frac{f}{E} \cdot \frac{c'^2}{y} = 0{\cdot}05025 \times \frac{f c^2}{E y} \text{.............} (5.)$$

The points of support D are at the same level with the points of contrary flexure B, being, in fact, points of no curvature themselves; and from this it is easily found that

$$v_D = \frac{4}{9}\,v_C = \frac{1}{18} \cdot \frac{f c'^2}{E y} = 0{\cdot}0223 \times \frac{f c^2}{E y} \text{..........} (6.)$$

CASE 2. *The girder partially loaded.* Let E B, in either of the figs. 169, 170, represent the length of the loaded part of the stiffening girder, and B D that of the unloaded part; let w be the uniform intensity of the load, and x the distance of the point where the load terminates from the middle of the beam; x being considered as a positive quantity when the loaded part is the longer, as in fig. 169, and as a negative quantity when the loaded part is the shorter, as in fig. 170.

The ends E and D of the beam being fastened so as to be incapable of vertical displacement, the loaded segment E B is convex downwards, and the unloaded segment B D convex upwards: the loaded segment is in the condition of a beam supported at E and B, and uniformly loaded with the excess of the weight sustained above the force exerted between the girder and the chain; and the unloaded segment is in the condition of a beam *held down* at B and D, and loaded with an uniformly distributed *upward* force, being that exerted between the girder and chain. The greatest moment of flexure of each segment is at its middle point, being A for the loaded part, and C for the unloaded part.

The length of the loaded segment being

$$\overline{\mathrm{E\,B}} = c + x,$$

its gross load is

$$\mathrm{W} = w\,(c + x);$$

and the intensity of the force exerted between the girder and chain,

$$w' = \frac{w\,(c + x)}{2\,c} \qquad (1.)$$

This is the intensity of the *upward* load on the segment B D, whose length is B D $= c - x$; and consequently, according to Articles 290 and 291, the greatest moment of flexure of that segment, at C, is

$$\mathrm{M_C} = \frac{w'\,(c - x)^2}{8} = \frac{w\,(c + x)\,(c - x)^2}{16\,c} \qquad (2.)$$

The *amount* of the upward force exerted between the chain and B D is

$$\mathrm{W}' = w'\,(c - x) = \frac{w\,(c^2 - x^2)}{2\,c}; \qquad (3.)$$

and this also is the amount of the *net* load on E B, being the excess of the gross load above the part borne by the chain. The half of this quantity,

$$F = \frac{W'}{2} = \frac{w\,(c^2 - x^2)}{4\,c} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4.)$$

is the value at once of the supporting force exerted by the pier against the girder at E, of the shearing force between the two divisions of the girder at B, and of the downward force by which the end D of the girder is held at its point of attachment to the pier.

The intensity of the net load on E B is

$$w - w' = \frac{w\,(c - x)}{2\,c}\,; \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5.)$$

and the length of that segment being $c + x$, its greatest moment of flexure, at A, according to Articles 290 and 291, is

$$M_A = \frac{(w - w')\,(c + x)^2}{8} = \frac{w\ \ (c + x)^2 \cdot (c - x)}{16\,c} \ldots\ldots (6.)$$

By the usual process of finding maxima and minima, it is easily ascertained, that the greatest moment of flexure of the *loaded* division of the girder occurs when $x = \frac{c}{3}$; or when *two-thirds of the beam are loaded;* and that the greatest moment of flexure of the *unloaded* division of the girder occurs when $x = -\frac{c}{3}$, or when *two-thirds of the beam are unloaded;* and further, that those two greatest moments are of equal magnitude though opposite in direction, viz.:—

$$\text{max. } M_A = -\text{max. } M_C = \frac{2\,w\,c^2}{27}\,; \ldots\ldots\ldots\ldots (7.)$$

and the stiffening girder must be made sufficiently strong to bear this bending moment safely in either direction. Now, the greatest moment of flexure which would arise from an uniform load of the given intensity w over the whole beam unsupported by the chain is

$$\frac{w\,c^2}{2}\,;$$

therefore the *transverse strength of the stiffening girder should be four twenty-seventh parts of that of a simple girder of the same span suited to bear an uniform load of the same intensity.*

The greatest value of the shearing force F in equation 4 occurs when *one-half* of the girder is loaded, or $x = 0$, and its amount is

$$\max. \mathrm{F} = \frac{w\,c}{4} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8.)$$

When two-thirds of the beam are loaded, the proof deflection of A below a straight line joining E and B, according to Article 300, is

$$v_{\mathrm{A}} = \frac{5}{12} \cdot \frac{f}{\mathrm{E}} \cdot \frac{(c+x)^2}{4\,y} = \frac{4}{9} \cdot \frac{5}{12} \cdot \frac{f\,c^2}{\mathrm{E}\,y} = \frac{5}{27} \cdot \frac{f\,c^2}{\mathrm{E}\,y}; \ldots (9.)$$

or *four-ninths* of the proof deflection of a beam of the same figure, uniformly loaded, of the span $2\,c$, unsupported by a chain. At the same time, the elevation of C above a straight line joining B and D is

$$v_{\mathrm{C}} = \frac{5}{12} \cdot \frac{f}{\mathrm{E}} \cdot \frac{(c-x)^2}{4\,y} = \frac{1}{9} \cdot \frac{5}{12} \cdot \frac{f\,c^2}{\mathrm{E}\,y} = \frac{5}{108} \cdot \frac{f\,c^2}{\mathrm{E}\,y} \ldots (10.)$$

The proof depression of the lowest point of the beam, A, below the highest, C, is given by the equation

$$v_{\mathrm{A}} + v_{\mathrm{C}} = \frac{5}{9} \cdot \frac{5}{12} \cdot \frac{f\,c^2}{\mathrm{E}\,y} = \frac{25}{108} \cdot \frac{f\,c^2}{\mathrm{E}\,y}; \ldots\ldots\ldots (11.)$$

or *five-ninths* of the proof deflection of an uniformly loaded beam. *

---

* In the preceding solution of Case 2, which appeared in the first edition of this work, the effect of the resistance of the chain to disfigurement upon the figure of the auxiliary girder is neglected; and hence the result is in almost every case an approximation only; but it can be shown that the error is always on the safe side, four twenty-sevenths of the strength of a simple girder being *somewhat more* than sufficient for the strength of the stiffening girder. In order to make the solution exact, the extensibility of the chain should be so great as to make its *proof central depression* nearly equal to the *proof deflection* of the stiffening girder; but in practice the proof depression of the chain is always much less.

The first solution in which the action of the chain just referred to is taken into account appeared in an editorial article of the *Civil Engineer and Architect's Journal* for November and December, 1860; and this is done by introducing into the conditions of the problem an equation, expressing that under all the alterations of the figure of the chain produced by the bending of the stiffening girder, the span continues constant.

Having applied the principle just stated to the problem of Case 2, the author of this work has arrived at the following results, supposing the chain to be *inextensible.*

The greatest bending moment of the stress on the stiffening girder takes place when 0·417, or about five-twelfths, of the span of the bridge are loaded, and 0·583, or about seven-twelfths, unloaded.

That moment is 0·138 of the bending moment which would be produced by an uniform load of the same intensity on a girder supported at the ends only.

Hence it appears that if the chain be supposed inextensible, the proportion borne by the strength of the stiffening girder to that of a simple girder of the same span, suited to bear an uniform load of the same intensity with the travelling load, ought to be..0·138 : 1;
while if the chain is supposed very extensible, as in the approximate solution, that proportion is found to be 4 : 27, or..............................0·148 : 1;
so that in the intermediate cases that occur in practice no material error will be committed if that proportion be made 1 : 7, or.........................0·143 : 1.

341. **Ribbed Arches.**—Bridges are frequently constructed whose arches consist of iron or timber ribs springing from stone abutments, as in fig. 171. In such cases it ought to be considered, that each rib fulfils at once the functions of an *equilibrated arch*, sustaining an uniform load of a certain intensity, and having a certain thrust along it, to be computed by the principles of Articles 169 and 178, and those of a *stiffening girder*, suited to produce an uniform distribution of a partial load, according to the principles of Article 340. Therefore, in designing the cross section of a rib for such a bridge, a provisional cross section ought first to be designed, suitable to bear a bending moment, upward or downward, of *four twenty-sevenths* of that which an uniform load of the given intensity would produce on a straight girder of the same span; and in the second place, it should be determined in what proportion the thrust along the rib, considered as an equilibrated arch, will increase the intensity of the greatest stress on the provisional section already designed, and the breadths of that section should be increased in that proportion, to obtain the final cross section.

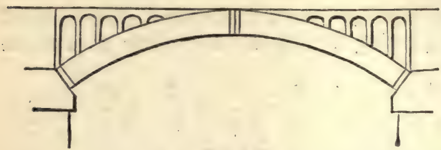

Fig. 171.

SECTION 10.—*Miscellaneous Remarks on Strength and Stiffness.*

342. **Effects of Temperature.**—At a temperature of 600° Fahrenheit, the tenacity of iron was found by Mr. Fairbairn not to be diminished. That of copper and brass, at the same temperature, is reduced to about two-thirds of its ordinary magnitude. Sudden cooling from a high temperature tends to make most substances hard, stiff, and brittle; gradual cooling tends to make them soft and tough; and if often repeated or performed slowly from a very high temperature, to weaken them. Various effects of temperature on the elasticity of solids have been ascertained by Dr. Joule, Dr. Thomson, and Professor Kupfer; but they are more important to the science of molecular physics than to the art of construction.

343. The **Effects of Repeated Meltings on Cast Iron** have been ascertained by Mr. Fairbairn. Up to and beyond the fourteenth melting the resistance to crushing increases; but the resistance to cross-breaking reaches its maximum about the twelfth melting, and afterwards diminishes, from the metal becoming brittle and crystalline.

344. The **Effects of Ductility** on strength form the subject of a

paper by Professor James Thomson in the *Cambridge and Dublin Mathematical Journal.* That author shows, that a bent bar or a twisted rod of a ductile material, by being slowly and gradually strained, may be brought into such a condition as to have nearly the whole of its cross section in the condition of proof or limiting stress instead of the outer layers only, and may thus have its strength increased much beyond that given by the ordinary formulæ.

345. **Internal Friction** is a term which may be used until a better shall be devised to express a phenomenon recently observed by Sir William Thomson in the extension of copper wire by a direct pull. The tension of the wire is increased, step by step, by successive augmentations of the load within the limits of permanent elasticity, and the elongation is observed at each step. Then by successive diminutions of the load, the tension is diminished by the same series of steps in the reverse order, and the elongation observed. When the load is completely removed, the wire recovers its original length without "set" or permanent elongation, but for each degree of tension the elongation is greater during the shortening of the wire than during the lengthening; as if there were some molecular force analogous to friction, in so far as it impedes motion both ways, making the elongation less than it would otherwise be while the wire is being elongated, and greater than it would otherwise be while the wire is returning to its original length. It appears also that the force in question must depend in some way on the stress, from its disappearing when the tension is removed.

346. It must be obvious that much of the subject of strength and stiffness is in a provisional state, both as to mathematical theory and as to experimental data. Considerable improvement in both these respects may be anticipated from researches now in progress.

CONDENSED SUMMARY OF EXPERIMENTS BY MESSRS. ROBERT NAPIER AND SONS ON THE TENACITY OF IRON AND STEEL.

(*For details, see Transactions of the Institution of Engineers in Scotland*, 1858-59.)

	Tenacity in lbs. per square inch.	
STEEL BARS.	Strongest Quality.	Weakest Quality.
Cast Steel, ...	132,909	92,015
Blistered Steel (one quality only)	104,298	
Bessemer's (do.) ..	111,460	
Homogeneous Metal, ...	90,647	89.724
Puddled Steel, ...	71,486	62,769
IRON BARS.		
Yorkshire, ...	66,392	60,075
Staffordshire, ...	62,231	56,715
Lanarkshire, ...	64,795	56,655
Lancashire, ...	60,110	53,775
Swedish, ...	48,232	47,855
Russian, ...	56,805	49,564
Hammered Scrap,........	55,878	53,420
Cut out of large forged crank, ...	47,582	44,758

	Tenacity in lbs. per square inch.	
STEEL PLATES.	Strongest Quality.	Weakest Quality.
Cast Steel, ...	95.299	72,338
Homogeneous Metal, ...	96.715	72,994
Puddled Steel, ...	93,979	72,366
IRON PLATES.		
Yorkshire, ...	56,735	49,338
Durham (one quality only)	48,979	
Staffordshire, ...	54,123	45,584
Lanarkshire, ...	51,349	41,743
IRON STRAPS, &c.		
Various districts, ...	55,937	41,386

The strength of each quality is the mean of at least four experiments, and sometimes of eight.

PART III.

PRINCIPLES OF CINEMATICS, OR THE COMPARISON OF MOTIONS.

347. **Division of the Subject.**—The science of cinematics, and the fundamental notions of rest and motion to which it relates, having already been defined in the Introduction, Articles 8, 9, 10, 11; it remains to be stated, that the principles of cinematics, or the comparison of motions, will be divided and arranged in the present part of this treatise in the following manner:—

I. Motions of Points.
II. ... ... Rigid Bodies or Systems.
III. ... ... Pliable Bodies and Fluids.
IV. ... ... Connected Bodies.

CHAPTER I.

MOTIONS OF POINTS.

SECTION 1.—*Motion of a Pair of Points.*

348. **Fixed and Nearly Fixed Directions.**—From the definition of motion given in Article 9, it follows, that in order to determine the relative motion of a pair of points, which consists in the change of length and direction of the straight line joining them, that line must be compared, at the beginning and end of the motion considered, with some fixed or standard length, and with at least two fixed directions. Standard lengths have already been considered in Article 7.

An *absolutely fixed direction* may be ascertained by means whose principles cannot be demonstrated until the subject of dynamics is considered. For the present it is sufficient to state, that when a solid body rotates free from the influence of any external force tending to change its rotation, there is an absolutely fixed direction called that of the *axis of angular momentum*, which bears certain relations to the successive positions of the body.

A *nearly fixed direction* is that of a straight line joining a pair of points in two bodies whose distance from each other is very great, such as the earth and a fixed star.

A *line fixed relatively to the earth* changes its absolute direction (unless parallel to the earth's axis) in a manner depending on the earth's rotation, and returns periodically to its original absolute direction at the end of each *sidereal day* of 86,164 seconds. This rate of change of direction is so slow compared with that which takes place in almost all pieces of mechanism to which cinematical and dynamical principles are applied, that in almost all questions of applied mechanics, directions fixed relatively to the earth may be treated as sufficiently nearly fixed for practical purposes.

When the motions of pieces of mechanism relatively to each other, or to the frame by which they are carried, are under consideration, directions fixed relatively to the frame, or to one of the pieces of the machine, may be considered provisionally as fixed for the purposes of the particular question.

9 349. **Motion of a Pair of Points.**—In fig. 172, let A_1 B_1 represent the relative situation of a pair of points at one instant, and A_2 B_2 the relative situation of the same pair of points at a later instant. Then the change of the straight line $\overline{A\,B}$ between those points, from the length and direction represented by $\overline{A_1\,B_1}$ to the length and direction represented by $\overline{A_2\,B_2}$, constitutes the *relative motion* of the pair of points A, B, during the interval between the two instants of time considered.

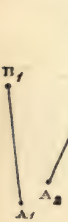

Fig. 172.

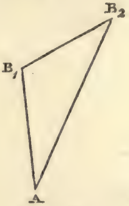

Fig. 173.

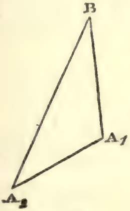

Fig. 174.

To represent that relative motion by one line, let there be drawn, from one point A, fig. 173, a pair of lines, $\overline{A\,B_1}$, $\overline{A\,B_2}$, equal and parallel to $\overline{A_1\,B_1}$, $\overline{A_2\,B_2}$, of fig. 172; then A represents one of the pair of points whose relative motion is under consideration, and B_1, B_2, represent the two successive positions of the other point B relatively to A; and the line $\overline{B_1\,B_2}$ represents *the motion of* B *relatively to* A.

Or otherwise, as in fig. 174, from a single point B let there be drawn a pair of lines, $\overline{B\,A_1}$, $\overline{B\,A_2}$, equal and parallel to $\overline{A_1\,B_1}$, $\overline{A_2\,B_2}$, of fig. 172; then A_1, A_2, represent the two successive positions of A relatively to B; and the line $\overline{A_1\,A_2}$, equal and parallel to $\overline{B_1\,B_2}$ of fig. 173, but *pointing in the contrary direction*, represents *the motion of* A *relatively to* B.

350. **Fixed Point and Moving Point.**—In fig. 173, A is treated as the fixed point, and B as the moving point; and in fig. 174, B is treated as the fixed point, and A as the moving point; and these are simply two different methods of representing to the mind the same relation between the points A and B (see Article 10).

351. **Component and Resultant Motions.**—Let O be a point assumed as fixed, and A and B two successive positions of a second point relatively to O. In order to express mathematically the amount and direction of $\overline{\mathrm{AB}}$, the motion of the second point relatively to O, that line may be compared with three *axes*, or lines in fixed directions, traversing the fixed point O, such as O X, O Y, O Z.

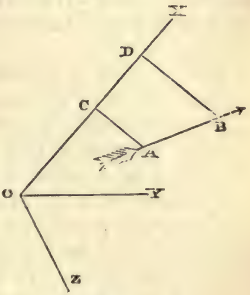

Fig 175.

Through A and B draw straight lines A C, B D, parallel to the plane of O Y and O Z, and cutting the axis O X in C and D. Then $\overline{\mathrm{CD}}$ is said to be the *component* of the motion of the second point relatively to O, *along* or *in the direction of* the axis O X; and by a similar process are found the components of the motion $\overline{\mathrm{AB}}$ along O Y and O Z. The entire motion $\overline{\mathrm{AB}}$ is said to be the *resultant* of these components, and is evidently the diagonal of a parallelopiped of which the components are the sides.

The three axes are usually taken at right angles to each other; in which case A C and B D are perpendiculars let fall from A and B upon O X; and if α be the angle made by the direction of the motion $\overline{\mathrm{AB}}$ with O X,

$$\overline{\mathrm{CD}} = \overline{\mathrm{AB}} \cdot \cos \alpha.$$

The relations between resultant and component motions are exactly analogous to those between the lines representing resultant and component couples, which have already been explained in Articles 32, 33, 34, 35, 36, and 37.

352. The **Measurement of Time** is effected by comparing the events, and especially the motions, which take place in intervals of time.

Equal times are the times occupied by the same body, or by equal and similar bodies, under precisely similar circumstances, in performing equal and similar motions. The *standard unit of time* is the period of the earth's rotation, or *sidereal day*, which has been proved by Laplace, from the records of celestial phenomena, not to have changed by so much as one *eight-millionth* part of its length in the course of the last two thousand years.

A subordinate unit is the *second*, being the time of one swing of a pendulum, so adjusted as to make 86,400 oscillations in 1·00273791 of a sidereal day; so that a sidereal day is 86164·09 seconds.

The length of a solar day is variable; but the *mean solar day*, being the exact mean of all its different lengths, is the period already mentioned of 1·00273791 of a sidereal day, or 86,400 seconds. The divisions of the mean solar day into 24 hours, of each hour into 60 minutes, and of each minute into 60 seconds, are familiar to all.

Fractions of a second are measured by the oscillations of small pendulums, or of springs, or by the rotations of bodies so contrived as to rotate through equal angles in equal times.

353. **Velocity** is the ratio of the number of units of length described by a point in its motion relatively to another point, to the number of units of time in the interval occupied in describing the length in question; and if that ratio is the same, whether it be computed for a longer or a shorter, an earlier or a later, part of the motion, the velocity is said to be UNIFORM. Velocity is expressed in *units of distance per unit of time.* For different purposes, there are employed various units of velocity, some of which, together with their proportions to each other, are given in the following table:—

Comparison of Different Measures of Velocity.

	Miles per hour.		Feet per second.		Feet per minute.		Feet per hour.
	1	=	$1{\cdot}4\dot{6}$	=	88	=	5280·
	$0{\cdot}68\dot{1}\dot{8}$	=	1	=	60	=	3600
	$0{\cdot}011\dot{3}\dot{6}$	=	$0{\cdot}01\dot{6}$	=	1	=	60
	$0{\cdot}00018\dot{9}\dot{3}$	=	$0{\cdot}0002\dot{7}$	=	$0{\cdot}01\dot{6}$	=	1
1 nautical mile per hour, or "knot,"……	= 1·1507	=	1·6877	=	101·262	=	6075·74

In treating of the general principles of mechanics, the *foot per second* is the unit of velocity commonly employed in Britain. The units of time being the same in all civilized countries, the proportions amongst their units of velocity are the same with those amongst their linear measures.

Component and resultant velocities are the velocities of component and resultant motions, and are related to each other in the same way with those motions, which have already been treated of in Article 351.

354. **Uniform Motion** consists in the combination of uniform velocity with uniform direction; that is, with motion along a straight line whose direction is fixed.

SECTION 2.—*Uniform Motion of Several Points.*

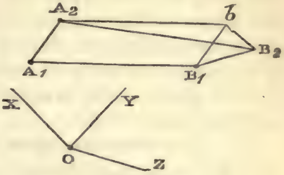

Fig. 176.

355. **Motion of Three Points.**—THEOREM. *The relative motions of three points in a given interval of time are represented in direction and magnitude by the three sides of a triangle.* Let O, A, B, denote the three points. Any one of them may be taken as a fixed point; let O be so chosen; and let O X, O Y, O Z, fig. 176, be axes traversing it in fixed directions. Let A_1 and B_1 be the positions of A and B relatively to O at the beginning of the given interval of time, and A_2 and B_2 their positions at the end of that interval. Then $\overline{A_1 A_2}$ and $\overline{B_1 B_2}$ are the respective motions of A and B relatively to O. Complete the parallelogram $A_1 B_1 b A_2$; then because $\overline{A_2 b}$ is parallel and equal to $A_1 B_1$, b is the position which B would have at the end of the interval, *if it had no motion relatively to* A; but B_2 is the actual position of B at the end of the interval; therefore, $\overline{b B_2}$ is the motion of B relatively to A. Then in the triangle $B_1 b B_2$,

$\overline{B_1 b} = \overline{A_1 A_2}$ is the motion of A relatively to O,
$\overline{b B_2}$ is the motion of B relatively to A,
$\overline{B_1 B_2}$ is the motion of B relatively to O;

so that those three motions are represented by the three sides of a triangle.—Q. E. D.

This Theorem might be otherwise expressed by saying, *that if three moving points be considered in any order, the motion of the third relatively to the first is the resultant of the motion of the third relatively to the second, and of the motion of the second relatively to the first;* the word "*resultant*" being understood as already explained in Article 351.

356. **Motions of a Series of Points.**—COROLLARY. *If a series of points be considered in any order, and the motion of each point determined relatively to that which precedes it in the series, and if the relative motion of the last point and the first point be also determined, then will those motions be represented by the sides of a closed polygon.* Let O be the first point, A, B, C, &c., successive points following it, M the last point but one, and N the last point; and, for brevity's sake, let the relative motion of two points, such as B and C, be denoted thus (B, C). Then by the Theorem of Article 355 (O, A), (A, B), and (O, B) are the three sides of a triangle; also (O, B), (B, C), and (O, C), are the three sides of a triangle; therefore

(O, A), (A, B), (B, C), and (O, C), are the four sides of a quadrilateral; and by continuing the same process, it is shown, that how great soever the number of points, (O, N), is the closing side of a polygon, of which (O, A), (A, B), (B, C), (C, D), &c., (M, N) are the other sides.—Q. E. D. In other words, *the motion of the last point relatively to the first is the resultant of the motions of each point of the series relatively to that preceding it.*

This proposition is exactly analogous to that of the "polygon of couples," Article 37.

357. The **Parallelopiped of Motions** is a case of the polygon of motions, analogous to the parallelopiped of forces in Article 54. In fig. 177, let there be four points, O, A, B, C, of which one, O, is assumed as fixed, and is traversed by three axes in fixed directions, O X, O Y, O Z. In a given interval of time, let A have the motion $\overline{A_1 A_2}$ along or parallel to OX; let B have, in the same interval, the motion $\overline{b B_2}$ parallel to O Y, and relatively to A; then $\overline{B_1 B_2}$, the diagonal of the parallelogram whose sides are $\overline{B_1 b} = \overline{A_1 A_2}$ and $\overline{b B_2}$, is the motion of B relatively to O. Let C have, relatively to B, the motion $\overline{c C_2}$ parallel to O Z; then $\overline{C_1 C_2}$, the diagonal of the parallelopiped whose edges are $\overline{A_1 A_2}$, $\overline{b B_2}$, and $\overline{c C_2}$, is the motion of C relatively to O, being the resultant of the motions represented by those three edges. This is a *mechanical* explanation of the composition of motions, leading to results corresponding with the *geometrical* explanation of Article 351.

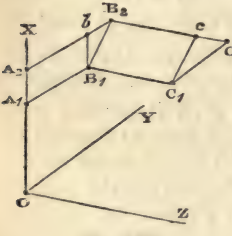

Fig. 177.

358. **Comparative Motion** is the relation which exists between the simultaneous motions of two points relatively to a third, which is assumed as fixed. The comparative motion of two points is expressed, in the most general case, by means of four quantities, viz. :—

(1.) The *velocity ratio,** or the proportion which their velocities bear to each other.

(2.) (3.) (4.) The *directional relation,** which requires, for its complete expression, three angles. Those three angles may be measured in different ways, and one of those ways is the following :—

(2.) The angle made by the directions of the compared motions with each other.

(3.) The angle made by a plane parallel to those two directions with a fixed plane.

---

* These terms are adopted from Mr. Willis's work on Mechanism.

(4.) The angle made by the intersection of those two planes with a fixed direction in the fixed plane.

Thus, the comparative motion of two points relatively to a third, is expressed by means of one of those groups of four elements which Sir William Rowan Hamilton has called "*quaternions.*" In most of the practical applications of cinematics, the motions to be compared are limited by conditions which render the comparison more simple than it is in the general case just described. In machines, for example, the motion of each point is limited to two directions, forward or backward in a fixed path; so that the comparative motion of two points is sufficiently expressed by means of the velocity ratio, together with a directional relation expressed by + or −, according as the motions at the instant in question are similar or contrary.

SECTION 3.—*Varied Motion of Points.*

359. **Velocity and Direction of Varied Motion.**—The motion of one point relatively to another may be varied, either by change of velocity, or by change of direction, or by both combined, which last case will now be considered, as being the most general.

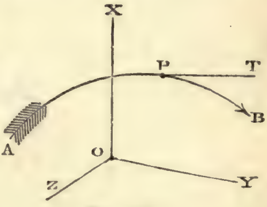

Fig. 178.

In fig. 178, let O represent a point assumed as fixed, O X, O Y, O Z, fixed directions, and A B part of the *path* or orbit traced by a second point in its varied motion relatively to O. At the instant when the second point reaches a given position, such as P, in its path, the *direction* of its motion is obviously that of $\overline{\mathrm{PT}}$, a tangent to the path at P.

To find the velocity at the instant of passing P, let Δt denote an interval of time which includes that instant, and Δs the distance traced in that interval. Then

$$\frac{\Delta s}{\Delta t}$$

is an *approximation* to the velocity at the instant in question, which will approach continually nearer and nearer to the exact velocity as the interval Δt and the distance Δs are made shorter and shorter; and the *limit* towards which $\frac{\Delta s}{\Delta t}$ converges, as Δs and Δt are indefinitely diminished, and which is denoted by

$$v = \frac{d s}{d t}, \qquad \text{(1.)}$$

is the *exact* velocity at the instant of passing P. This is the process called "*differentiation.*"

Should the velocity at each instant of time be known, then the distance $s_1 - s_0$, described during an interval of time $t_1 - t_0$, is found by *integration* (see Article 81), as follows :—

$$s_1 - s_0 = \int_{t_0}^{t_1} v \, d \, t \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.)$$

360. **Components of Varied Motion.**—All the propositions of the two preceding sections, respecting the composition and resolution of motions, are applicable to the velocities of varied motions at a given instant, each such velocity being represented by a line, such as $\overline{\mathrm{P\,T}}$, in the direction of the tangent to the path of the point which moves with that velocity, at the instant in question. For example, if the axes O X, O Y, O Z, are at right angles to each other, and if the tangent $\overline{\mathrm{P\,T}}$ makes with their directions respectively the angles α, β, γ, then the three rectangular components of the velocity of the point parallel to those three axes are

$$v \cos \alpha; \; v \cos \beta; \; v \cos \gamma.$$

Let x, y, z, be the co-ordinates of any point, such as P, in the path A P B, as referred to the three given axes. Then it is well known that

$$\cos \alpha = \frac{d\,x}{d\,s}; \; \cos \beta = \frac{d\,y}{d\,s}; \; \cos \gamma = \frac{d\,z}{d\,s};$$

and consequently the three components of the velocity v are

$$v \cos \alpha = \frac{d\,x}{d\,t}; \; v \cos \beta = \frac{d\,y}{d\,t}; \; v \cos \gamma = \frac{d\,z}{d\,t}; \ldots\ldots (3.)$$

and these are related to their resultant by the equation

$$\left(\frac{d\,x}{d\,t}\right)^2 + \left(\frac{d\,y}{d\,t}\right)^2 + \left(\frac{d\,z}{d\,t}\right)^2 = v^2 \ldots\ldots\ldots\ldots\ldots (4.)$$

361. **Uniformly-Varied Velocity.**—Let the velocity of a point either increase or diminish at an uniform rate; so that if t represents the time elapsed from a fixed instant when the velocity was v_0, the velocity at the end of that time shall be

$$v = v_0 + a\,t; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

a being a constant quantity, which is the *rate of variation* of the velocity, and is called *acceleration* when positive, and *retardation* when negative. Then the *mean* velocity during the time t is

$$\frac{v_0 + v}{2} = v_0 + \frac{a\,t}{2}; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.)$$

and the distance described is

$$s = v_0 t + \frac{a t^2}{2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.)$$

To find the velocity of a point, whose velocity is uniformly varied, at a given instant, and the rate of variation of that velocity, let the distances, Δs_1, Δs_2, described in two equal intervals of time, each equal to Δt, before and after the instant in question, be observed. Then the velocity at the instant between those intervals is

$$v = \frac{\Delta s_1 + \Delta s_2}{2 \Delta t} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4.)$$

and its rate of variation is

$$a = \frac{\Delta v}{\Delta t} = \frac{\Delta s_2 - \Delta s_1}{(\Delta t)^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5.)$$

362. **Varied Rate of Variation of Velocity.**—When the velocity of a point is neither constant nor uniformly-varied, its rate of variation may still be found by applying to the velocity the same operation of *differentiation*, which, in Article 359, was applied to the distance described in order to find the velocity. The result of this operation is expressed by the symbols,

$$\frac{d v}{d t} = \frac{d^2 s}{d t^2};$$

and is the limit to which the quantity obtained by means of the formula 5 of Article 361 continually approximates, as the interval denoted by Δt is indefinitely diminished.

363. **Uniform Deviation** is the change of motion of a point which moves with uniform velocity in a circular path. The *rate* at which uniform deviation takes place is determined in the following manner.

Let C, fig. 179, be the centre of the circular path described by a point A with an uniform velocity v, and let the radius $\overline{CA}$ be denoted by r. At the beginning and end of an interval of time Δt, let A_1 and A_2 be the positions of the moving point. Then

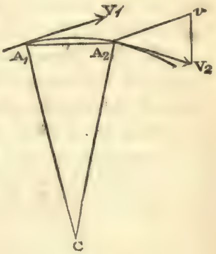

Fig. 179.

$$\text{the arc } A_1 A_2 = v \Delta t; \text{ and}$$

$$\text{the chord } A_1 A_2 = v \Delta t \cdot \frac{\text{chord}}{\text{arc}}.$$

The velocities at A_1 and A_2 are represented by the equal lines

$\overline{A_1 V_1} = \overline{A_2 V_2} = v$, touching the circle at A_1 and A_2 respectively. From A_2 draw $\overline{A_2 v}$ equal and parallel to $\overline{A_1 V_1}$, and join $\overline{V_2 v}$. Then the velocity $\overline{A_2 V_2}$ may be considered as compounded of $\overline{A_2 v}$ and $\overline{v V_2}$; so that $\overline{v V_2}$ is the *deviation* of the motion during the interval Δt; and because the isosceles triangles $A_2 v V_2$, $C A_1 A_2$, are similar :—

$$\overline{v V_2} = \frac{\overline{A_2 V_2} \cdot \overline{A_1 A_2}}{\overline{C A}} = \frac{v^2 \cdot \Delta t}{r} \cdot \frac{\text{chord}}{\text{arc}};$$

and the *approximate rate* of that deviation is

$$\frac{v^2}{r} \cdot \frac{\text{chord}}{\text{arc}};$$

Divide by Δt [illegible]

but the deviation does not take place by instantaneous changes of velocity, but by insensible degrees; so that the true rate of deviation is to be found by finding the limit to which the approximate rate continually approaches as the interval Δt is diminished indefinitely. Now the factor $\frac{v^2}{r}$ remains unaltered by that diminution; and the ratio of the chord to the arc approximates continually to equality; so that the limit in question, or *true rate of deviation*, is expressed by

$$\frac{v^2}{r} \quad \text{..................................} \quad (1.)$$

364. **Varying Deviation.**—When a point moves with a varying velocity, or in a curve not circular, or has both these variations of motion combined, the *rate of deviation* at a given instant is still represented by equation 1 of Article 363, provided v be taken to denote the velocity, and r the radius of curvature of the path, of the point at the instant in question.

365. The **Resultant Rate of Variation** of the motion of a point is found by considering the rate of variation of velocity and the rate of deviation as represented by two lines, the former in the direction of a tangent to the path of the point, and the latter in the direction of the radius of curvature at the instant in question, and taking the diagonal of the rectangle of which those two lines are the sides, which has the following value :—

$$\sqrt{\left(\frac{dv}{dt}\right)^2 + \frac{v^4}{r^2}} = \sqrt{\left\{\left(\frac{d^2 s}{dt^2}\right)^2 + \frac{1}{r^2}\left(\frac{ds}{dt}\right)^4\right\}} \ldots (1.)$$

366. The **Rates of Variation of the Component Velocities** of a point parallel to three rectangular axes, are represented as follows :—

[illegible]

$$\frac{d^2 x}{d t^2};\ \frac{d^2 y}{d t^2};\ \frac{d^2 z}{d t^2};\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.)$$

See Art 162

and if a rectangular parallelopiped be constructed, of which the edges represent these quantities, its diagonal, whose length is

$$\sqrt{\left\{\left(\frac{d^2 x}{d t^2}\right)^2 + \left(\frac{d^2 y}{d t^2}\right)^2 + \left(\frac{d^2 z}{d t^2}\right)^2\right\}}\ \ldots\ldots\ldots\ldots(2.)$$

will represent the *resultant rate of variation*, already given in another form in equation 1 of Article 365.

367. The **Comparison of the Varied Motions** of a pair of points relatively to a third point assumed as fixed, is made by finding the ratio of their velocities, and the directional relation of the tangents of their paths, at the same instant, in the manner already described in Article 358 as applied to uniform motions. It is evident that the comparative motions of a pair of points may be so regulated as to be constant, although the motion of each point is varied, provided the variations take place for both points at the same instant, and at rates proportional to their velocities.

X

See Calc

$$\cos\alpha\ \frac{d^2 s}{dt^2} = \frac{d^2 x}{dt^2}$$

$$\cos\beta\ \frac{d^2 s}{dt^2} = \frac{d^2 y}{dt^2}$$

$$\cos\gamma\ \frac{d^2 s}{dt^2} = \frac{d^2 z}{dt^2}$$

CHAPTER II.

MOTIONS OF RIGID BODIES.

SECTION 1.—*Rigid Bodies, and their Translation.*

368. The term **Rigid Body** is to be understood to denote a body, or an assemblage of bodies, or a system of points, whose figure undergoes no alteration during the motion which is under consideration.

369. **Translation** or **Shifting** is the motion of a rigid body relatively to a fixed point, when the points of the rigid body have no motion relatively to each other; that is to say, when they all move with the same velocity and in the same direction at the same instant, so that no line in the rigid body changes its direction.

It is obvious that if three points in the rigid body, not in the same straight line, move in parallel directions with equal velocities at each instant, the body must have a motion of translation.

The paths of the different points of the body, provided they are all equal and similar, and at each instant parallel, may have any figure whatsoever.

SECTION 2.—*Simple Rotation.*

370. **Rotation** or **Turning** is the motion of a rigid body when lines in it change their direction. Any point in or rigidly attached to the body may be assumed as a fixed point to which to refer the motions of the other points. Such a point is called *centre of rotation.*

371. **Axis of Rotation.**—THEOREM. *In every possible change of position of a rigid body, relatively to a fixed centre, there is a line traversing that centre whose direction is not changed.* In fig. 180, let O be the centre of rotation, and let A and B denote any two other points in the body, whose situations relatively to O are, before the turning, A_1, B_1, and after the turning, A_2, B_2. Join $\overline{A_1 A_2}$, $\overline{B_1 B_2}$, forming the isosceles triangles O A_1 A_2, O B_1 B_2. Bisect the bases of those triangles in C and

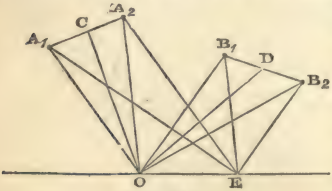

Fig. 180.

D respectively, and through the points of bisection draw two planes perpendicular to the respective bases, intersecting each other in the straight line $\overline{OE}$, which must traverse O. Let E be any point in the line O E; then $E A_1 A_2$, and $E B_1 B_2$, are isosceles triangles; and E is at the same distance from O, A, and B, before and after the turning; therefore E is one and the same point in the body, whose place is unchanged by the turning; and this demonstration applies to every point in the straight line $\overline{OE}$; therefore that line is unchanged in direction.—Q. E. D.

COROLLARY. It is evident that every line in the body, parallel to the axis, has its direction unchanged.

372. The **Plane of Rotation** is any plane perpendicular to the axis. The **Angle of Rotation**, or angular motion, is the angle made by the two directions, before and after the turning, of a line perpendicular to the axis.

373. The **Angular Velocity** of a turning body is the ratio of the angle of rotation, expressed in terms of radius, to the number of units of time in the interval of time occupied by the angular motion. Speed of turning is sometimes expressed also by the number of turns or fractions of a turn in a given time. The relation between these two modes of expression is the following:—Let a be the angular velocity, as above defined, and T the turns in the same unit of time; then

$$T = \frac{a}{2\pi};$$

$$a = 2\pi T;$$

$$(2\pi = 6{\cdot}2831852).$$

374. **Uniform Rotation** consists in uniformity of the angular velocity of the turning body, and constancy of the direction of its axis of rotation.

375. **Rotation common to all Parts of Body.**—Since the angular motion of rotation consists in the change of direction of a line in a plane of rotation, and since that change of direction is the same how short soever the line may be, it is evident that the condition of rotation, like that of translation, is common to every particle, how small soever, of the turning rigid body, and that the angular velocity of turning of each particle, how small soever, is the same with that of the entire body This is otherwise evident by considering, that each part into which a rigid body can be divided turns completely about in the same time with every other part, and with the entire body.

376. **Right and Left-Handed Rotation.**—The *direction* of rotation round a given axis is distinguished in an arbitrary manner into

right-handed and *left-handed.* One end of the axis is chosen, as that from which an observer is supposed to look along the direction of the axis towards the rotating body. Then if the body seems to the observer to turn in the same direction in which the sun seems to revolve to an observer north of the tropics, the rotation is said to be *right-handed;* if in the contrary direction, *left-handed:* and it is usual to consider the angular velocity of right-handed rotation to be positive, and that of left-handed rotation to be negative; but this is a matter of convenience. It is obvious that the same rotation which seems right-handed when looked at from one end of the axis, seems left-handed when looked at from the other end.

377. **Relative Motion of a Pair of Points in a Rotating Body.**—Let O and A denote any two points in a rotating body; and considering O as fixed, let it be required to determine the motion of A relatively to an axis of rotation drawn through O. On that axis let fall a perpendicular from A; let r be the length of that perpendicular. Then the motion of A relatively to the axis traversing O is one of *revolution,* or *translation in a circular path of the radius* r; the centre of that circular path being at the point where the perpendicular from A meets the axis. If a be the angular velocity of the body, then the *velocity* of A relatively to the axis traversing O is

$$v = a\,r; \quad \text{.............................(1.)}$$

and the *direction* of that velocity is at each instant perpendicular to the plane drawn through A and the axis. The *rate of deviation* of A in its motion relatively to the given axis is

$$\frac{v^2}{r} = a^2\,r; \quad \text{..........................(2.)}$$

in which the first expression is that already found in Article 363, and the second is deduced from the first by the aid of equation 1 of this Article. It is evident that for a given rotation the motion of O relatively to an axis of rotation traversing A is exactly the same with that of A relatively to a parallel axis traversing O; for it depends solely on the angular velocity a, the perpendicular distance r of the moving point from the axis, and the direction of the axis; all which are the same in either case.

r is called the *radius-vector* of the moving point.

378. **Cylindrical Surface of Equal Velocities.**—If a cylindrical surface of circular cross section be described about an axis of rotation, all the points in that surface have equal velocities relatively to the axis, and the direction of motion of each point in the cylin-

drical surface relatively to the axis is a tangent to the surface in a plane perpendicular to the axis.

379. **Comparative Motions of Two Points relatively to an Axis.**—Let O, A, B, denote three points in a rotating rigid body; let O be considered as fixed, and let an axis of rotation be drawn through it. Then the *comparative* motions of A and B relatively to that axis are expressed as follows:—*the velocity-ratio is that of the radii-vectores of the points, and the directional relation consists in the angle between their directions of motion being the same with that between their radii-vectores.* Or symbolically: Let r_1, r_2, be the perpendicular distances of A and B from the axis traversing O, and v_1 and v_2 their velocities; then

$$\frac{v_2}{v_1} = \frac{r_2}{r_1}; \text{ and } \overset{\wedge}{v_1 v_2} = \overset{\wedge}{r_1 r_2}.$$

380. **Components of Velocity of a Point in a Rotating Body.**—The component parallel to an axis of rotation, of the velocity of a point in a rotating body relatively to that axis, is null. That velocity may be resolved into components in the plane of rotation. Thus let O, in fig. 181, represent an axis of rotation of a body whose plane of rotation is that of the figure; and let A be any point in the body whose radius-vector is $\overline{OA} = r$. The velocity of that point being $v = a r$, let that velocity be represented by the line $\overline{AV}$ perpendicular to OA. Let BA be any direction in the plane of rotation, along which it is desired to find the component of the velocity of A; and let $\angle VAU = \theta$ be the angle made by that line with AV From V let fall VU perpendicular to BA; then $\overline{AU}$ represents the component in question, and denoting it by u,

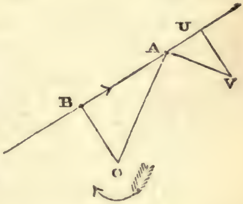

Fig. 181.

$$u = v \cdot \cos \theta = a r \cdot \cos \theta \ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

From O let fall OB perpendicular to BA. Then $\angle AOB = \angle VAU = \theta$; and the right-angled triangles OBA and AUV are similar; so that

$$\overline{AV} : \overline{AU} :: \overline{OA} : \overline{OB} = r \cos \theta \ldots\ldots\ldots\ldots (2.)$$

Now the *entire* velocity of B relatively to the axis O is

$$a r \cos \theta = u, \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.)$$

so that *the component, along a given straight line in the plane of rotation, of the velocity of any point in that line, is equal to the velocity of the point where a perpendicular from the axis meets that line.*

SECTION 3.—*Combined Rotations and Translations.*

381. **Property of all Motions of Rigid Bodies.**—The foregoing proposition may be regarded as a particular case of the following, which is true of all motions of a rigid body.

The components, along a given straight line in a rigid body, of the velocities of the points in that line relatively to any point, whether in or attached to the body or otherwise, are all equal to each other; for otherwise, the distances between points in the given straight line must alter, which is inconsistent with the idea of rigidity.

382. **Helical Motion.**—Rotation is the only movement which a rigid body as a whole can have relatively to a point belonging to it or attached to it. But if the motion of the body be determined relatively to a point not attached to it, a translation may be combined with the rotation. When that translation takes place in the direction of the axis of rotation, the motion of the rigid body is said to be *helical*, or *screw-like*, because each point in the rigid body describes a helix or screw, or a part of a helix or screw.

Let v_1 denote the velocity of translation, parallel to the axis of rotation, which is common to all points of the body; this is called the *velocity of advance.* The advance during one complete turn of the rotating body is the *pitch* of each of the helical or screw-like paths described by its particles; that is, the distance, in a direction parallel to the axis, between one turn of each such helix and the next; and a being the angular velocity, so that $\frac{2\pi}{a}$ is the time of one turn, the value of the pitch is

$$p = \frac{2\pi v_1}{a};\ \text{whence } v_1 = \frac{a p}{2\pi} \ldots\ldots\ldots\ldots\ldots\ldots(1.)$$

Let r, as before, be the radius-vector of any point in the body, and let

$$v_2 = a r \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.)$$

denote its *velocity of revolution*, or velocity relatively to the axis, due to the rotation alone. Then the *resultant* velocity of that point is

$$v = \sqrt{v_1^2 + v_2^2} = a \cdot \sqrt{\left\{\frac{p^2}{4\pi^2} + r^2\right\}} \ldots\ldots\ldots\ldots(3.)$$

The *inclination* of the helix described by that point to the *plane of rotation* is given by the equation

$$i = \text{arc} \cdot \tan \cdot \frac{v_1}{v_2} = \text{arc} \cdot \tan \cdot \frac{p}{2\pi r};\ldots\ldots\ldots\ldots(4.)$$

the tangent of that angle being the ratio of the pitch to the circumference of the circle described by the point relatively to the axis of rotation.

383. PROBLEM. **To Find the Motion of a Rigid Body from the Motions of Three of its Points.**—Let A, B, C, fig. 182, be three points in a rigid body, and at a given instant let them have motions relatively to a point independent of the body, which motions are represented in velocity and direction by the three lines $\overline{\mathrm{A\,V}_a}$, $\mathrm{B\,V}_b$, $\mathrm{C\,V}_c$. It is required to find the motion of the entire rigid body relatively to the same fixed point.

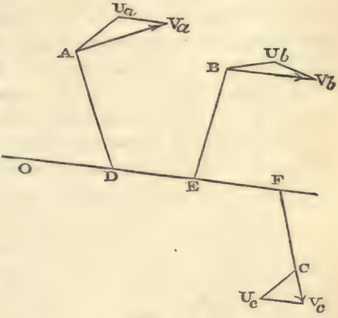

Fig. 182.

Through any point o, fig. 183, draw three lines $o\,a$, $o\,b$, $o\,c$, equal and parallel to the three lines $\overline{\mathrm{A\,V}_a}$, $\overline{\mathrm{B\,V}_b}$, $\mathrm{C\,V}_c$. Through a, b, and c, draw a plane $a\,b\,c$, on which let fall a perpendicular $o\,n$ from o. Then $o\,n$ represents a component, which is common to the velocities of all the three points A, B, C, and must therefore be common to all the points in the body; that is, it is a *velocity of translation.*

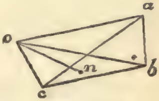

Fig. 183.

From the points V_a, V_b, V_c, draw lines $\overline{\mathrm{V}_a\,\mathrm{U}_a}$, $\overline{\mathrm{V}_b\,\mathrm{U}_b}$, $\overline{\mathrm{V}_c\,\mathrm{U}_c}$, equal and parallel to $o\,n$, but opposite in direction to it; and join $\overline{\mathrm{A\,U}_a}$, $\overline{\mathrm{B\,U}_b}$, $\overline{\mathrm{C\,U}_c}$, which will all be parallel to the same plane; that is, to the plane $a\,b\,c$. The last three lines will represent the component velocities which, along with the common velocity of translation parallel to $o\,n$, make up the resultant velocities of the three points. Through any two of the points A, B, draw planes perpendicular to the respective components of their motions which are parallel to $a\,b\,c$. These two planes will intersect each other in a line O D E, which will be parallel to $o\,n$. The perpendicular distances of that line from the points A, B, being unchanged by the motion, it represents one and the same line in or attached to the rigid body, and it is therefore the axis of rotation. A plane drawn through the third point C, perpendicular to $\overline{\mathrm{C\,U}_c}$, will cut the other two planes in the same axis: the three revolving component velocities

$$\overline{\mathrm{A\,U}_a},\ \overline{\mathrm{B\,U}_b},\ \overline{\mathrm{C\,U}_c},$$

will be respectively proportional to the perpendicular distances, or *radii-vectores,*

$$\overline{\mathrm{A\,D}}, \overline{\mathrm{B\,E}}, \overline{\mathrm{C\,F}},$$

of the three points from that axis; and the angular velocity will be equal to each of the three quotients made by dividing the revolving component velocities of the points by their respective radii-vectores. This rotation, combined with a translation parallel to the axis, with a velocity represented by *o n,* constitutes a *helical motion,* being the required motion of the rigid body.—Q. E. I.

384. **Special Cases** of the preceding problem occur, in which either a more simple method of solution is sufficient, or the general method fails, and a special method has to be employed.

I. *When the motions of the points of the body are known to be all parallel to one plane,* it is sufficient to know the motions of two points, such as A, B, fig. 184. Let A O, B O, be two planes traversing A and B, and perpendicular to the respective directions of the simultaneous velocities of those points; if those planes cut each other, the entire motion is a rotation; the line of intersection of the planes O, being the axis of rotation, and the angular velocity, are found as in the last Article. If the two planes are parallel, the motion is a translation.

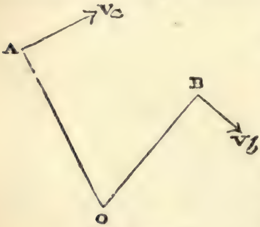

Fig. 184.

II. *If three points, not in the same plane, have parallel motions, or if three points in the same plane have parallel motions oblique to the plane,* the motion is a translation.

III. *If three points in the same plane move perpendicularly to the plane,* as A B C, fig. 184 *a,* then if their velocities are equal, the motion is a translation; and if their velocities are unequal, the motion is a rotation about the axis which is the intersection of the

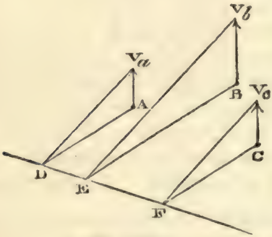

Fig. 184 *a.*

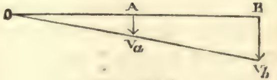

Fig. 184 *b.*

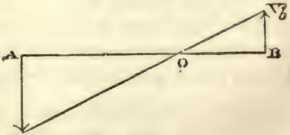

Fig. 184 *c.*

plane of the three points with the plane drawn through the extremities V_a, V_b, V_c, of the three lines which represent their velocities; the angular velocity being found as in Article 383.

If the plane of rotation is known, then the simultaneous velocities of two points, as A and B in figs. 184 *b* and 184 *c*, are sufficient to determine the axis O.

385. **Rotation Combined with Translation in the Same Plane.**—Let a body rotate about an axis C (fig. 185), fixed relatively to the body, with an angular velocity *a*, and at the same time let that axis have a motion of translation in a straight path perpendicular to the direction of the axis, with the velocity *u*, represented by the line $\overline{\mathrm{C\,U}}$. It is required to find the velocity and direction of motion of any point in the body. From the moving axis draw a straight line $\overline{\mathrm{C\,T}}$ perpendicular to that axis and to $\overline{\mathrm{C\,U}}$, and in that direction into which the rotation (as represented by the feathered arrow) tends to turn $\overline{\mathrm{C\,U}}$, and make

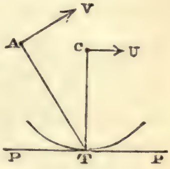

Fig. 185.

$$\overline{\mathrm{C\,T}} = \frac{u}{a} \text{..............................} (1.)$$

Then the point T has, in virtue of *translation along with* the axis C, a *forward* motion with the velocity *u*; and in virtue of *rotation about* that axis, it has a *backward* motion with the velocity

$$a \cdot \overline{\mathrm{C\,T}} = u,$$

equal and opposite to the former; and its resultant velocity is 0. Hence every point in the body, which comes in succession into the position T, situated at the distance $\frac{u}{a}$ from the axis C in the direction above described, is *at rest at the instant of its arriving at that position;* that is, it has just ceased to move in one direction, and is about to move in another direction; and this is true of every point which arrives at *a line traversing* T *parallel to* C. Consequently the resultant motion of the body, at any given instant, is the same as if it were rotating about the line which *at the instant in question* occupies the position T, parallel to C, at the distance $\frac{u}{a}$; and that line is called THE INSTANTANEOUS AXIS. To find the motion of any point A in the body at a given instant, let fall the perpendicular $\overline{\mathrm{A\,T}}$ from that point on the instantaneous axis; then the motion of A is in the direction A V perpendicular to the plane of the instantaneous axis and of the *instantaneous radius-vector* $\overline{\mathrm{A\,T}}$, and the velocity of that motion is

$$v = a \cdot \overline{\mathrm{A\,T}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.)$$

386. **Rolling Cylinder; Trochoid.**—Every straight line parallel to the moving axis C, in a cylindrical surface described about C with the radius $\frac{u}{a}$, becomes in turn the instantaneous axis. Hence the motion of the body is the same with that produced by the rolling of such a cylindrical surface on a plane P T P parallel to C and to $\overline{\mathrm{C\,U}}$, at the distance $\frac{u}{a}$.

The path described by any point in the body, such as A, which is not in the moving axis C, is a curve well known by the name of *trochoid.* The particular form of trochoid called the *cycloid,* is described by each of the points in the rolling cylindrical surface.

387. **Plane Rolling on Cylinder; Spiral Paths.**—Another mode of representing the combination of rotation with translation in the same plane is as follows:—Let O be an axis assumed as fixed, about which let the plane O C (containing the axis O) rotate (right-handedly, in the figure), with the angular velocity *a.* Let a rigid body have, *relatively to the rotating plane,* and in a direction perpendicular to it, a translation with the velocity *u.* In the plane O C, and at right angles to the axis O, take $\mathrm{O\,T} = \frac{u}{a}$, in such a direction that the velocity

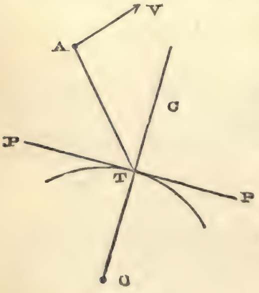

Fig. 186.

$$u = a \cdot \overline{\mathrm{O\,T}},$$

which the point T *in the rotating plane* has at a given instant, shall be in the contrary direction to the equal velocity of translation *u,* which the rigid body has relatively to the rotating plane. Then each point *in the rigid body* which arrives at the position T, or at any position in a line traversing T parallel to the fixed axis O, is at rest *at the instant* of its occupying that position; therefore the line traversing T parallel to the fixed axis O is *the instantaneous axis;* the motion at a given instant of any point in the rigid body, such as A, is at right angles to the radius-vector A T drawn perpendicular to the instantaneous axis; and the velocity of that motion is given by the equation

$$v = a \cdot \overline{\mathrm{A\,T}}.$$

All the lines in the rigid body which successively occupy the position of instantaneous axis are situated in a plane of that body, P T P, perpendicular to O C; and all the positions of the instantaneous axis are situated in a cylinder described about O with the radius $\overline{\mathrm{O\,T}}$; so that the motion of the rigid body is such as is produced by the *rolling of the plane* P P *on the cylinder whose radius is* $\overline{\mathrm{O\,T}} = \frac{u}{a}$. Each point in the rigid body, such as A, describes a plane spiral about the fixed axis O. For each point in the *rolling plane*, P P, that spiral is the involute of the circle whose radius is $\overline{\mathrm{O\,T}}$. For each point whose path of motion traverses the fixed axis O, that is, for each point in a plane of the rigid body traversing O parallel to P P, the spiral is Archimedean, having a radius-vector increasing by the length u for each angle a through which it rotates.

388. **Combined Parallel Rotations.**—In figs. 187, 188, and 189,

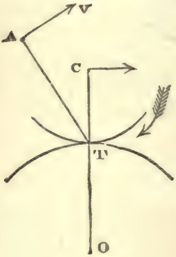

Fig. 187.

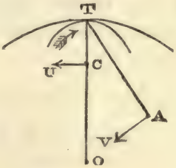

Fig. 188.

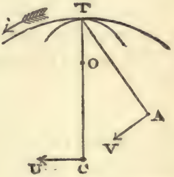

Fig. 189.

let O be an axis assumed as fixed, and O C a plane traversing that axis, and rotating about it with the angular velocity a. Let C be an axis in that plane, parallel to the fixed axis O; and about the moving axis C let a rigid body rotate with the angular velocity b *relatively to the plane* O C; and let the directions of the rotations a and b be distinguished by positive and negative signs. The body is said to have the rotations about the parallel axes O and C *combined* or *compounded*, and it is required to find the result of that combination of parallel rotations.

Fig. 187 represents the case in which a and b are similar in direction; fig. 188, that in which a and b are in opposite directions, and b is the greater; and fig. 189, that in which a and b are in opposite directions, and a is the greater.

Let a common perpendicular $\overline{\mathrm{O\,C}}$ to the fixed and moving axes be intersected in T by a straight line parallel to both those axes, in such a manner that the distances of T from the fixed and moving axes respectively shall be inversely proportional to the angular velocities of the component rotations about them, as is expressed by the following proportion:—

$$a : b :: \overline{\mathrm{C\,T}} : \overline{\mathrm{O\,T}} \ldots\ldots\ldots\ldots\ldots\ldots(1.)$$

When a and b are similar in direction, let T fall between O and C, as in fig. 187; when they are contrary, beyond, as in figs. 188 and 189. Then the velocity of the line T *of the plane* O C is $a \cdot \overline{\mathrm{O\,T}}$; and the velocity of the line T *of the rigid body, relatively to the plane* O C, is $b \cdot \overline{\mathrm{C\,T}}$, equal in amount and contrary in direction to the former; therefore each line of the rigid body which arrives at the position T is at rest at the instant of its occupying that position, and is then *the instantaneous axis.* The *resultant angular velocity* is given by the equation

$$c = a + b; \ldots\ldots\ldots\ldots\ldots\ldots(2.)$$

regard being had to the directions or signs of a and b; that is to say, if we now take a and b to represent *arithmetical* magnitudes, and affix explicit signs to denote their directions, the direction of c will be the same with that of the greater; the case of fig. 187 will be represented by the equation 2, already given; and those of figs. 188 and 189 respectively by

$$c = b - a;\ c = a - b \ldots\ldots\ldots\ldots\ldots\ldots(2\ \mathrm{A}.)$$

The relative proportions of a, b, and c, and of the distances between the fixed, moving, and instantaneous axes, are given by the equation

$$a : b : c :: \overline{\mathrm{C\,T}} : \overline{\mathrm{O\,T}} : \overline{\mathrm{O\,C}} \ldots\ldots\ldots\ldots\ldots\ldots(3.)$$

The motion of any point, such as A, in the rigid body, is at each instant at right angles to the radius-vector $\overline{\mathrm{A\,T}}$ drawn from the point perpendicular to the instantaneous axis; and the velocity of that motion is

$$v = c \cdot \overline{\mathrm{A\,T}} \ldots\ldots\ldots\ldots\ldots\ldots(4.)$$

389. **Cylinder Rolling on Cylinder; Epitrochoids.**—All the lines in the rigid body which successively occupy the position of instantaneous axis are situated in a cylindrical surface described about C with the radius $\overline{\mathrm{C\,T}}$; and all the positions of the instantaneous

axis are contained in a cylindrical surface described about O with the radius $\overline{\mathrm{O\,T}}$; therefore the resultant motion of the rigid body is that which is produced by rolling the former cylinder, attached to the body, on the latter cylinder, considered as fixed.

In fig. 187, a convex cylinder rolls on a convex cylinder; in fig. 188, a smaller convex cylinder rolls in a larger concave cylinder; in fig. 189, a larger concave cylinder rolls on a smaller convex cylinder.

Each point in the rolling rigid body traces, relatively to the fixed axis, a curve of the kind called *epitrochoids.* The epitrochoid traced by a point in the surface of the rolling cylinder is an *epicycloid.*

In certain cases, the epitrochoids become curves of a more simple class. For example, each point in the *moving axis* C traces a circle.

When a cylinder, as in fig. 188, rolls within a concave cylinder of *double its radius*, each point in the surface of the rolling cylinder moves backwards and forwards in a straight line, being a diameter of the fixed cylinder; each point in the axis of the rolling cylinder traces a circle of the same radius with that cylinder, and each other point in or attached to the rolling cylinder traces an ellipse of greater or less eccentricity, having its centre in the fixed axis O. This principle has been made available in instruments for drawing and turning ellipses.

390. **Curvature of Epitrochoids.**—The following being given:—

the radius of the fixed cylinder, $\overline{\mathrm{O\,T}} = r_1$;

the radius of the rolling cylinder, $\overline{\mathrm{C\,T}} = r_2$;

the instantaneous radius-vector of a tracing-point A, $\overline{\mathrm{A\,T}} = r$;

the angle made by that radius-vector with the rotating plane, $\angle \mathrm{C\,T\,A} = \theta$;

it is required to find the radius of curvature, ρ, of the path of the tracing-point A, at the instant under consideration.

The radius of a convex cylinder is to be considered as positive, and that of a concave cylinder as negative; and regard is to be paid to the principle, that $\cos\theta$ is $\left\{\begin{matrix}\text{positive}\\\text{negative}\end{matrix}\right\}$ according as θ is $\left\{\begin{matrix}\text{acute}\\\text{obtuse}\end{matrix}\right\}$.

Let $d\,t$ be an indefinitely short interval of time; then during that interval the tracing-point A moves through the distance $c\,r\,d\,t$. Let the direction of the radius-vector r, which is perpendicular to the path traced by A, alter in the same time by the angle $d\,i$. Then the radius of curvature of the path of A is

$$\rho = \frac{c\,r\,d\,t}{d\,i} \dots\dots\dots\dots\dots\dots\dots\dots (1.)$$

To determine the angular motion $d\,i$ of the radius-vector, it has to be considered, that the absolute angular velocity of the rolling cylinder is c, which gives that cylinder an angular motion, $c\,d\,t$, in the given interval; and also that, in the course of the same interval, a *new line* comes to occupy the position of instantaneous axis, distant from the original line by the length $b\,r_2\,d\,t$, in a direction *opposite* to that of the rotation of the rolling cylinder. The effect of this shifting of the instantaneous axis is, to turn the angular position of the radius-vector r, in a *negative* direction relatively to the rolling cylinder, through the angle

$$-\frac{b\,r_2 \cos\theta \cdot d\,t}{r},$$

which being combined with the angular motion of the cylinder, $c\,d\,t$, gives as the resultant angular motion of the radius-vector,

$$d\,i = \left(c - \frac{b\,r_2 \cos\theta}{r}\right) d\,t;$$

which being substituted in equation 1, gives for the radius of curvature of the path traced by A,

$$\rho = \frac{c\,r}{c - \frac{b\,r_2 \cos\theta}{r}} = \frac{r}{1 - \frac{b\,r_2 \cos\theta}{c\,r}} \dots\dots\dots\dots\dots\dots (2.)$$

Now,

$$\frac{b}{c} = \frac{r_1}{r_1 + r_2};$$

(attention being paid to the implicit signs of r_1 and r_2); and consequently,

$$\rho = r \cdot \frac{r_1 + r_2}{r_1 + r_2 - \frac{r_1\,r_2 \cos\theta}{r}} \dots\dots\dots\dots\dots\dots (3.)$$

The sign of this result, when $\left\{\begin{matrix}\text{positive}\\ \text{negative}\end{matrix}\right\}$, shows that the curve traced by A is $\left\{\begin{matrix}\text{concave}\\ \text{convex}\end{matrix}\right\}$ towards T. The following are some limited cases:—

I. *When the tracing-point is the surface of the rolling cylinder,* $r = 2\,r_2 \cos\theta$; and therefore,

$$\rho = 2\, r_2 \cos \theta \cdot \frac{r_1 + r_2}{\frac{1}{2} r_1 + r_2}; \ldots\ldots\ldots\ldots\ldots\ldots (4.)$$

which is the radius of curvature of an *epicycloid.*

II. *When a cylinder rolls on a plane,* r_1 becomes infinitely great as compared with r_2, and thus reduces equation 3 to

$$\rho = \frac{r}{1 - \frac{r_2 \cos \theta}{r}} \ldots\ldots\ldots\ldots\ldots\ldots (5.)$$

which is the radius of curvature of a *trochoid.*

III. *When a cylinder rolls on a plane, and the tracing-point is in the surface of the cylinder,* $r = 2\, r_2 \cos \theta$, and

$$\rho = 2\, r = 4\, r_2 \cos \theta, \ldots\ldots\ldots\ldots\ldots\ldots (6.)$$

which is the radius of curvature of a *cycloid.*

IV. *When a plane rolls on a cylinder,* r_2 becomes infinitely great as compared with r_1 and r; and equation 3 becomes

$$\rho = \frac{r}{1 - \frac{r_1 \cos \theta}{r}}, \ldots\ldots\ldots\ldots\ldots\ldots (7.)$$

which is the radius of curvature of a spiral of the class mentioned in Article 387.

V. *When a plane rolls on a cylinder, and the tracing-point is in the plane,* $\cos \theta = 0$; and equation 7 becomes

$$\rho = r, \ldots\ldots\ldots\ldots\ldots\ldots (8.)$$

which is the radius of curvature of the *involute of a circle.*

VI. *When a plane rolls on a cylinder, and the tracing-point is at the distance* r_1 *from the plane on the side next the cylinder,* $\cos \theta = -\frac{r_1}{r}$; and equation 7 takes the following form :—

$$\rho = \frac{r^3}{r^2 + r_1^2}, \ldots\ldots\ldots\ldots\ldots\ldots (9.)$$

which is the radius of curvature of an *Archimedean spiral.* Let R be the distance of a point in that spiral from the fixed axis O; then $r^2 = \mathrm{R}^2 + r_1^2$, and

$$\rho = \frac{(\mathrm{R}^2 + r_1^2)^{\frac{3}{2}}}{\mathrm{R}^2 + 2\, r_1^2} \ldots\ldots\ldots\ldots\ldots\ldots (9\ \mathrm{A}.)$$

As to rolling curves in general, see Professor Clerk Maxwell's paper in the *Transactions of the Royal Society of Edinburgh,* vol. xvi.

391. **Equal and Opposite Parallel Rotations Combined.**—Let a plane O C rotate with an angular velocity *a* about an axis O contained in the plane, and let a rigid body rotate about the axis C in that plane parallel to O, with an angular velocity $-a$, equal and opposite to that of the plane. Then the angular velocity of the rigid body is nothing; that is, its motion is one of *translation* only, all its points moving in equal circles of the radius $\overline{\mathrm{O\,C}}$, with the velocity $a \cdot \overline{\mathrm{O\,C}}$. This case is not capable of being represented by a rolling action.

392. **Rotations about Intersecting Axes Combined.**—In fig. 190, let O A be an axis assumed as fixed; and about it let the plane A O C rotate with the angular velocity *a*. Let O C be an axis in the rotating plane; and about that axis let a rigid body rotate with the angular velocity *b* relatively to the rotating plane.

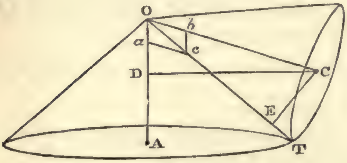

Fig. 190.

Because the point O in the rigid body is fixed, the instantaneous axis must traverse that point. The direction of that axis is determined, as before, by considering that each point which arrives at that line must have, in virtue of the rotation about O C, a velocity relatively to the rotating plane, equal and directly opposed to that which the coincident point of the rotating plane has. Hence it follows, that the ratio of the perpendicular distances of each point in the instantaneous axis from the fixed and moving axes respectively—that is, the ratio of the sines of the angles which the instantaneous axis makes with the fixed and moving axes—must be the reciprocal of the ratio of the component angular velocities about those axes; or symbolically, if O T be the instantaneous axis,

$$\sin \mathrm{A\,O\,T} : \sin \mathrm{C\,O\,T} :: b : a \text{ ....(1.)}$$

This determines the direction of the instantaneous axis, which may also be found by graphic construction as follows:—On O A take $\overline{\mathrm{O}\,a}$ proportional to *a*; and on O C take $\overline{\mathrm{O}\,b}$ proportional to *b*. Let those lines be taken in such directions, that to an observer looking from their extremities towards O, the component rotations seem both right-handed. Complete the parallelogram O *b c a*; the diagonal $\overline{\mathrm{O}\,c}$ will be the instantaneous axis.

The resultant angular velocity about this instantaneous axis is found by considering, that if C be any point in the moving axis, the linear velocity of that point must be the same, whether computed from the angular velocity *a* of the rotating plane about the fixed axis O A, or from the resultant angular velocity c of the rigid body about the instantaneous axis. That is to say, let C D, C E, be perpendiculars from C upon O A, O T, respectively; then

$$a \cdot \overline{\mathrm{C\,D}} = c \cdot \overline{\mathrm{C\,E}};$$

but $\overline{\mathrm{C\,D}} : \overline{\mathrm{C\,E}} :: \sin \angle \mathrm{A\,O\,C} : \sin \angle \mathrm{C\,O\,T}$; and therefore

$$\sin \angle \mathrm{C\,O\,T} : \sin \angle \mathrm{A\,O\,C} :: a : c;$$

and, combining this proportion with that given in equation 1, we obtain the following proportional equation:—

$$\left.\begin{array}{ccccccc} & \sin \angle \mathrm{C\,O\,T} & : & \sin \angle \mathrm{A\,O\,T} & : & \sin \angle \mathrm{A\,O\,C} \\ :: & a & : & b & : & c \\ :: & \overline{\mathrm{O}a} & : & \overline{\mathrm{O}b} & : & \overline{\mathrm{O}c} \end{array}\right\} \text{.....(2.)}$$

that is to say, *the angular velocities of the component and resultant rotations are each proportional to the sine of the angle between the axes of the other two;* and *the diagonal of the parallelogram* O b c a *represents both the direction of the instantaneous axis and the angular velocity about that axis.*

393. **Rolling Cones.**—All the lines which successively come into the position of instantaneous axis are situated in the surface of a cone described by the revolution of O T about O C; and all the positions of the instantaneous axis lie in the surface of a cone described by the revolution of O T about O A. Therefore the motion of the rigid body is such as would be produced by the rolling of the former of those cones upon the latter.

It is to be understood, that either of the cones may become a flat disc, or may be hollow, and touched internally by the other. For example, should ∠ A O T become a right angle, the fixed cone would become a flat disc; and should ∠ A O T become obtuse, that cone would be hollow, and would be touched internally by the rolling cone; and similar changes may be made in the rolling cone.

The path described by a point in or attached to the rolling cone is a *spherical epitrochoid;* but for the purposes of the present treatise, it is unnecessary to enter into details respecting the properties of that class of curves.

394. **Analogy of Rotations and Single Forces.**—If the proportional equation 3 of Article 388, which shows the relations between the component angular velocities of rotation about a pair of parallel axes, the resultant angular velocity, and the position of the instantaneous axis, be compared with the proportional equation of Article 39, by means of which, as explained in Article 40, the magnitude and position of the resultant of a pair of parallel forces are found, it will be evident that those equations are exactly analogous.

The result of the combination of a rotation with a translation in the same plane, in producing a rotation of equal angular velocity about an instantaneous axis at a certain distance to one side of the moving axis, as explained in Article 385, is exactly analogous to the result of the combination of a single force with a couple in producing an equal single force transferred laterally to a certain distance, as explained in Article 41.

The result of the combination of two equal and opposite rotations about parallel axes, in producing a translation with a velocity which is the product of the angular velocity into the distance between the axes, as explained in Article 391, is exactly analogous to the production of a couple by means of a pair of equal and opposite forces, as explained in Article 25.

The result of the combination of two rotations about intersecting axes, as explained in Article 392, is exactly analogous to the result of the combination of a pair of inclined forces acting through one point, as explained in Article 51.

The combination of a rotation about a given axis with a translation parallel to the same axis, as explained in Article 382, is exactly analogous to the combination of a force acting in a given line with a couple whose axis is parallel to the same line, as explained in Article 60, cases 4 and 5.

It thus appears, that just as the composition and resolution of translations are exactly analogous to the composition and resolution of couples, so the composition and resolution of rotations are exactly analogous to the composition and resolution of single forces; that is to say, if lines be taken, representing in direction axes of rotation, and in length the angular velocities of rotation about such axes, all mathematical theorems which are true of lines representing single forces are true of such lines representing rotations: and if with this be combined the principle, that all mathematical theorems which are true of lines representing in direction the axes and in length the moments of couples are true also of lines representing the velocities and directions of translations, all problems of the resolution and composition of motions may be solved by referring to the solutions of analogous problems of statics.

395. **Comparative Motions in Compound Rotation.**—The velocity-ratio of two points in a rotating rigid body at any instant is that of their perpendicular distances from its instantaneous axis; and the angle between the directions of motion of the two points is equal to that between the two planes which traverse the points and the instantaneous axis.

SECTION 4.—*Varied Rotation.*

396. **Variation of Angular Velocity** is measured like variation of linear velocity, by comparing the change which takes place in the

angular velocity of a rotating body, Δa, during a given interval of time, with the length of that interval, Δt, and the *rate of variation* is the value towards which the ratio of the change of angular velocity to the interval of time, $\frac{\Delta a}{\Delta t}$, converges, as the length of the interval is indefinitely diminished; being represented by

$$\frac{d a}{d t},$$

and found by the operation of differentiation.

397. **Change of the Axis of Rotation** has been already considered, so far as it is consistent with uniform angular velocity, in the preceding section. All the propositions of that section are applicable also to cases in which the *angular velocity is varied*, so long as the ratio of each pair of component angular velocities, such as $a : b$, is constant.

When that ratio varies, the propositions are true also, provided it be understood, that the *rolling cylinders and cones with circular bases*, spoken of in section 3, are simply the *osculating cylinders and cones* at the lines of contact of rolling cylinders and cones with bases not circular; and that r_1, r_2, in each case, represent the values of the variable radii of curvature of non-circular cylinders at their lines of contact, and $\angle$ A O T, $\angle$ C O T, the variable angles of obliquity of the osculating circular cones of non-circular cones.

398. **Components of Varied Rotation.**—The most convenient way, in most cases, of expressing the mode of variation of a rotatory motion, is to resolve the angular velocity at each instant into three component angular velocities about three rectangular axes fixed in direction. The values of those components, at any instant, show at once the resultant angular velocity, and the direction of the instantaneous axis. For example, let a_x, a_y, a_z, be the rectangular components of the angular velocity of a rigid body at a given instant.

rotation about x from y towards z,
about y from z towards x,
and about z from x towards y,

being considered as positive; then

$$a = \sqrt{(a_x^2 + a_y^2 + a_z^2)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

is the resultant angular velocity, and

$$\cos \alpha = \frac{a_x}{a};\ \cos \beta = \frac{a_y}{a};\ \cos \gamma = \frac{a_z}{a};\ldots\ldots\ldots\ldots (2.)$$

are the cosines of the angles which the instantaneous axis makes with the axes of x, y and z, respectively.

CHAPTER III.

MOTIONS OF PLIABLE BODIES, AND OF FLUIDS.

399. **Division of the Subject.**—The subject of the present chapter, so far as it comprehends the relative motions of the points of pliable solids, has been already treated of in those portions of the Third Chapter of Part II. which relate to strains. There remain now to be considered the following branches :—

I. The Motions of Flexible Cords.
II. The Motions of Fluids not altering in Volume.
III. The Motions of Fluids altering in Volume.

SECTION 1.—*Motions of Flexible Cords.*

400. **General Principles.**—As those relative motions of the points of a cord which may arise from its extensibility, belong to the subject of resistance to tension, which is a branch of that of strength and stiffness, the present section is confined to those motions of which a flexible cord is capable when the length, not merely of the whole cord, but of each part lying between two points fixed in the cord, is invariable, or sensibly invariable.

In order that the figure and motions of a flexible cord may be determined from cinematical considerations alone, independently of the magnitude and distribution of forces acting on the cord, its weight must be insensible compared with the tension on it, and it must everywhere be *tight* ; and when that is the case, each part of the cord which is not straight is maintained in a curved figure by passing over a *convex* surface. The line in which a tight cord lies on a convex surface is the *shortest line* which it is possible to draw on that surface between each pair of points in the course of the cord. (It is a well known principle of the geometry of curved surfaces, that the *osculating plane* at each point of such a line is perpendicular to the curved surface.)

Hence it appears, that the motions of a tight flexible cord of invariable length and insensible weight are regulated by the following principles :—

I. *The length between each pair of points in the cord is constant.*

II. *That length is the shortest line which can be drawn between its extremities over the surfaces by which the cord is guided.*

401. **Motions Classed.**—The motions of a cord are of two kinds—

I. Travelling of a cord along a track of invariable form; in which case the velocities of all points of the cord are equal.

II. Alteration of the figure of the track by the motion of the guiding surfaces.

Those two kinds of motion may be combined.

The most usual problems in practice respecting the motions of cords are those in which cords are the means of transmitting motion between two pieces in a train of mechanism. Such problems will be considered in Part IV. of this treatise.

Next in point of frequency in practice are the problems to be considered in the ensuing Article.

402. **Cord Guided by Surfaces of Revolution.**—Let a cord in some portions of its course be straight, and in others guided by the surfaces of circular drums or pulleys, over each of which its track is a circular arc in a plane perpendicular to the axis of the guiding surface. Let r be the radius of any one of the guiding surfaces, i the angle of inclination which the two straight portions of the cord contiguous to that surface make with each other, expressed in length of arc to radius unity. Then the length of the portion of the cord which lies on that surface is $r\,i$; and if s be the length of any straight portion of the cord, the total length between two given points fixed in the cord may be expressed thus:—

$$\mathrm{L} = \Sigma \cdot s + \Sigma \cdot r\,i \ldots\ldots\ldots\ldots(1.)$$

Let c be the distance between the centres of a given adjacent pair of guiding surfaces, s the length of the straight portion of cord which lies between them, and r, r', their respective radii; then evidently

$$s = \sqrt{c^2 \mp (r \pm r')^2} \ldots\ldots\ldots\ldots(2.)$$

the upper signs being employed when the cord crosses, and the lower when it does not cross the line of centres c.

Now let a given point in the cord, A, be considered as fixed, and let L be the constant length of cord between A and another point in the cord, B. Let one of the guiding surfaces between A and B be moved through an indefinitely short distance, $d\,x$, in a direction which makes angles, j, j', with the two contiguous straight divisions of the cord respectively. Then, in order to keep the cord tight, B must be drawn longitudinally through the distance,

$$d\,x \cdot (\cos j + \cos j')\,; \ldots\ldots\ldots\ldots(3.)$$

and consequently, if u represent the velocity of translation of the guiding surface in the given direction, and v the longitudinal velocity of the point B in the cord,

$$v = u\,(\cos j + \cos j')\,; \qquad (4.)$$

and if any number of guiding surfaces between A and B be translated, each in its own direction,

$$v = \Sigma \cdot u\,(\cos j + \cos j') \qquad (5.)$$

The case most common in practice is that in which the *plies*, or straight parts of the cord, are all parallel to each other; so that $i = 180°$ in each case, while a certain number, n, of the guiding bodies or pulleys all move simultaneously in a direction parallel to the plies of the cord with the same velocity, u. Then $\cos j = \cos j' = 1$; and

$$v = 2\,n\,u \qquad (6.)$$

SECTION 2.—*Motions of Fluids of Constant Density.*

403. **Velocity and Flow.**—The density of a moving fluid mass may be either exactly invariable, from the constancy or the adjustment of its temperature and pressure, or sensibly invariable, from the smallness of the alterations of volume which the actual alterations of pressure and temperature are capable of producing. The latter is the case in most problems of practical mechanics affecting liquids.

Conceive an ideal surface of any figure, and of the area A, to be situated within a fluid mass, the parts of which have motion relatively to that surface; and let u denote, as the case may be, the *uniform* velocity, or the *mean* value of the varying velocity, resolved in a direction perpendicular to A, with which the particles of the fluid pass A. Then

$$Q = u\,A \qquad (1.)$$

is the volume of fluid which passes from one side to the other of the surface A in an unit of time, and is called the *flow*, or *rate of flow*, through A.

When the particles of fluid move obliquely to A, let θ denote the angle which the direction of motion of any particle passing A makes with a normal to A, and v the velocity of that particle; then

$$u = v \cdot \cos\theta \qquad (2.)$$

When the velocity normal to A varies at different points, either from the variation of v, or of θ, or of both, the flow may also be expressed as follows:—Let A be divided into indefinitely small elements, each of which is represented by $d\,A$; then

$$Q = \int u \, d\,A = \int v \cos \theta \cdot d\,A \,; \;\ldots\ldots\ldots\ldots\ldots(3.)$$

and if we now distinguish the *mean normal velocity* from the velocity at any particular point by the symbol u_0, we have,

$$u_0 = \frac{Q}{A} = \frac{\int u \, d\,A}{\int d\,A} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4.)$$

404. **Principle of Continuity.**—AXIOM. *When the motion of a fluid of constant density is considered relatively to an enclosed space of invariable volume which is always filled with the fluid, the flow into the space and the flow out of it, in any one given interval of time, must be equal*—a principle expressed symbolically by

$$\Sigma \cdot Q = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5.)$$

The preceding self-evident principle regulates all the motions of fluids of constant density, when considered in a purely cinematical manner. The ensuing articles of this section contain its most usual applications.

405. **Flow in a Stream.**—A stream is a moving fluid mass, indefinitely extended in length, and limited transversely, and having a continuous longitudinal motion. At any given instant, let A, A′, be the areas of any two of its transverse sections, considered as fixed; u, u', the mean normal velocities through them; Q, Q′, the rates of flow through them; then in order that the principle of continuity may be fulfilled, those rates of flow must be equal; that is,

$u\,A = u'\,A' = Q = Q'$ = constant for all cross sections of the channel at the given instant; ………………… (1.)

consequently,

$$\frac{u'}{u} = \frac{A}{A'} \,; \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.)$$

or, *the normal velocities at a given instant at two fixed cross sections are inversely as the areas of these sections.*

406. **Pipes, Channels, Currents, and Jets.**—When a stream of fluid completely fills a *pipe* or *tube*, the area of each cross section is given by the figure and dimensions of the pipe, and for similar forms of section varies as the square of the diameter. Hence the mean normal velocities of a stream flowing in a full pipe, at different cross sections of the pipe, are inversely as the squares of the diameters of those sections.

A *channel* partially encloses the stream flowing in it, leaving the upper surface free; and this description applies not only to channels commonly so called, but to pipes partially filled. In this case the area of a cross section of the stream depends not only on the figure and dimensions of the channel, but on the figure and elevation of the free upper surface of the stream.

A *current* is a stream bounded by other portions of fluid whose motions are different.

A *jet* is a stream whose surface is either free all round, or is touched by a solid body in a small portion of its extent only.

407. **A Radiating Current** is a part of a stream which moves towards or from an axis. It is evident that such a stream cannot extend to the axis itself, but must turn aside into a different course at some finite distance from the axis. Conceive a radiating current to be cut by a cylindrical surface of the radius r described about the axis, and let h be the depth, parallel to the axis, of the portion of that surface which is traversed by the current; then the *mean radial component*, u, of the velocity of the current at that surface has the value,

$$u = \frac{Q}{2\pi r h} \text{ ....} (1.)$$

408. **A Vortex, Eddy, or Whirl,** is a stream which either returns into itself, or moves in a spiral course towards or from an axis. In the latter case two or more successive turns of the same vortex may touch each other laterally without the intervention of any solid partition.

409. **Steady Motion** of a fluid relatively to a given space considered as fixed is that in which the velocity and direction of the motion of the fluid at each *fixed point* is uniform at every instant of the time under consideration; so that although the velocity and direction of the motion of a given particle of the fluid may vary while it is transferred from one point to another, that particle assumes, at each fixed point at which it arrives, a certain definite velocity and direction depending on the position of that point alone; which velocity and direction are successively assumed by each particle which successively arrives at the same fixed point.

The steady motion of a stream is expressed by the two conditions, that the area of each fixed cross section is constant, and that the flow through each cross section is constant; that is to say,

$$\frac{dA}{dt} = 0; \frac{dQ}{dt} = 0 \text{ ....} (1.)$$

If u represents the normal velocity of a fluid moving steadily, ***at a given fixed point***, then

$$\frac{du}{dt} = 0; \text{ ....} (2.)$$

expresses the condition of steady motion. Next, let u represent the normal velocity, not at *a given fixed point,* but *of a given identical particle of fluid;* then the variation undergone by u in an indefinitely small interval of time, $d\,t$, is that arising from its being transferred from one cross section to another, whose distance down the stream from the former is $d\,s = u \cdot d\,t$. Hence, denoting by $\frac{d\,u}{d\,s} \cdot d\,s$, the indefinitely small variation of velocity which takes place from this cause, and by $\frac{d \cdot u}{d\,t}$ $= \frac{ds}{dt}$, the *rate* at which that variation takes place, we have

$u = \frac{ds}{dt}$ Substitute value of u for

$$\frac{d \cdot u}{d\,t} = \frac{d\,u}{d\,s} \cdot \frac{d\,s}{d\,t} = u \cdot \frac{d\,u}{d\,s} \text{..................(3.)}$$

Most of the problems respecting streams which occur in practice have reference to steady motion.

§ 410. In **Unsteady Motion,** the velocity at each *fixed point* varies, at a rate denoted by $\frac{d\,u}{d\,t}$; and the total rate of variation of the velocity of *an individual particle* in a stream, being found by adding together the rates of variation due to lapse of time and to change of position, is expressed by

$$\frac{d \cdot u}{d\,t} = \frac{d\,u}{d\,t} + \frac{d\,u}{d\,s} \cdot \frac{d\,s}{d\,t} = \frac{d\,u}{d\,t} + u \cdot \frac{d\,u}{d\,s} \text{..........(1.)}$$

§ 411. **Motion of Pistons.**—Let a mass of fluid of invariable volume be enclosed in a vessel, two portions of the boundary of which (called *pistons*) are moveable inwards and outwards, the rest of the boundary being fixed. Then, if motion be transmitted between the pistons by moving one inwards and the other outwards, it follows, from the invariability of the volume of the enclosed fluid, that the velocities of the two pistons at each instant will be to each other in the inverse ratio of the areas of the respective projections of the pistons on planes normal to their directions of motion. This is the principle of the transmission of motion in the *hydraulic press* and *hydraulic crane.*

The *flow* produced by a piston whose velocity is u, and the area of whose projection on a plane perpendicular to the direction of its motion is A, is given, as in other cases, by the equation

$$\mathrm{Q} = u\,\mathrm{A} \text{..............................(1.)}$$

412. **General Differential Equations of Continuity.**—When the motions of a fluid of invariable density are considered in the most

general way, the principle of continuity stated in Article 404 is expressed symbolically in the following manner. The space assumed as fixed, to which the motion of the fluid is referred, is conceived to be divided into indefinitely small rectangular elementary spaces, each having for its linear dimensions, dx, dy, dz, and for the areas of its three pairs of faces, $dy\,dz$, $dz\,dx$, $dx\,dy$. Let

x, $x + dx$, be the co-ordinates of the pair of faces, $dy\,dz$;
y, $y + dy$, „ „ „ „ $dz\,dx$;
z, $z + dz$, „ „ „ „ $dx\,dy$.

Let the velocity of the particles of water at any point be resolved into three rectangular components, u, v, w, parallel respectively to x, y, z, with proper algebraical signs. Let outward flow be positive, and inward flow negative. The values of the flow for the six faces are as follows :—

Through the first face $dy\,dz$, $-u \cdot dy\,dz$;
„ „ second face $dy\,dz$, $\left(u + \frac{du}{dx}dx\right)dy\,dz$;
„ „ first face $dz\,dx$, $-v \cdot dz\,dx$;
„ „ second face $dz\,dx$, $\left(v + \frac{dv}{dy}dy\right)dz\,dx$;
„ „ first face $dx\,dy$, $-w \cdot dx\,dy$;
„ „ second face $dx\,dy$, $\left(w + \frac{dw}{dz}dz\right)dx\,dy$.

Adding those six parts of the flow together, and equating the result, in virtue of the principle of continuity, to nothing, we find the following equation :—

$$\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right)dx\,dy\,dz = 0;$$

and, striking out the common factor,

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.)$$

This is the *general differential equation of continuity* in a fluid of invariable volume.

413. **General Differential Equations of Steady Motion.**—If each particle which arrives successively at a given point assumes a velocity and direction of motion depending on the position of the point

alone, and not on the lapse of time, that state of steady motion is represented by the equations,

$$\frac{du}{dt}=0;\ \frac{dv}{dt}=0;\ \frac{dw}{dt}=0;\ldots\ldots\ldots\ldots\ldots\ldots(1.)$$

where u, v, w, are the *component velocities at a fixed point.* Next, instead of the velocities at a fixed point, let u, v, w, be the *component velocities of an individual particle;* then in the indefinitely short interval dt, the co-ordinates of that particle alter by the lengths $dx = u\,dt$, $dy = v\,dt$, $dz = w\,dt$; and it assumes the component velocities proper to its new position, differing from its original velocities by quantities, which, being divided by dt, give the rates of variation of the component velocities of *an individual particle*, viz. :—

$$\left.\begin{aligned}\frac{d\cdot u}{dt}&=u\frac{du}{dx}+v\frac{du}{dy}+w\frac{du}{dz};\\ \frac{d\cdot v}{dt}&=u\frac{dv}{dx}+v\frac{dv}{dy}+w\frac{dv}{dz};\\ \frac{d\cdot w}{dt}&=u\frac{dw}{dx}+v\frac{dw}{dy}+w\frac{dw}{dz}.\end{aligned}\right\}\ldots\ldots\ldots\ldots(2.)$$

414. **General Differential Equations of Unsteady Motion.**—When the motion is not steady, each of the three rates of variation in the equations 2 of Article 413 requires the addition of a term representing the rate of variation of velocity due to *lapse of time independently of change of position,* as follows :—

$$\frac{d\cdot u}{dt}=\frac{du}{dt}+u\frac{du}{dx}+v\frac{du}{dy}+w\frac{du}{dz};\ldots\ldots\ldots(1.)$$

and similar equations for $\frac{d\cdot v}{dt}$ and $\frac{d\cdot w}{dt}$: the presence of the dot denoting that the velocities are those of an individual particle, and its absence, that they are those at a fixed point.

415. **Equations of Displacement.**—In all the preceding Articles, x, y, and z, denote the co-ordinates of a real or ideal *fixed point* in the space to which the motions of the fluid are referred; and the differentials $\frac{du}{dx}$, &c., refer to the differences amongst the conditions of the fluid at different points in that space. Let ξ, η, ζ, represent the co-ordinates of an individual particle; then the three components of the velocity of that particle have the values

$$u=\frac{d\xi}{dt};\ v=\frac{d\eta}{dt};\ w=\frac{d\zeta}{dt};\ldots\ldots\ldots\ldots\ldots(1.)$$

and the three components of the *rate of variation* of its motion, as defined in Article 366, are

$$\frac{d^2 \xi}{d t^2} = \frac{d \cdot u}{d t}; \ \frac{d^2 \eta}{d t^2} = \frac{d \cdot v}{d t}; \ \frac{d^2 \zeta}{d t^2} = \frac{d \cdot w}{d t}; \ldots\ldots\ldots(2.)$$

the values of $\frac{d \cdot u}{d t}$, $\frac{d \cdot v}{d t}$, and $\frac{d \cdot w}{d t}$, being taken from Article 413 for steady motion, and from Article 414 for unsteady motion.

416. **A Wave** is a state of unsteady motion of a mass, whether solid or fluid, such, that the state of motion which at a given instant of time takes place amongst the particles occupying a certain space, is transmitted to other particles occupying a certain other space, along a continuous course, it may be unchanged, or it may be with modifications which still leave a certain similarity between the motions of the particles originally affected, and of those affected in succession.

For example, let a given fixed point O be taken as the origin, and let the particle which is at that point, at an instant of time denoted by 0, have a certain velocity and direction of motion. After the lapse of the time *t*, let another particle which is at a point A, distant from O by the length x, have either the same velocity and direction of motion, or a velocity and direction bearing a definite relation to those of the original particle; the motion so communicated having been transmitted in succession to all the particles between O and A.

The *velocity* of *transmission* or *propagation* of a wave, when constant, is the ratio, $\frac{x}{t}$, of the distance between two points to the time which elapses between the instants when the motions at those points are similar. Let a denote that velocity; then the condition of motion at any point whose distance from the origin is x, at the instant t, *depends upon*, or is *a function of*, $a\,t - x$; which quantity, or a quantity bearing some definite proportion to it, is called the *phase* of the wave motion. Wave motion in fluids of invariable density is regulated by *the principle of continuity* already stated.

417. **Oscillation** in a fluid, is a motion in which each individual particle of the fluid returns over and over again to the same position, and repeats over and over again the same motions. The *period* of an oscillation is the interval of time which elapses between the commencement of a series of movements, and the commencement of the repetition of the same movements. The most usual kind of oscillation in a fluid is that of a series of *oscillatory waves*, in which a certain state of motion is transmitted onward from particle to particle, that motion being oscillatory.

SECTION 3.—*Motions of Fluids of Varying Density.*

418. **Flow of Volume and Flow of Mass.**—In the case of a fluid of varying density, the *volume*, which in an unit of time flows through a given area A, with a normal velocity u, is still represented, as for a fluid of constant density, by

$$Q = A\,u\,;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.)$$

but the *absolute quantity*, or *mass* of fluid which so flows, bears no longer a constant proportion to that volume, but is proportional to the volume multiplied by the density. The density may be expressed, either in units of weight per unit of volume, or in arbitrary units suited to the particular case. Let ϱ be the density; then the *flow of mass* may be thus expressed :—

$$\varrho\,Q = \varrho\,A\,u\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.)$$

419. The **Principle of Continuity**, as applied to fluids of varying density, takes the following form :—*the flow into or out of any fixed space of constant volume is that due to the variation of density alone.*

To express this symbolically, let there be a fixed space of the constant volume V, and in a given interval of time let the density of the fluid in it, which in the first place may be supposed uniform at each instant, change from ϱ_1 to ϱ_2. Then the mass of fluid which at the beginning of the interval occupied the volume V, occupies at the end of the interval the volume $\frac{V\,\varrho_1}{\varrho_2}$; and the difference of those volumes is the volume which flows through the surface bounding the space, *outward* if ϱ_2 is less than ϱ_1, *inward* if ϱ_2 is greater then ϱ_1. Let $t_2 - t_1$ be the length of the interval of time; then the rate of flow of volume is expressed as follows :—

$$Q = \frac{V\left(\frac{\varrho_1}{\varrho_2} - 1\right)}{t_2 - t_1}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.)$$

If the rate of flow is variable during the instant in question, the above equation gives its mean value; and in that case the exact rate of *flow of volume* at a given instant is the value towards which the result of equation 1 converges as the interval of time is indefinitely diminished, viz. :—

$$Q = \frac{-V\,d\,\varrho}{\varrho\,d\,t}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.)$$

The *flow of mass* at the same instant is

$$Q\,\varrho = -\frac{V\,d\,\varrho}{d\,t}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.)$$

Next let it be supposed that the density of the fluid varies at different points of the space. Then on the right-hand side of equation 3, ϱ is to be held to represent the *mean density throughout the space* at the given instant; while on the left-hand side, ϱ must be held to represent the *mean density at the surface through which the flow takes place.* Let that surface be divided into parts, over each of which the density is uniform at a given instant; let Q′ represent the part of the flow of volume which takes place through one of those parts of the surface, and ϱ' the density of the fluid so flowing, so that $Q' \varrho'$ is the part of the flow of mass which takes place through the part of the surface in question; then for equation 3 is to be substituted

$$\Sigma \cdot Q' \varrho' = -\frac{V\, d\varrho}{dt} \dots\dots\dots\dots\dots\dots (4.)$$

420. **Stream.**—To apply the preceding principles to a *stream* of fluid of varying density, let the *axis* of the stream be a line, straight or curved, which traverses the centres of gravity of all the cross sections of the stream made at right angles to that axis, and let distances from a fixed point in that axis, measured *down-stream*, be denoted by s, and the area of any cross section by A. Let s_1, s_2, be the positions of two cross sections of the stream whose distance apart along the axis is $s_2 - s_1$; then the volume of the space between those cross sections is

$$V = \int_{s_1}^{s_2} A\, ds \dots\dots\dots\dots\dots\dots (1.)$$

Let Q_1 be the rate of flow of volume through the first cross section; Q_2 that through the second; u_1, u_2, the corresponding mean velocities normal to the respective cross sections; ϱ the mean density of the fluid in the space V; ϱ_1 the mean density at the first cross section, and ϱ_2 that at the second. Then equation 4 of Article 419 becomes

$$Q_2 \varrho_2 - Q_1 \varrho_1 = \frac{-V\, d\varrho}{dt} = -\frac{d\varrho}{dt} \cdot \int_{s_1}^{s_2} A\, ds \dots\dots\dots (2.)$$

The *rate* at which the *flow of mass varies*, in passing from one cross section of the stream to another, is the limit to which the ratio

$$\frac{Q_2 \varrho_2 - Q_1 \varrho_1}{s_2 - s_1}$$

converges as the distance $s_2 - s_1$ is indefinitely diminished; that is to say,

$$\frac{d \cdot Q\varrho}{ds} = Q \cdot \frac{d\varrho}{ds} + \varrho \frac{dQ}{ds} = -\frac{A\, d\varrho}{dt} \dots\dots\dots\dots (3.)$$

The *mean normal velocity* at a given cross section of a stream having the value $u = \frac{Q}{A}$, is subject to the equation

$$\frac{d \cdot \mathrm{A}\, u\, \varrho}{d s} = - \frac{\mathrm{A}\, d\, \varrho}{d t} \text{......................} (4.)$$

421. **Steady Motion.**—In the case of steady motion in a fluid of varying density, the density, velocity, and direction of motion at each fixed point of the space to which the motion is referred, are constant, and are assumed successively by each particle which arrives at the given point. Hence in this case, equation 4 of Article 419 becomes

$$\Sigma \cdot \mathrm{Q}'\, \varrho' = 0 \text{..........................} (1.)$$

The case of a stream is expressed by the forms assumed by equations 3 and 4 of Article 420, viz. :—

$$\frac{d \cdot \mathrm{Q}\, \varrho}{d s} = \frac{d \cdot \mathrm{A}\, u\, \varrho}{d s} = 0\,; \text{..................} (2.)$$

that is to say, *the flow of mass is uniform for all cross sections of the stream;* and being also constant for all instants of time, is therefore absolutely constant.

422. **Pistons and Cylinders.**—Let a mass of fluid of variable density be enclosed in a space whose volume is capable of being varied by the motion of one or more pistons. Let A be the area of the projection of a piston on a plane perpendicular to its direction of motion; u its normal velocity, positive if outward, negative if inward; ϱ' the density of the fluid in contact with it; V the whole volume of fluid enclosed; ϱ its mean density. Then equation 4 becomes

$$\Sigma \cdot \mathrm{A}\, u\, \varrho' = - \frac{\mathrm{V}\, d\, \varrho}{d t} = \frac{d\, \mathrm{V}}{d t}; \text{.................} (1.)$$

the last expression being introduced because $\varrho\, \mathrm{V}$ = the mass enclosed, is constant. If the density is uniform, then

$$\Sigma \cdot \mathrm{A}\, u = \frac{d\, \mathrm{V}}{d t}, \text{.....................} (1\ \text{A}.)$$

as is otherwise evident.

If the space is not completely enclosed, but has an opening whose cross section is A″, and at which the mean normal velocity of the stream is u'' (positive outward), and the density ϱ'', then the flow of mass through that opening, $\mathrm{A}''\, u''\, \varrho''$, is to be included in the summation at the left side of equation 1.

423. **General Differential Equations.**—As in Article 412 and the subsequent Articles, let u, v, and w, be the rectangular components of the velocity of the fluid at any given fixed point in the space to which the motion is referred, and $d\,x$, $d\,y$, $d\,z$, the dimensions of an indefinitely small fixed rectangular portion of that space. Then considering the pair of faces of that space whose common area is

$d\,y\,d\,z$, the flow of mass in at the first face is $-u\,\varrho \cdot d\,y\,d\,z$, and the flow of mass out at the second face is $(u\,\varrho + \frac{d \cdot u\,\varrho}{d\,x}\,d\,x)\,d\,y\,d\,z$; the resultant of which pair of flows is

$$\frac{d \cdot u\,\varrho}{d\,x} \,.\, d\,x\,d\,y\,d\,z.$$

Taking the corresponding resultant for the other two pairs of faces, adding the three quantities thus found together, observing that $\mathrm{V} = d\,x\,d\,y\,d\,z$, and dividing by that common factor, the equation 4 of Article 419, which expresses the principle of continuity, becomes the following :—

$$\frac{d \cdot u\,\varrho}{d\,x} + \frac{d \cdot v\,\varrho}{d\,y} + \frac{d \cdot w\,\varrho}{d\,z} = -\frac{d\,\varrho}{d\,t}; \ldots\ldots\ldots\ldots (1.)$$

which is the *equation of continuity for a fluid of varying density.* This equation may be otherwise expressed as follows :—

$$\varrho\left(\frac{d\,u}{d\,x} + \frac{d\,v}{d\,y} + \frac{d\,w}{d\,z}\right) + \left(u\frac{d}{d\,x} + v\frac{d}{d\,y} + w\frac{d}{d\,z} + \frac{d}{d\,t}\right)\rho = 0\,; \;(2.)$$

or dividing by ϱ,

$$\frac{d\,u}{d\,x} + \frac{d\,v}{d\,y} + \frac{d\,w}{d\,z} + \left(u\frac{d}{d\,x} + v\frac{d}{d\,y} + w\frac{d}{d\,z} + \frac{d}{d\,t}\right)\text{hyp. log. } \varrho = 0. \;(2\ \text{A.})$$

The first three terms of the last equation are identical with the three terms of the equation of continuity for a fluid of uniform density.

The conditions of *steady motion* are the following :—

$$\frac{d\,u}{d\,t} = 0;\; \frac{d\,v}{d\,t} = 0;\; \frac{d\,w}{d\,t} = 0;\; \frac{d\,\varrho}{d\,t} = 0\,; \ldots\ldots\ldots\ldots (3.)$$

which conditions apply to a *fixed point in space,* and not to an individual particle of fluid. The rates of variation of the component velocities and of the density of an individual particle of fluid are expressed as follows :—

$$\frac{d \cdot u}{d\,t} = \frac{d\,u}{d\,t} + u\frac{d\,u}{d\,x} + v\frac{d\,u}{d\,y} + w\frac{d\,u}{d\,z}\,; \ldots\ldots\ldots\ldots (4.)$$

and similar equations for $\frac{d \cdot v}{d\,t}$, $\frac{d \cdot w}{d\,t}$, and $\frac{d \cdot \varrho}{d\,t}$.

424. The **Motions of Connected Bodies** form the subject of the Theory of Mechanism, to which the Fourth Part of this treatise relates.

PART IV.

THEORY OF MECHANISM.

CHAPTER I.

DEFINITIONS AND GENERAL PRINCIPLES.

425. **Theory of Pure Mechanism Defined.**—*Machines* are bodies, or assemblages of bodies, which transmit and modify motion and force. The word "machine," in its widest sense, may be applied to every material substance and system, and to the material universe itself; but it is usually restricted to works of human art, and in that restricted sense it is employed in this treatise. A machine transmits and modifies motion when it is the means of making one motion cause another; as when the mechanism of a clock is the means of making the descent of the weight cause the rotation of the hands. A machine transmits and modifies force when it is the means of making a given kind of physical energy perform a given kind of work; as when the furnace, boiler, water, and mechanism of a marine steam engine are the means of making the energy of the chemical combination of fuel with oxygen perform the work of overcoming the resistance of water to the motion of a ship. The acts of transmitting and modifying motion, and of transmitting and modifying force, take place together, and are connected by a certain law; and until lately, they were always considered together in treatises on mechanics; but recently great advantage in point of clearness has been gained by first considering separately the act of transmitting and modifying motion. The principles which regulate this function of machines constitute a branch of Cinematics, called the *theory of pure mechanism.* The principles of the theory of pure mechanism having been first established and understood, those of the *theory of the work of machines*, which regulate the act of transmitting and modifying force, are much more readily demonstrated and apprehended than when the two departments of the theory of machines are mingled. The establishment of the theory of pure mechanism as an independent subject has been mainly accomplished by the labours of Mr. Willis, whose nomenclature and methods are, to a great extent, followed in this treatise.

426. **The General Problem** of the theory of pure mechanism may be stated as follows :—*Given the mode of connection of two or more moveable points or bodies with each other, and with certain fixed bodies; required the comparative motions of the moveable points or bodies:* and conversely, *when the comparative motions of two or more moveable points are given, to find their proper mode of connection.*

The term "comparative motion" is to be understood as in Articles 358, 367, 379, and 395. In those Articles, the comparative motions of points belonging to one body have already been considered. In order to constitute *mechanism,* two or more bodies must be so connected that their motions depend on each other through cinematical principles alone.

427. **Frame; Moving Pieces; Connectors.**—The *frame* of a machine is a structure which supports the *moving pieces,* and regulates the path or kind of motion of most of them directly. In considering the movements of machines mathematically, the frame is considered as fixed, and the motions of the moving pieces are referred to it. The frame itself may have (as in the case of a ship or of a locomotive engine) a motion relatively to the earth, and in that case the motions of the moving pieces relatively to the earth are the resultants of their motions relatively to the frame, and of the motion of the frame relatively to the earth; but in all problems of pure mechanism, and in many problems of the work of machines, the motion of the frame relatively to the earth does not require to be considered.

The *moving pieces* may be distinguished into *primary* and *secondary;* the former being those which are directly carried by the frame, and the latter those which are carried by other moving pieces. The motion of a secondary moving piece relatively to the frame is the resultant of its motion relatively to the primary piece which carries it, and of the motion of that primary piece relatively to the frame.

Connectors are those secondary moving pieces, such as links, belts, cords, and chains, which transmit motion from one moving piece to another, when that transmission is not effected by immediate contact.

428. **Bearings** are the surfaces of contact of primary moving pieces with the frame, and of secondary moving pieces with the pieces which carry them. Bearings guide the motions of the pieces which they support, and their figures depend on the nature of those motions. The bearings of a piece which has a motion of translation in a straight line, must have plane or cylindrical surfaces, *exactly straight* in the direction of motion. The bearings of rotating pieces must have surfaces accurately turned to *figures of revolu-*

tion, such as cylinders, spheres, conoids, and flat discs. The bearing of a piece whose motion is helical, must be an *exact screw*, of a pitch equal to that of the helical motion (Article 382). Those parts of moving pieces which touch the bearings, should have surfaces accurately fitting those of the bearings. They may be distinguished into *slides*, for pieces which move in straight lines, *gudgeons*, *journals*, *bushes*, and *pivots*, for those which rotate, and *screws* for those which move helically.

The accurate formation and fitting of bearing surfaces is of primary importance to the correct and efficient working of machines. Surfaces of revolution are the most easy to form accurately, screws are more difficult, and planes the most difficult of all. The success of Mr. Whitworth in making true planes, is regarded as one of the greatest achievements in the construction of machinery.

429. **The Motions of Primary Moving Pieces** are limited by the fact, that in order that different portions of a pair of bearing surfaces may accurately fit each other during their relative motion, those surfaces must be either straight, circular, or helical; from which it follows, that the motions in question can be of three kinds only, viz :—

I. *Straight translation*, or *shifting*, which is necessarily of limited extent, and which, if the motion of the machine is of indefinite duration, must be *reciprocating;* that is to say, must take place alternately in opposite directions. (See Part III., Chapter II., Section 1.)

II. *Simple rotation*, or *turning* about a fixed axis, which motion may be either continuous or reciprocating, being called in the latter case *oscillation*. (See Part III., Chapter II., Section 2.)

III. *Helical* or *screw-like motion*, to which the same remarks apply as to straight translation. (See Part III., Chapter II., Section 3, Article 382.)

430. **The Motions of Secondary Moving Pieces** relatively to the pieces which carry them, are limited by the same principles which apply to the motions of primary pieces relatively to the frame. But the motions of secondary moving pieces relatively to the frame may be any motions which can be compounded of straight translations and simple rotations according to the principles already explained in Part III., Chapter II., Section 3.

431. An **Elementary Combination** in mechanism consists of a pair of *primary moving pieces*, so connected that one transmits motion to the other.

The piece whose motion is the cause is called the *driver;* that whose motion is the effect, the *follower*. The *connection* between the driver and the follower may be—

I. By *rolling contact* of their surfaces, as in *toothless wheels*.

II. By *sliding contact* of their surfaces, as in *toothed wheels, screws, wedges, cams,* and *escapements.*

III. By *bands* or *wrapping connectors,* such as *belts, cords,* and *gearing-chains.*

IV. By *link-work,* such as *connecting rods, universal joints,* and *clicks.*

V. By *reduplication of cords,* as in the case of ropes and pulleys.

VI. By an *intervening fluid,* transmitting motion between two pistons.

The various cases of the transmission of motion from a driver to a follower are further classified, according as the relation between their *directions* of *motion* is constant or changeable, and according as the ratio of their *velocities* is constant or variable. This latter principle of classification is employed by Mr. Willis as the foundation of a primary division of the subject of elementary combinations in mechanism into classes, which are subdivided according to the mode of connection of the pieces. In the present treatise, elementary combinations will be classed primarily according to the mode of connection.

432. **Line of Connection.**—In every class of elementary combinations, except those in which the connection is made by reduplication of cords, or by an intervening fluid, there is at each instant a certain straight line, called the *line of connection,* or *line of mutual action* of the driver and follower. In the case of rolling contact, this is any straight line whatsoever traversing the point of contact of the surfaces of the pieces; in the case of sliding contact, it is a line perpendicular to those surfaces at their point of contact; in the case of wrapping connectors, it is the centre line of that part of the connector by whose tension the motion is transmitted; in the case of link-work, it is the straight line passing through the points of attachment of the link to the driver and follower.

433. **Principle of Connection.**—The line of connection of the driver and follower at any instant being known, their comparative velocities are determined by the following principle:—*The respective linear velocities of a point in the driver, and a point in the follower, each situated anywhere in the line of connection, are to each other inversely as the cosines of the respective angles made by the paths of the points with the line of connection.* This principle might be otherwise stated as follows:—*The components, along the line of connection, of the velocities of any two points situated in that line, are equal.*

434. **Adjustments of Speed.**—The velocity-ratio of a driver and its follower is sometimes made capable of being changed at will, by means of apparatus for varying the position of their line of connection; as when a pair of rotating cones are embraced by a belt

which can be shifted so as to connect portions of their surfaces of different diameters.

435. A **Train of Mechanism** consists of a series of moving pieces, each of which is follower to that which drives it, and driver to that which follows it.

436. **Aggregate Combinations** in mechanism are those by which compound motions are given to secondary pieces.

CHAPTER II.

ON ELEMENTARY COMBINATIONS AND TRAINS OF MECHANISM.

SECTION 1.—*Rolling Contact.*

437. **Pitch Surfaces** are those surfaces of a pair of moving pieces, which touch each other when motion is communicated by rolling contact. The LINE OF CONTACT is that line which at each instant traverses all the pairs of points of the pair of pitch surfaces which are in contact.

438. **Smooth Wheels, Rollers, Smooth Racks.**—Of a pair of primary moving pieces in rolling contact, both may rotate, or one may rotate and the other have a motion of sliding, or straight translation. A rotating piece, in rolling contact, is called a *smooth wheel*, and sometimes a *roller;* a sliding piece may be called a *smooth rack.*

439. **General Conditions of Rolling Contact.**—The whole of the principles which regulate the motions of a pair of pieces in rolling contact follow from the single principle, *that each pair of points in the pitch surfaces, which are in contact at a given instant, must at that instant be moving in the same direction with the same velocity.*

The direction of motion of a point in a rotating body being perpendicular to a plane passing through its axis, the condition, that each pair of points in contact with each other must move in the same direction leads to the following consequences :—

I. That when both pieces rotate, their axes, and all their points of contact, lie in the same plane.

II. That when one piece rotates and the other slides, the axis of the rotating piece, and all the points of contact, lie in a plane perpendicular to the direction of motion of the sliding piece.

The condition, that the velocities of each pair of points of contact must be equal, leads to the following consequences :—

III. That the angular velocities of a pair of wheels, in rolling contact, must be inversely as the perpendicular distances of any pair of points of contact from the respective axes.

IV. That the linear velocity of a smooth rack in rolling contact with a wheel, is equal to the product of the angular velocity of the wheel by the perpendicular distance from its axis to a pair of points of contact.

Respecting the line of contact, the above principles III. and IV. lead to the following conclusions :—

V. That for a pair of wheels with parallel axes, and for a wheel and rack, the line of contact is straight, and parallel to the axes or axis; and hence that the pitch surfaces are either plane or cylindrical (the term "cylindrical" including all surfaces generated by the motion of a straight line parallel to itself).

VI. That for a pair of wheels, with intersecting axes, the line of contact is also straight, and traverses the point of intersection of the axes; and hence that the rolling surfaces are conical, with a common apex (the term "conical" including all surfaces generated by the motion of a straight line which traverses a fixed point).

440. **Circular Cylindrical Wheels** are employed when an uniform velocity-ratio is to be communicated between parallel axes. Figs. 187, 188, and 189, of Article 388, may be taken to represent pairs of such wheels; C and O, in each figure, being the parallel axes of the wheels, and T a point in their line of contact. In fig. 187, both pitch surfaces are convex, the wheels are said to be in *outside gearing*, and their directions of rotation are contrary. In figs. 188 and 189, the pitch surface of the larger wheel is concave, and that of the smaller convex; they are said to be in *inside gearing*, and their directions of rotation are the same.

To represent the comparative motions of such pairs of wheels symbolically, let

$$\overline{\mathrm{O\,T}} = r_1,\ \overline{\mathrm{C\,T}} = r_2,$$

be their radii: let $\overline{\mathrm{O\,C}} = c$ be the *line of centres*, or perpendicular distance between the axes, so that for

$$\left.\begin{matrix}\text{outside}\\ \text{inside}\end{matrix}\right\}\text{gearing, } c = r_1 \pm r_2 \ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

Let a_1, a_2, be the angular velocities of the wheels, and v the common linear velocity of their pitch surfaces; then

$$\left.\begin{matrix} v = a_1 r_1 = a_2 r_2\,, \\ c : r_1 : r_2 : : a_2 \pm a_1 : a_2 : a_1\,; \end{matrix}\right\} \ldots\ldots\ldots\ldots\ldots (2.)$$

the sign $\pm$ applying to $\left\{\begin{matrix}\text{outside}\\ \text{inside}\end{matrix}\right\}$ gearing.

441. **A Straight Rack and Circular Wheel,** which are used when an uniform velocity-ratio is to be communicated between a sliding piece and a turning piece, may be represented by fig. 185 of Article 385, C being the axis of the wheel, P T P the plane surface of the rack, and T a point in their line of contact. Let r be the radius of the wheel, a its angular velocity, and v the linear velocity of the rack; then

$$v = r\,a.$$

442. **Bevel Wheels,** whose pitch surfaces are frustra of regular cones, are used to transmit an uniform angular velocity-ratio between a pair of axes which intersect each other. Fig. 190 of Article 392 will serve to illustrate this case; O A and O C being the pair of axes, intersecting each other in O, O T the line of contact, and the cones described by the revolution of O T about O A and O C respectively being the pitch surfaces, of which narrow zones or frustra are used in practice.

Let a_1, a_2, be the angular velocities about the two axes respectively; and let $i_1 = \angle \mathrm{A\,O\,T}$, $i_2 = \angle \mathrm{C\,O\,T}$, be the angles made by those axes respectively with the line of contact; then from the principle III. of Article 439 it follows, that the angular velocity-ratio is

$$\frac{a_2}{a_1} = \frac{\sin i_1}{\sin i_2};\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.)$$

Which equation serves to find the angular velocity-ratio when the axes and the line of contact are given.

Conversely, let the angle between the axes,

$$\angle \mathrm{A\,O\,C} = i_1 + i_2 = j,$$

be given, and also the ratio $\frac{a_2}{a_1}$; then the position of the line of contact is given by either of the two following equations :—

$$\left.\begin{aligned} \sin i_1 &= \frac{a_2 \sin j}{\sqrt{(a_1^2 + a_2^2 + 2\, a_1\, a_2 \cos j)}}; \\ \sin i_2 &= \frac{a_1 \sin j}{\sqrt{(a_1^2 + a_2^2 + 2\, a_1\, a_2 \cos j)}}; \end{aligned}\right\} \ldots\ldots\ldots(2.)$$

Graphically, the same problem is solved as follows :—On the two axes respectively, take lengths to represent the angular velocities of their respective wheels. Complete the parallelogram of which those lengths are the sides, and its diagonal will be the line of contact. As in the case of the rolling cones of Article 393, one of a pair of bevel wheels may be a flat disc, or a concave cone.

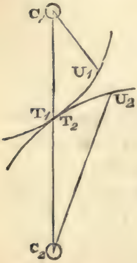

Fig. 191.

443. **Non-Circular Wheels** are used to transmit a variable velocity-ratio between a pair of parallel axes. In fig. 191, let C_1, C_2, represent the axes of such a pair of wheels; T_1, T_2, a pair of points which at a given instant touch each other in the line of contact (which line is parallel to the axes and in the same plane with them); and U_1, U_2, another pair of points, which touch each other at another instant of the motion; and let the four points, T_1,

T_2, U_1, U_2, be in one plane perpendicular to the two axes, and to the line of contact. Then for every such set of four points, the two following equations must be fulfilled :—

$$\left.\begin{array}{c}\overline{C_1 U_1} + \overline{C_2 U_2} = \overline{C_1 T_1} + \overline{C_2 T_2} = \overline{C_1 C_2}; \\ \text{arc } T_1 U_1 = \text{arc } T_2 U_2;\end{array}\right\} \text{........} (1.)$$

and those equations show the geometrical relations which must exist between a pair of rotating surfaces in order that they may move in rolling contact round fixed axes.

The same conditions are expressed differentially in the following manner :—Let r_1, r_2, be the *radii vectores* of a pair of points which touch each other; $d s_1$, $d s_2$, a pair of elementary arcs of the cross sections $T_1 U_1$, $T_2 U_2$, of the pitch surfaces, and c the line of centres or distance between the axes. Then

$$\left.\begin{array}{c} r_1 + r_2 = c; \\ \dfrac{d s_1}{d r_1} = -\dfrac{d s_2}{d r_2}. \end{array}\right\} \text{........................} (2.)$$

If one of the wheels be fixed and the other be rolled upon it, a point in the axis of the rolling wheel describes a circle of the radius c round the axis of the fixed wheel.

The equations 1 and 2 are made applicable to *inside gearing* by putting $-$ instead of $+$ and $+$ instead of $-$.

The angular velocity-ratio at a given instant has the value

$$\frac{a_2}{a_1} = \frac{r_1}{r_2} \text{..............................} (3.)$$

As examples of non-circular wheels, the following may be mentioned :—

I. An ellipse rotating about one focus rolls completely round in outside gearing with an equal and similar ellipse also rotating about one focus, the distance between the axes of rotation being equal to the major axis of the ellipses, and the velocity-ratio varying from

$$\frac{1 - \text{excentricity}}{1 + \text{excentricity}} \text{ to } \frac{1 + \text{excentricity}}{1 - \text{excentricity}}.$$

II. A hyperbola rotating about its farther focus, rolls in inside gearing, through a limited arc, with an equal and similar hyperbola rotating about its nearer focus, the distance between the axes of rotation being equal to the axis of the hyperbolas, and the velocity-ratio varying between

$$\frac{\text{excentricity} + 1}{\text{excentricity} - 1} \text{ and unity.}$$

III. Two logarithmic spirals of equal obliquity rotate in rolling contact with each other through an indefinite angle. (For further examples of non-circular wheels, see Professor Clerk Maxwell's paper on Rolling Curves, *Trans. Roy. Soc. Edin.*, vol. xvi., and Professor Willis's work on Mechanism.)

Section 2.—*Sliding Contact.*

444. **Skew-Bevel Wheels** are employed to transmit an uniform velocity-ratio between two axes which are neither parallel nor intersecting. The pitch surface of a skew-bevel wheel is a frustrum or zone of a *hyperboloid of revolution.* In fig. 192, a pair of large portions of such hyperboloids are shown, rotating about axes A B, C D. In fig. 193 are shown a pair of narrow zones of the same figures, such as are employed in practice.

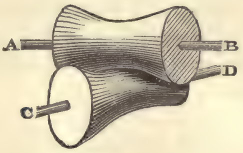

Fig. 192.

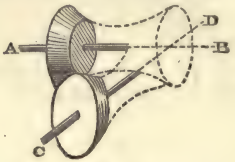

Fig. 193.

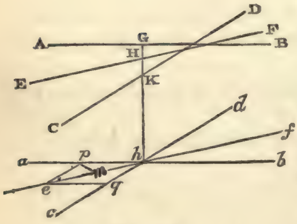

Fig. 194.

A hyperboloid of revolution is a surface resembling a sheaf or a dice box, generated by the rotation of a straight line round an axis from which it is at a constant distance, and to which it is inclined at a constant angle. If two such hyperboloids, equal or unequal, be placed in the closest possible contact, as in fig. 192, they will touch each other along one of the generating straight lines of each, which will form their line of contact, and will be inclined to the axes A B, C D, in opposite directions. The axes will neither be parallel, nor will they intersect each other.

The motion of two such hyperboloids, rotating in contact with each other, has sometimes been classed amongst cases of rolling contact; but that classification is not strictly correct; for although the component velocities of a pair of points of contact in a direction at right angles to the line of contact are equal, still, as the axes are neither parallel to each other nor to the line of contact, the velocities of a pair of points of contact have components along the line of contact, which are unequal, and their difference constitutes a *lateral sliding.*

The directions and positions of the axes being given, and the required angular velocity-ratio, $\frac{a_2}{a_1}$, it is required to find the *obliquities* of the generating line to the two axes, and its *radii vectores,* or least perpendicular distances from these axes.

In fig. 194, let A B, C D, be the two axes, and G K their common perpendicular.

On any plane normal to the common perpendicular G K h, draw $a\,b \parallel$ A B, $c\,d \parallel$ C D, in which take lengths in the following proportions:—

$$a_1 : a_2 : : \overline{h\,p} : \overline{h\,q};$$

complete the parallelogram $h\,p\,e\,q$, and draw its diagonal $e\,h\,f$; the line of contact E H F will be parallel to that diagonal.

From p let fall $p\,m$ perpendicular to $h\,e$. Then divide the common perpendicular G K in the ratio given by the proportional equation

$$\overline{h\,e} : \overline{e\,m} : \overline{m\,h} : : \overline{\mathrm{G\,K}} : \overline{\mathrm{G\,H}} : \overline{\mathrm{K\,H}};$$

then the two segments thus found will be the least distances of the line of contact from the axes.

The first pitch surface is generated by the rotation of the line E H F about the axis A B with the radius vector $\overline{\mathrm{G\,H}} = r_1$; the second, by the rotation of the same line about the axis C D with the radius vector $\overline{\mathrm{H\,K}} = r_2$.

To draw the hyperbola which is the longitudinal section of a skew-bevel wheel whose generating line has a given radius vector and obliquity, let A G B, fig. 195, represent the axis, G H ⊥ A G B, the radius vector of the generating line, and let the straight line E G F make with the axis an angle equal to the obliquity of the generating line. H will be the vertex, and E G F one of the asymptotes, of the required hyperbola. To find any number of points in that hyperbola, proceed as follows:—Draw X W Y parallel to G H, cutting G E in W, and make $\overline{\mathrm{X\,Y}} = \sqrt{(\overline{\mathrm{G\,H}}^2 + \overline{\mathrm{X\,W}}^2)}$. Then will Y be a point in the hyperbola.

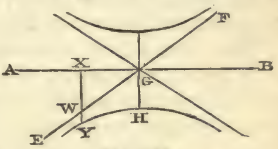

Fig. 195.

445. **Grooved Wheels.**—To increase the friction or adhesion between a pair of wheels, which is the means of transmitting force and motion from one to the other, their surfaces of contact are sometimes formed into alternate circular ridges and grooves, constituting what is called *frictional gearing.* Fig. 196 is a cross

section illustrating the kind of frictional gearing invented by Mr. Robertson. The comparative motion of a pair of wheels thus ridged and grooved is nearly the same with that of a pair of smooth wheels in rolling contact, having cylindrical or conical pitch surfaces lying midway between the tops of the ridges and bottoms of the grooves.

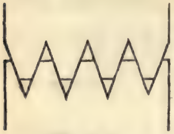

Fig. 196.

The relative motion of the faces of contact of the edges and grooves is a rotatory sliding, about the line of contact of the ideal pitch surfaces as an instantaneous axis.

The angle between the sides of each groove is about 40° ; and it is stated that the mutual friction of the wheels is about once and a-half the force with which their axes are pressed towards each other.

446. **Teeth of Wheels.**—The most usual method of communicating motion between a pair of wheels, or a wheel and a rack, and the only method which, by preventing the possibility of the rotation of one wheel unless accompanied by the other, insures the preservation of a given velocity-ratio exactly, is by means of the projections called *teeth.*

The *pitch surface* of a wheel is an ideal smooth surface, intermediate between the crests of the teeth and the bottoms of the spaces between them, which, by rolling contact with the pitch surface of another wheel, would communicate the same velocity-ratio that the teeth communicate by their sliding contact. In designing wheels, the forms of the ideal pitch surfaces are first determined, and from them are deduced the forms of the teeth.

Wheels with cylindrical pitch surfaces are called *spur wheels;* those with conical pitch surfaces, *bevel wheels;* and those with hyperboloidal pitch surfaces, *skew-bevel wheels.*

The *pitch line* of a wheel, or, in circular wheels, the *pitch circle,* is a transverse section of the pitch surface made by a surface perpendicular to it and to the axis ; that is, in spur wheels, by a plane perpendicular to the axis ; in bevel wheels, by a sphere described about the apex of the conical pitch surface ; and in skew-bevel wheels, by any oblate spheroid generated by the rotation of an ellipse whose foci are the same with those of the hyperbola that generates the pitch surface.

The *pitch point* of a pair of wheels is the point of contact of their pitch lines ; that is, the transverse section of the line of contact of the pitch surfaces.

Similar terms are applied to racks.

That part of the acting surface of a tooth which projects beyond the pitch surface is called the *face;* that which lies within the pitch surface, the *flank.*

The radius of the pitch circle of a circular wheel is called the *geometrical radius;* that of a circle touching the crests of the teeth is called the *real radius;* and the difference between those radii, the *addendum.*

447. **Pitch and Number of Teeth.**—The distance, measured along the pitch line, from the face of one tooth to the face of the next, is called the PITCH.

The pitch, and the number of teeth in circular wheels, are regulated by the following principles:—

I. In wheels which rotate continuously for one revolution or more, it is obviously necessary *that the pitch should be an aliquot part of the circumference.*

In wheels which reciprocate without performing a complete revolution, this condition is not necessary. Such wheels are called *sectors.*

II. In order that a pair of wheels, or a wheel and a rack, may work correctly together, it is in all cases essential *that the pitch should be the same in each.*

III. Hence, in any pair of circular wheels which work together, the numbers of teeth in a complete circumference are directly as the radii, and inversely as the angular velocities.

IV. Hence also, in any pair of circular wheels which rotate continuously for one revolution or more, the ratio of the numbers of teeth, and its reciprocal, the angular velocity-ratio, must be expressible in whole numbers.

V. Let n, N, be the respective numbers of teeth in a pair of wheels, N being the greater. Let t, T, be a pair of teeth in the smaller and larger wheel respectively, which at a particular instant work together. It is required to find, first, how many pairs of teeth must pass the line of contact of the pitch surfaces before t and T work together again (let this number be called a); secondly, with how many different teeth of the larger wheel the tooth t will work at different times (let this number be called b); and thirdly, with how many different teeth of the smaller wheel the tooth T will work at different times (let this be called c).

CASE 1. If n is a divisor of N,

$$a = \mathrm{N}\,;\ b = \frac{\mathrm{N}}{n};\ c = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

CASE 2. If the greatest common divisor of N and n be d, a number less than n, so that $n = m\,d$, $\mathrm{N} = \mathrm{M}\,d$, then

$$a = m\,\mathrm{N} = \mathrm{M}\,n = \mathrm{M}\,m\,d\,;\ b = \mathrm{M}\,;\ c = m \ldots\ldots\ldots (2.)$$

CASE 3. If N and n be prime to each other,

$$a = \mathrm{N}n;\ b = \mathrm{N};\ c = n \ldots\ldots\ldots\ldots\ldots\ldots (3.)$$

It is considered desirable by millwrights, with a view to the preservation of the uniformity of shape of the teeth of a pair of wheels, that each given tooth in one wheel should work with as many different teeth in the other wheel as possible. They, therefore, study to make the numbers of teeth in each pair of wheels which work together such as to be either prime to each other, or to have their greatest common divisor as small as is possible consistently with the purposes of the machine.

VI. The *smallest* number of teeth which it is practicable to give to a pinion (that is, a small wheel), is regulated by the principle, that in order that the communication of motion from one wheel to another may be continuous, at least *one* pair of teeth should always be in action; and that in order to provide for the contingency of a tooth breaking, a *second* pair, at least, should be in action also. For reasons which will appear when the forms of teeth are considered, this principle gives the following as the least numbers of teeth which can be *usually* employed in pinions having teeth of the three classes of figures named below, whose properties will be explained in the sequel:—

I.	Involute teeth,...............................	25.
II.	Epicycloidal teeth,.........................	12.
III.	Cylindrical teeth, or *staves*, ...	6.

448. **Hunting Cog.**—When the ratio of the angular velocities of two wheels, being reduced to its least terms, is expressed by small numbers, less than those which can be given to wheels in practice, and it becomes necessary to employ multiples of those numbers by a common multiplier, which becomes a common divisor of the numbers of teeth in the wheels, millwrights and engine-makers avoid the evil of frequent contact between the same pairs of teeth, by giving one additional tooth, called a *hunting cog*, to the larger of the two wheels. This expedient causes the velocity-ratio to be not exactly but only approximately equal to that which was at first contemplated; and therefore it cannot be used where the exactness of certain velocity-ratios amongst the wheels is of importance, as in clockwork.

449. A **Train of Wheelwork** consists of a series of axes, each having upon it two wheels, one of which is *driven* by a wheel on the preceding axis, while the other *drives* a wheel on the following axis. If the wheels are all in outside gearing, the direction of rotation of each axis is contrary to that of the adjoining axes. In some cases, a single wheel upon one axis answers the purpose both of receiving motion from a wheel on the preceding axis and giving

motion to a wheel on the following axis. Such a wheel is called an *idle wheel:* it affects the direction of rotation only, and not the velocity-ratio.

Let the series of axes be distinguished by numbers 1, 2, 3, &c. .... m; let the numbers of teeth in the *driving wheels* be denoted by N's, each with the number of its axis affixed; thus, N_1, N_2, &c. .... N_{m-1}; and let the numbers of teeth in the *driven* or *following* wheels be denoted by n's, each with the number of its axis affixed; thus, n_2, n_3, &c. .... n_m. Then the ratio of the angular velocity a_m of the m^{th} axis to the angular velocity a_1 of the first axis is the product of the $m-1$ velocity-ratios of the successive elementary combinations, viz.:—

$$\frac{a_m}{a_1} = \frac{N_1 \cdot N_2 \cdot \&c. \ldots N_{m-1}}{n_2 \cdot n_3 \cdot \&c. \ldots n_m}; \ldots\ldots\ldots\ldots (1.)$$

that is to say, the velocity-ratio of the last and first axes is the ratio of the product of the numbers of teeth in the drivers to the product of the numbers of teeth in the followers; and it is obvious, that so long as the same drivers and followers constitute the train, the *order* in which they succeed each other does not affect the resultant velocity-ratio.

Supposing all the wheels to be in outside gearing, then as each elementary combination reverses the direction of rotation, and as the number of elementary combinations, $m-1$, is one less than the number of axes, m, it is evident that if m is odd, the direction of rotation is preserved, and if even, reversed.

It is often a question of importance to determine the numbers of teeth in a train of wheels best suited for giving a determinate velocity-ratio to two axes. It was shown by Young, that to do this with the *least total number of teeth*, the velocity-ratio of each elementary combination should approximate as nearly as possible 3·59. This would in many cases give too many axes; and as a useful practical rule it may be laid down, that from 3 to 6 ought to be the limit of the velocity-ratio of an elementary combination in wheelwork.

Let $\frac{B}{C}$ be the velocity-ratio required, reduced to its least terms, and let B be greater than C.

If $\frac{B}{C}$ is not greater than 6, and C lies between the prescribed minimum number of teeth (which may be called t), and its double $2t$, then one pair of wheels will answer the purpose, and B and C will themselves be the numbers required. Should B and C be inconveniently large, they are if possible to be resolved into factors, and those factors, or if they are too small, multiples of them, used for the numbers of teeth. Should B or C, or both, be at once inconveniently large, and prime, then instead of the exact ratio $\frac{\mathrm{B}}{\mathrm{C}}$, some ratio approximating to that ratio, and capable of resolution into convenient factors, is to be found by the method of continued fractions.

Should $\frac{\mathrm{B}}{\mathrm{C}}$ be greater than 6, the best number of elementary combinations, $m - 1$, will lie between

$$\frac{\log \mathrm{B} - \log \mathrm{C}}{\log 6} \text{ and } \frac{\log \mathrm{B} - \log \mathrm{C}}{\log 3} \ldots\ldots\ldots\ldots (2.)$$

Then, if possible, B and C themselves are to be resolved each into $m - 1$ factors (counting 1 as a factor), which factors, or multiples of them, shall be not less than t, nor greater than $6\,t$; or if B and C contain inconveniently large prime factors, an approximate velocity-ratio, found by the method of continued fractions, is to be substituted for $\frac{\mathrm{B}}{\mathrm{C}}$ as before.

So far as the resultant velocity-ratio is concerned, the *order* of the drivers N and of the followers n is immaterial; but to secure equable wear of the teeth, as explained in Article 447, Principle V., the wheels ought to be so arranged that for each elementary combination the greatest common divisor of N and n shall be either 1, or as small as possible.

450. **Principle of Sliding Contact.**—The *line of action*, or *of connection*, in the case of sliding contact of two moving pieces, is the common perpendicular to their surfaces at the point where they touch; and the principle of their comparative motion is, that *the components, along that perpendicular, of the velocities of any two points traversed by it, are equal.*

CASE 1. *Two shifting pieces*, in sliding contact, have linear velocities proportional to the secants of the angles which their directions of motion make with their line of action.

CASE 2. *Two rotating pieces*, in sliding contact, have angular velocities inversely proportional to the perpendicular distances from their axes of rotation to their line of action, each multiplied by the sine of the angle which the line of action makes with the particular axis on which the perpendicular is let fall.

In fig. 197, let C_1, C_2, represent the axes of rotation of the two pieces; A_1, A_2, two portions of their respective surfaces; and T_1, T_2, a pair of points in those surfaces, which, at the instant under consideration, are in contact with each other. Let $P_1\,P_2$ be the common perpendicular of the surfaces at the pair of points T_1, T_2;

that is, the *line of action;* and let $\overline{C_1 P_1}$, $\overline{C_2 P_2}$, be the common perpendiculars of the line of action and of the two axes respectively. Then at the given instant, the components along the line $P_1 P_2$ of the velocities of the points P_1, P_2, are equal. Let i_1, i_2, be the angles which that line makes with the directions of the axes respectively. Let a_1, a_2, be the respective angular velocities of the moving pieces; then

$$a_1 \cdot \overline{C_1 P_1} \cdot \sin i_1 = a_2 \cdot \overline{C_2 P_2} \cdot \sin i_2;$$

consequently,

$$\frac{a_2}{a_1} = \frac{\overline{C_1 P_1} \sin i_1}{\overline{C_2 P_2} \sin i_2}; \text{..........} (1.)$$

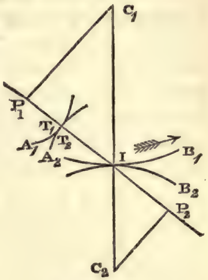

Fig. 197.

which is the principle stated above.

When the line of action is perpendicular in direction to both axes, then $\sin i_1 = \sin i_2 = 1$; and equation 1 becomes

$$\frac{a_2}{a_1} = \frac{\overline{C_1 P_1}}{\overline{C_2 P_2}} \text{........................} (1 \text{ A.})$$

When the axes are parallel, $i_1 = i_2$. Let I be the point where the line of action cuts the plane of the two axes; then the triangles $P_1 C_1 I$, $P_2 C_2 I$, are similar; so that equation 1 A is equivalent to the following :—

$$\frac{a_2}{a_1} = \frac{\overline{I C_1}}{\overline{I C_2}} \text{........................} (1 \text{ B.})$$

CASE 3. *A rotating piece and a shifting piece,* in sliding contact, have their comparative motion regulated by the following principle :—Let $\overline{C P}$ denote the perpendicular distance from the axis of the rotating piece to the line of action; i the angle which the direction of the line of action makes with that axis; a the angular velocity of the rotating piece; v the linear velocity of the sliding piece; j the angle which its direction of motion makes with the line of action; then

$$v = a \cdot \overline{C P} \cdot \sin i \cdot \sec j \text{....................} (2.)$$

When the line of action is perpendicular in direction to the axis of the rotating piece, $\sin i = 1$; and

$$v = a \cdot \overline{C P} \cdot \sec \cdot j = a \cdot \overline{I C}; \text{..............} (2 \text{ A.})$$

where $\overline{I C}$ denotes the distance from the axis of the rotating piece to the point where the line of action cuts a perpendicular from that axis on the direction of motion of the shifting piece.

451. **Teeth of Spur-Wheels and Racks. General Principle.**—The figures of the teeth of wheels are regulated by the principle, *that the teeth of a pair of wheels shall give the same velocity-ratio by their sliding contact, which the ideal smooth pitch surfaces would give by their rolling contact.* Let B_1, B_2, in fig. 197, be parts of the pitch lines (that is, of cross sections of the pitch surfaces) of a pair of wheels with parallel axes, and I the pitch point (that is, a section of the line of contact). Then the angular velocities which would be given to the wheels by the rolling contact of those pitch lines are inversely as the segments $\overline{I C_1}$, $\overline{I C_2}$, of the line of centres; and this also is the proportion of the angular velocities given by a pair of surfaces in sliding contact whose line of action traverses the point I (Article 450, case 2, equation 1 B). Hence the condition of correct working for the teeth of wheels with parallel axes is, *that the line of action of the teeth shall at every instant traverse the line of contact of the pitch surfaces;* and the same condition obviously applies to a rack sliding in a direction perpendicular to that of the axis of the wheel with which it works.

452. **Teeth Described by Rolling Curves.**—From the principle of the preceding Article it follows, that at every instant, the position of the point of contact T_1 in the cross section of the acting surface of a tooth (such as the line $A_1 T_1$ in fig. 197), and the corresponding position of the pitch point I in the pitch line $I B_1$ of the wheel to which that tooth belongs, are so related, that the line $I T_1$ which joins them is normal to the outline of the tooth $A_1 T_1$ at the point T_1. Now this is the relation which exists between the *tracing-point* T_1, and the *instantaneous axis or line of contact* I, in a rolling curve of such a figure, that being rolled upon the pitch surface B_1, its tracing-point T_1 traces the outline of the tooth. (As to rolling curves, see Articles 386, 387, 389, 390, 393, 396, 397, and Professor Clerk Maxwell's paper there referred to).

In order that a pair of teeth may work correctly together, it is necessary and sufficient that the *instantaneous radii vectores* from the pitch point to the points of contact of the two teeth should coincide at each instant, as expressed by the equation

$$\overline{I T_1} = \overline{I T_2}; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

and this condition is fulfilled, *if the outlines of the two teeth be traced by the motion of the same tracing-point, in rolling the same rolling curve on the same side of the pitch surfaces of the respective wheels.*

The *flank* of a tooth is traced while the rolling curve rolls *inside* of the pitch line; the *face*, while it rolls *outside.* Hence it is evident that the *flanks* of the teeth of the driving wheel drive the *faces* of the teeth of the driven wheel; and that the *faces* of the teeth of the driving wheel drive the *flanks* of the teeth of the driven wheel. The former takes place while the point of contact of the teeth is *approaching* the pitch point, as in fig. 197, supposing the motion to be from P_1 towards P_2; the latter, after the point of contact has passed, and while it is *receding from*, the pitch point. The pitch point divides the path of the point of contact of the teeth into two parts, called the *path of approach* and the *path of recess;* and the lengths of those paths must be so adjusted, that two pairs of teeth at least shall be in action at each instant.

It is evidently necessary that the surfaces of contact of a pair of teeth should either be both convex, or that if one is convex and the other concave, the concave surface should have the flatter curvature.

The equations of Article 390 give the relations which exist between the radius of curvature of a pitch line at the pitch point (r_1), the radius of curvature of the rolling curve at the same point (r_2), the radius vector of the tracing-point ($r = \mathrm{I\,T}$), the angle made by that line with the line of centres of the fixed and rolling curves ($\theta = \angle \mathrm{T\,I\,C}$), and the radius of curvature of the curve traced by the point T (ϱ), all at a given instant.

When a pair of tooth surfaces are both convex absolutely, that which is a face is concave, and that which is a flank is convex, towards the pitch point; and this is indicated by the values of ϱ having contrary signs for the two teeth, being positive for the face and negative for the flank. The *face* of a tooth is always convex absolutely, and concave towards the pitch point, ϱ being positive; so that if it works with a concave flank, the value of ϱ for that flank is positive also, and greater than for the face with which it works.

453. The **Sliding of a Pair of Teeth on Each Other,** that is, their relative motion in a direction perpendicular to their line of action, is found by supposing one of the wheels, such as 1, to be fixed, the line of centres $C_1 C_2$ to rotate backwards round C_1 with the angular velocity a_1, and the wheel 2 to rotate round C_2 as before with the angular velocity a_2 relatively to the line of centres $C_1 C_2$, so as to have the same motion as if its pitch surface *rolled* on the pitch surface of the first wheel. Thus the *relative* motion of the wheels is unchanged; but 1 is considered as fixed, and 2 has the resultant motion given by the principles of Article 388; that is, a rotation about the instantaneous axis I with the angular velocity $a_1 + a_2$. Hence the *velocity of sliding* is that due to this rotation about I, with the radius $\overline{\mathrm{I\,T}} = r$; that is to say, its value is

$$r\,(a_1 + a_2);\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.)$$

so that it is greater, the farther the point of contact is from the line of centres; and at the instant when that point, passing the line of centres, coincides with the *pitch point*, the velocity of sliding is null, and the action of the teeth is, for the instant, that of rolling contact.

The roots of the teeth slide towards each other during the approach, and from each other during the recess. To find the *amount* or *total distance* through which the sliding takes place, let t_1 be the time occupied by the approach, and t_2 that occupied by the recess; then the distance of sliding is

$$s = \int_0^{t_1} r\,(a_1 + a_2)\,d\,t + \int_0^{t_2} r\,(a_1 + a_2)\,d\,t\,;\ldots\ldots\ldots(2.)$$

or in another form, if $d\,i$ denote an element of the change of angular position of one wheel relatively to the other, i_1 the amount of that change during the approach, and i_2 during the recess, then

$$(a_1 + a_2)\,d\,t = d\,i\,;\ \text{and}$$

$$s = \int_0^{i_1} r\,d\,i + \int_0^{i_2} r\,d\,i\ldots\ldots\ldots\ldots\ldots\ldots(3.)$$

(See also Article 455.)

454. **The Arc of Contact on the Pitch Lines** is the length of that portion of the pitch lines which passes the pitch point during the action of one pair of teeth; and in order that two pairs of teeth at least may be in action at each instant, its length should be at least double of the pitch. It is divided into two parts, the arc of approach and the arc of recess. In order that the teeth may be of length sufficient to give the required duration of contact, the distance moved over by the point I upon the pitch line during the rolling of a rolling curve to describe the face and flank of a tooth, must be in all equal to the length of the required arc of contact. It is usual to make the arcs of approach and recess equal.

455. **The Length of a Tooth** may be divided into two parts, that of the face and that of the flank. For teeth in the driving wheel, the length of the flank depends on the arc of approach,—that of the face, on the arc of recess; for those in the following wheel, the length of the flank depends on the arc of recess,—that of the face, on the arc of approach.

Let q_1 be the arc of approach, q_2 that of recess; l_1 the length of the flank, l'_1 the length of the face of a tooth in the driving wheel. Let r_1 be the radius of curvature of the pitch line, r_0 that of the rolling curve, r the radius vector of the tracing-point, at any instant. The angular velocity of the rolling curve relatively to the wheel is

$$\frac{d\,q}{d\,t}\cdot\left(\frac{1}{r_0} \pm \frac{1}{r_1}\right),$$

the positive sign applying to rolling outside, or describing the face, and the negative sign to rolling inside, or describing the flank. Hence the velocity of the tracing-point at a given instant is

$$\frac{d\,q}{d\,t}\cdot\left(\frac{r}{r_0}\pm\frac{r}{r_1}\right);$$

and consequently

$$\left.\begin{aligned} l_1 &= \int_0^{q_1}\left(\frac{r}{r_0}-\frac{r}{r_1}\right)d\,q; \\ l'_1 &= \int_0^{q_2}\left(\frac{r}{r_0}+\frac{r}{r_1}\right)d\,q. \end{aligned}\right\}\ \text{..................}(1.)$$

For the following wheel, q_1 and q_2 have to be interchanged, so that, if r_2 be the radius of that wheel,

$$\left.\begin{aligned} l_2 &= \int_0^{q_2}\left(\frac{r}{r_0}-\frac{r}{r_2}\right)d\,q; \\ l'_2 &= \int_0^{q_1}\left(\frac{r}{r_0}+\frac{r}{r_2}\right)d\,q. \end{aligned}\right\}\ \text{..................}(2.)$$

The equations 2 and 3 evidently give the means of finding the distance of sliding between a pair of teeth, in a different form from that given in Article 453; for that distance is

$$\begin{aligned} s &= (l'_2-l_1)+(l'_1-l_2) \\ &= \int_0^{q_1}\left(\frac{r}{r_1}+\frac{r}{r_2}\right)d\,q+\int_0^{q_2}\left(\frac{r}{r_1}+\frac{r}{r_2}\right)d\,q\text{..........}(3.) \end{aligned}$$

456. To **Inside Gearing** all the preceding principles apply, observing that the radius of the greater, or concave pitch surface, is to be considered as negative, and that in Article 453, the difference of the angular velocities is to be taken instead of their sum.

457. **Involute Teeth for Circular Wheels,** being the first of the three kinds mentioned in Article 447, are of the form of the involute of a circle, of a radius less than the pitch circle in a ratio which may be expressed by the sine of a certain angle θ, and may be traced by the pole of a logarithmic spiral rolling on the pitch circle, the angle made by that spiral at each point with its own radius vector being the complement of the given angle θ. But this mode of describing involutes of circles, being more complex than the ordinary method, is mentioned merely to show that they fall under the general description of curves described by rolling.

In fig. 198, let C_1, C_2, be the centres of two circular wheels, whose pitch circles are B_1, B_2. Through the pitch point I draw the intended *line of action* $P_1\,P_2$, making the angle $C\,I\,P = \theta$ with the line of centres. From C_1, C_2, draw

$$\left.\begin{aligned}\overline{C_1\,P_1} &= \overline{I\,C_1}\cdot\sin\theta,\\ \overline{C_2\,P_2} &= \overline{I\,C_2}\cdot\sin\theta,\end{aligned}\right\}\text{.....................}(1.)$$

perpendicular to $P_1\,P_2$, with which two perpendiculars as radii, describe circles (called *base circles*) D_1, D_2.

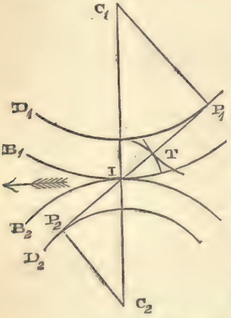

Fig. 198.

Suppose the base circles to be a pair of circular pulleys, connected by means of a cord whose course from pulley to pulley is $P_1\,I\,P_2$. As the line of connection of those pulleys is the same with that of the proposed teeth, they will rotate with the required velocity-ratio. Now suppose a tracing-point T to be fixed to the cord, so as to be carried along the path of contact $P_1\,I\,P_2$. That point will trace, on a plane rotating along with the wheel 1, part of the involute of the base circle D_1, and on a plane rotating along with the wheel 2, part of the involute of the base circle D_2, and the two curves so traced will always touch each other in the required point of contact T, and will therefore fulfil the condition required by Article 451.

All involute teeth of the same pitch work smoothly together.

To find the length of the path of contact on either side of the pitch point I, it is to be observed that the distance between the fronts of two successive teeth as measured along $P_1\,I\,P_2$, is less than the pitch in the ratio $\sin\theta : 1$, and consequently that if distances not less than the pitch $\times \sin\theta$ be marked off either way from I towards P_1 and P_2 respectively, as the extremities of the path of contact, and if the addendum circles be described through the points so found, there will always be at least two pairs of teeth in action at once. In practice, it is usual to make the path of contact somewhat longer, viz., about $2\frac{1}{4}$ times the pitch; and with this length of path and the value of θ which is usual in practice, viz., $75\frac{1}{2}^\circ$, the addendum is about $\frac{3}{10}$ of the pitch.

The teeth of a *rack*, to work correctly with wheels having involute teeth, should have plane surfaces, perpendicular to the line of connection, and consequently making, with the direction of motion of the rack, angles equal to the before-mentioned angle θ.

458. **Sliding of Involute Teeth.**—The distance through which a pair of involute teeth slide on each other, is found by observing that the distance from the point of contact of the teeth to the pitch point is given by the equation

$$r = q \cdot \frac{\overline{\mathrm{CP}}}{\overline{\mathrm{CI}}} = q \cdot \sin \theta ; \dots\dots\dots\dots\dots\dots (1.)$$

which reduces equation 3 of Article 455 to the following :—

$$s = \left(\frac{1}{r_1} + \frac{1}{r_2}\right) \cdot \frac{q_1^2 + q_2^2}{2} \cdot \sin \theta \dots\dots\dots\dots\dots\dots (2.)$$

This distance may also be expressed in terms of the extreme distances of the point of contact from the pitch point. Let these be denoted by t_1, t_2 ; then

$$t_1 = q_1 \sin \theta ;\ t_2 = q_2 \sin \theta ; \text{ and } s = \left(\frac{1}{r_1} + \frac{1}{r_2}\right) \cdot \frac{t_1^2 + t_2^2}{2 \sin \theta} .. (2\,\mathrm{A}.)$$

For *inside gearing*, the difference of the reciprocals of the radii of the wheels is to be taken instead of their sum.

The preceding formulæ, which are exact for involute teeth, are approximately correct for all teeth, if θ be taken to represent the mean value of the angle C I P between the line of centres and the line of action.

The usual value of θ being $75\frac{1}{2}°$, $\sin \theta = \frac{31}{32}$ nearly.

459. The **Addendum of Involute Teeth,** that is, their projection beyond the pitch circle, is found by considering, that for one of the wheels in fig. 198, such as the wheel 1, the *real radius*, or radius of the addendum circle, is the hypothenuse of a right-angled triangle, of which one side is the radius of the base circle $\overline{\mathrm{CP}}$, and the other is $\overline{\mathrm{PI}}$ + the portion of the path of contact beyond I. Now $\overline{\mathrm{CP}} = r_1 \cdot \sin \theta$; $\overline{\mathrm{PI}} = r_1 \,.\, \cos \theta$. Let t_2 be the portion of the path of contact above mentioned ($= q_2 \cdot \sin \theta$), and d_1 the addendum of the wheel 1 ; then

$$(r_1 + d_1)^2 = r_1^2 \cdot \sin^2 \theta + (r_1 \cos \theta + t_2)^2 ; \dots\dots\dots\dots (1.)$$

and for the wheel 2 the suffixes 1 and 2 are to be interchanged.

The usual value of $\sin \theta$ is about $\frac{31}{32}$, and that of $\cos \theta$ about $\frac{1}{4}$.

The same formulæ apply to teeth of any figure, if θ be taken to represent the *extreme* value of the angle C I P.

460. The **Smallest Pinion with Involute Teeth** of a given pitch p, has its size fixed by the consideration that the path of contact of the flanks of its teeth, which must not be less than $p \cdot \sin \theta$, cannot

be greater than the distance along the line of action from the pitch point to the base circle, $\overline{\mathrm{I\,P}} = r \cdot \cos \theta$. Hence the *least radius* is

$$r = p \tan \theta ; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

which, for $\theta = 75\frac{1}{2}°$, gives for the radius $r = 3{\cdot}867\,p$, and for the circumference of the pitch circle, $p \times 3{\cdot}867 \times 2\,\pi = 24{\cdot}3\,p$; to which the next greater integer multiple of p is $25\,p$; and therefore *twenty-five*, as formerly stated, in Article 447, is the least number of *involute teeth* to be employed in a pinion.

461. **Epicycloidal Teeth.**—For tracing the figures of teeth, the most convenient rolling curve is the circle. The path of contact which a point in its circumference traces is identical with the circle itself; the flanks of the teeth are internal, and their faces external epicycloids, for wheels; and both flanks and faces are cycloids for a rack.

Wheels of the same pitch, with epicycloidal teeth traced by the same rolling circle, all work correctly with each other, whatsoever may be the numbers of their teeth; and they are said to belong to *the same set.*

For a pitch circle of twice the radius of the rolling or *describing* circle (as it is called), the internal epicycloid is a straight line, being in fact a diameter of the pitch circle; so that the flanks of the teeth for such a pitch circle are planes radiating from the axis. For a smaller pitch circle, the flanks would be convex, and *incurved* or *under-cut,* which would be inconvenient; therefore the smallest wheel of a set should have its pitch circle of twice the radius of the describing circle, so that the flanks may be either straight or concave.

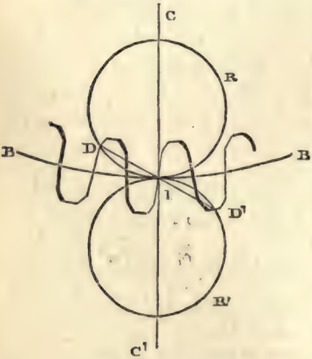

Fig. 199.

In fig. 199, let B be part of the pitch circle of a wheel, C C the line of centres, I the pitch-point, R the internal, and R′ the equal external describing circles, so placed as to touch the pitch circle and each other at I; let D I D′ be the path of contact, consisting of the path of approach D I, and the path of recess I D′. In order that there may always be at least two pairs of teeth in action, each of those arcs should be equal to the pitch.

The angle θ, on passing the line of centres, is 90°; the least value of that angle is $\theta = \angle\,\mathrm{C\,I\,D} = \angle\,\mathrm{C'\,I\,D'}$.

It appears from experience that the least value of θ should be about 60°; therefore the arcs D I = I D′ should each be one-sixth of a circumference; therefore the circumference of the describing circle should be *six times the pitch.*

It follows that the smallest pinion of a set, in which pinion the flanks are straight, should have *twelve teeth,* as has already been stated in Article 447.

462. The **Addendum for Epicycloidal Teeth** is found from the formula already given in Article 459, equation 1, by putting for θ the angle C I D, and for t_2 the chord $\overline{\mathrm{I\,D}}' = 2\, r_0 \cdot \cos \theta$, r_0 being the radius of the rolling circle. Hence

$$(r_1 + d_1)^2 = r_1^2 \;\sin^2 \theta + (r_1 + 2\, r_0)^2 \cdot \cos^2 \theta \text{..........(1.)}$$

For the usual value of θ, 60°, $\sin^2 \theta = \frac{3}{4}$, and $\cos^2 \theta = \frac{1}{4}$; whence

$$(r_1 + d_1)^2 = r_1^2 + r_1 r_0 + r_0^2 \text{..................(2.)}$$

462 A. The **Sliding of Epicycloidal Teeth** is deduced from equation 3 of Article 455, by observing, that the radius vector of the point of contact is

$$r = 2\, r_0 \cdot \sin \frac{q}{2\, r_0}, \text{........................(1.)}$$

and that the extreme values of q are the arcs of approach and recess,

$$q = q_2 = 2\, r_0 \left(\frac{\pi}{2} - \theta\right), \text{..................(2.)}$$

whence we have

$$\begin{aligned} s &= 2 \left(\frac{1}{r_1} + \frac{1}{r_2}\right) \int_0^{q_1} r \, d\, q \\ &= 8\, r_0^2 \left(\frac{1}{r_1} + \frac{1}{r_2}\right) \int_0^{\frac{\pi}{2} - \theta} \sin \frac{q}{2\, r_0} \cdot d\, \frac{q}{2\, r_0} \\ &= 8\, (1 - \sin\, \theta)\, r_0^2 \left(\frac{1}{r_1} + \frac{1}{r_2}\right); \text{................(3.)} \end{aligned}$$

which, for $\theta = 60°$, has the value

$$s = 1{\cdot}07\, r_0^2 \left(\frac{1}{r_1} + \frac{1}{r_2}\right) \text{..................(3 A.)}$$

463. **Approximate Epicycloidal Teeth.**—Mr. Willis has shown how to approximate to the figure of an epicycloidal tooth by means of two circular arcs, one concave, for the flank, the other convex, for the face, and each having for its radius, the *mean* radius of curvature of the epicycloidal arc. Mr. Willis's formulæ are deduced in his own work from certain propositions respecting the transmission of motion by linkwork. In the present treatise they will be deduced from the values already given for the radii of curvature of

epicycloids in Article 390, case 1, equation 4: viz., let r_1 be the radius of the pitch circle, r_0 that of the rolling circle, ϱ the radius of curvature required; then

$$\varrho = 2\, r_0 \cdot \cos\theta \frac{r_1 \pm r_0}{\frac{r_1}{2} \pm r_0} = 4\, r_0 \cdot \cos\theta \, \frac{r_1 \pm r_0}{r_1 \pm 2\, r_0}; \ldots\ldots (1.)$$

the sign + applying to an *external epicycloid*, that is, to the *face* of a tooth, and the sign – to an *internal epicycloid*, that is, to the *flank* of a tooth.

To find the distances of the centres of curvature of the given point in an epicycloid from the point of contact I of the pitch circle and rolling circle, there is to be subtracted from the radius of curvature, the instantaneous radius vector, $r = 2\, r_0 \cos\theta$; that is to say,

$$\varrho - r = 2\, r_0 \cos\theta \cdot \frac{r_1}{r_1 \pm 2\, r_0} \ldots\ldots\ldots\ldots\ldots\ldots (2.)$$

The value to be assumed for θ is its mean value, that is, $75\frac{1}{2}°$; and $\cos\theta = \frac{1}{4}$ nearly: r_0 is nearly equal to the pitch, p; and if n be the number of teeth in the wheel,

$$6 : n :: r_0 : r_1.$$

Therefore, for the proportions approved of by Mr. Willis, equation 2 becomes

$$\varrho - r = \frac{p}{2} \cdot \frac{n}{n \pm 12}; \ldots\ldots\ldots\ldots\ldots\ldots (3.)$$

+ being used for the face, and – for the flank; also

$$r = \frac{p}{2} \text{ nearly} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4.)$$

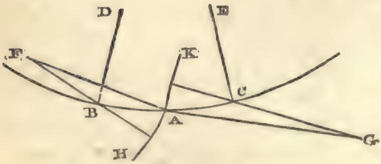

Fig. 200.

Hence the following construction. In fig. 200, let B C be part of the pitch circle, A the point where a tooth is to cross it. Set off $AB = AC = \frac{p}{2}$. Draw radii of the pitch circle, D B, E C. Draw F B, C G, making angles of $75\frac{1}{2}°$ with those radii, in which take

$$\overline{BF} = \frac{p}{2} \cdot \frac{n}{n + 12}; \ \overline{CG} = \frac{p}{2} \cdot \frac{n}{n - 12} \ldots\ldots\ldots (5.)$$

Round F, with the radius F A, draw the circular arc A H; this will be the face of the tooth. Round G, with the radius G A, draw the circular arc G K; this will be the flank of the tooth.

To facilitate the application of this rule, Mr. Willis has published tables of the values of $\varrho - r$, and invented an instrument called the "*odontograph.*"

464. **Teeth of Wheel and Trundle.**—A *trundle*, as in fig. 201, has cylindrical pins called *staves* for teeth. The face of the teeth of a wheel suitable for driving it, in outside gearing, are described by first tracing external epicycloids by rolling the pitch circle B_2 of the trundle on the pitch circle B_1 of the driving wheel, with the centre of a stave for a tracing-point, as shown by the dotted lines, and then drawing curves parallel to and within the epicycloids, at a distance from them equal to the radius of a stave. Trundles having only six staves will work with large wheels.

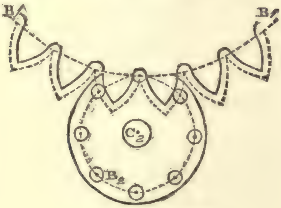

Fig. 201.

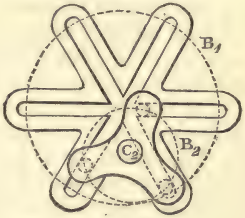

Fig. 202.

To drive a trundle in *inside gearing*, the outlines of the teeth of the wheel should be curves parallel to internal epicycloids. A peculiar case of this is represented in fig. 202, where the radius of the pitch circle of the trundle is exactly one-half of that of the pitch circle of the wheel; the trundle has three equi-distant staves; and the internal epicycloids described by their centres while the pitch circle of the trundle is rolling within that of the wheel, are three straight lines, diameters of the wheel, making angles of 60° with each other. Hence the surfaces of the teeth of the wheel form three straight grooves intersecting each other at the centre, each being of a breadth equal to the diameter of a stave of the trundle.

465. **Dimensions of Teeth.**—Toothed wheels being in general intended to rotate either way, the *backs* of the teeth are made similar to the fronts. The *space* between two teeth, measured on the pitch circle, is made about one-fifth part wider than the thickness of the tooth on the pitch circle; that is to say,

$$\text{thickness of tooth} = \frac{5}{11} \text{ pitch},$$

$$\text{width of space} = \frac{6}{11} \text{ pitch.}$$

The difference of $\frac{1}{11}$ of the pitch is called the *back-lash.*

The clearance allowed between the points of teeth and the bottoms of the spaces between the teeth of the other wheel, is about one-tenth of the pitch.

The *thickness* of a tooth is fixed according to the principles already stated in Article 326; and the *breadth* is so adjusted, that when multiplied by the pitch, the product shall contain *one square inch* for each 160 lbs. of force transmitted by the teeth.

466. **Mr. Sang's Process.**—Mr. Sang has published an elaborate work on the teeth of wheels, in which a process is followed differing in some respects from any of those before described. A form is selected for the path of the point of contact of the teeth, and from that form the figures of the teeth are deduced. For details, the reader is referred to Mr. Sang's work.

467. **The Teeth of a Bevel-Wheel** have acting surfaces of the conical kind, generated by the motion of a line traversing the apex of the conical pitch surface, while a point in it is carried round the outlines of the cross section of the teeth made by a sphere described about that apex.

The operations of describing the exact figures of the teeth of bevel-wheels, whether by involutes or by rolling curves, are in every respect analogous to those for describing the figures of the teeth of spur-wheels, except that in the case of bevel-wheels, all those operations are to be performed on the surface of a sphere described about the apex, instead of on a plane, substituting *poles* for *centres,* and *great circles* for *straight lines.*

In consideration of the practical difficulty, especially in the case of large wheels, of obtaining an accurate spherical surface, and of drawing upon it when obtained, the following *approximate* method, proposed originally by Tredgold, is generally used:—Let O, fig. 203, be the apex, and O C the axis of the pitch cone of a bevel-wheel; and let the largest pitch circle be that whose radius is C B. Perpendicular to O B draw B A cutting the axis produced in A, let the outer rim of the pattern and of the wheel be made a portion of the surface of the cone whose apex is A and side A B. The narrow zone of that cone thus employed will approach sufficiently near to a zone of the sphere described about O with the radius O B, to be used in its stead. On

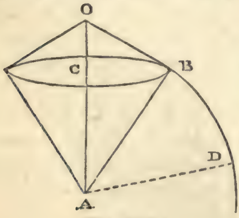

Fig. 203.

a plane surface, with the radius A B, draw a circular arc B D; a sector of that circle will represent a portion of the surface of the cone A B C *developed*, or *spread out flat*. Describe the figures of teeth of the required pitch, suited to the pitch circle B D, as if it were that of a spur-wheel of the radius A B; those figures will be the required cross sections of the teeth of the bevel-wheel, made by the conical zone whose apex is A.

468. **Teeth of Skew-Bevel Wheels.**—The cross sections of the teeth of a skew-bevel wheel at a given pitch circle are similar to those of a bevel wheel whose pitch surface is a cone touching the hyperboloidal pitch surface of the skew-bevel wheel at the given pitch circle; and the surfaces of the teeth of the skew-bevel wheel are generated by a straight line which moves round the outlines of the cross section and at the same time is kept always in the position of the generating line of a hyperboloïdal surface similar to the pitch-surface (see Article 444, pages 430, 431).

469. **The Teeth of Non-Circular Wheels** are described by rolling circles or other curves on the pitch surfaces, like the teeth of circular wheels; and when they are small compared with the wheels to which they belong, each tooth is nearly similar to the tooth of a circular wheel having the same radius of curvature with the pitch surface of the actual wheel at the point where the tooth is situated.

470. A **Cam** or **Wiper** is a single tooth, either rotating continuously or oscillating, and driving a sliding or turning piece, either constantly or at intervals. All the principles which have been stated in Article 450, as being applicable to sliding contact, are applicable to cams; but in designing cams, it is not usual to determine or take into consideration the form of the ideal pitch surface which would give the same comparative motion by rolling contact that the cam gives by sliding contact.

471. **Screws. Pitch.**—The figure of a screw is that of a convex or concave cylinder with one or more helical projections called *threads* winding round it. Convex and concave screws are distinguished technically by the respective names of *male* and *female*, or *external* and *internal;* a short internal screw is called a *nut;* and when a *screw* is not otherwise specified, *external* is understood.

The relation between the *advance* and the *rotation*, which compose the motion of a screw working in contact with a fixed nut or helical guide, has already been demonstrated in Article 382, equation 1; and the same relation exists between the rotation of a screw about an axis fixed longitudinally relatively to the framework, and the advance of a nut in which that screw rotates, the nut being free to shift longitudinally, but not to turn. The advance of the nut in the latter case is in the direction opposite to that of the advance of the screw in the former case.

A screw is called *right-handed* or *left-handed*, according as its advance in a fixed nut is accompanied by right-handed or left-handed rotation, when viewed by an observer *from* whom the advance takes place. Fig. 204 represents a right-handed screw, and fig. 205 a left-handed screw.

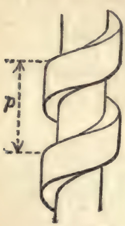

Fig. 204.

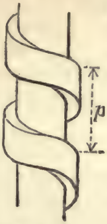

Fig. 205.

The *pitch* of a screw of one thread, and the *total pitch* of a screw of any number of threads, is the pitch of the helical motion of that screw, as explained in Article 382, and is the distance (marked p in figs. 204 and 205) measured parallel to the axis of the screw, between the corresponding points in two consecutive turns *of the same thread.*

In a screw of two or more threads, the distance measured parallel to the axis, between the corresponding points in *two adjacent threads,* may be called the *divided pitch.*

472. **Normal and Circular Pitch.**—When the pitch of a screw is not otherwise specified, it is always understood to be measured parallel to the axis. But it is sometimes convenient for particular purposes to measure it in other directions; and for that purpose a *cylindrical pitch surface* is to be conceived as described about the axis of the screw, intermediate between the crests of the threads and the bottoms of the grooves between them.

If a helix be now described upon the pitch cylinder, so as to cross each turn of each thread at right angles, the distance between two corresponding points on two successive turns of the same thread, measured along this *normal helix,* may be called the *normal pitch;* and when the screw has more than one thread, the normal pitch from thread to thread may be called the *normal divided pitch.*

The distance from thread to thread measured on a circle described on the pitch cylinder, and called the *pitch circle,* may be called the *circular pitch;* for a screw of one thread it is one circumference; for a screw of n threads

$$\frac{\text{one circumference}}{n}.$$

The following set of formulæ show the relations amongst the different modes of measuring the pitch of a screw. The *pitch,* properly speaking, as originally defined, is distinguished as the *axial pitch,* and is the same for all parts of the same screw: the normal and circular pitch depend on the radius of the pitch cylinder.

Let r denote the radius of the pitch cylinder;

n, the number of threads;

i, the obliquity of the threads to the pitch circles, and of the normal helix to the axis;

$\left.\begin{matrix} \mathrm{P}_a \\ \frac{\mathrm{P}_a}{n} = p_a \end{matrix}\right\}$ the axial $\left\{\begin{matrix}\text{pitch;} \\ \text{divided pitch;}\end{matrix}\right.$

$\left.\begin{matrix} \mathrm{P}_n \\ \frac{\mathrm{P}_n}{n} = p_n \end{matrix}\right\}$ the normal $\left\{\begin{matrix}\text{pitch;} \\ \text{divided pitch;}\end{matrix}\right.$

p_c, the circular pitch;

Then

$$p_c = p_a \cdot \operatorname{cotan} i = p_n \cdot \operatorname{cosec} i = \frac{2 \pi r}{n};$$

$$p_a = p_n \cdot \sec i = p_c \cdot \tan i = \frac{2 \pi r \cdot \tan i}{n};$$

$$p_n = p_c \cdot \sin i = p_a \cdot \cos i = \frac{2 \pi r \cdot \sin i}{n}.$$

473. **Screw Gearing.**—A pair of convex screws, each rotating about its axis, are used as an elementary combination, to transmit motion by the sliding contact of their threads. Such screws are commonly called *endless screws.* At the point of contact of the screws, their threads must be parallel; and their line of connection is the common perpendicular to the acting surfaces of the threads at their point of contact. Hence the following principles:—

I. If the screws are both right-handed or both left-handed, the angle between the directions of their axes is the sum of their obliquities:—if one is right-handed and the other left-handed, that angle is the difference of their obliquities.

II. The normal pitch, for a screw of one thread, and the normal divided pitch, for a screw of more than one thread, must be the same in each screw.

III. The angular velocities of the screws are inversely as their number of threads.

474. **Hooke's Gearing** is a case of screw gearing, in which the axes of the screws are parallel, one screw being right-handed and the other left-handed, and in which, from the shortness and great diameter of the screws, and their large number of threads, they are in fact *wheels,* with teeth whose crests, instead of being parallel to the line of contact of the pitch cylinders, cross it obliquely, so as to be of a screw-like or helical form. In wheelwork of this kind, the contact of each pair of teeth commences at the foremost end of

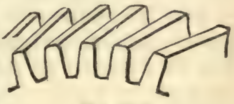

Fig. 206.

the helical front and terminates at the aftermost end; and the helix is of such a pitch that the contact of one pair of teeth does not terminate until that of the next pair has commenced. The object of this is to increase the smoothness of motion.

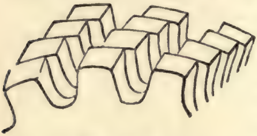

Fig. 207.

With the same object, Dr. Hooke invented the making of the fronts of teeth in a series of steps. A wheel thus formed resembles in shape a series of equal and similar toothed discs placed side by side, with the teeth of each a little behind those of the preceding disc. In such a wheel, let p be the circular pitch, and n the number of steps. Then the arc of contact, the addendum, and the extent of sliding, are those due to the smaller pitch $\frac{p}{n}$, while the strength of the teeth is that due to the thickness corresponding to the entire pitch p; so that the smooth action of small teeth and the strength of large teeth are combined. Stepped teeth being more expensive and difficult to execute than common teeth, are used for special purposes only.

475. **The Wheel and Screw** is an elementary combination of two screws, whose axes are at right angles to each other, both being right-handed or both left-handed. As the usual object of this combination is to produce a change of angular velocity in a ratio greater than can be obtained by any single pair of ordinary wheels, one of the screws is commonly wheel-like, being of large diameter and many-threaded, while the other is short and of few threads; and the angular velocities are inversely as the number of threads.

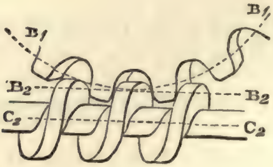

Fig. 208.

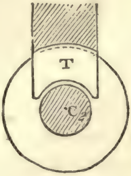

Fig. 209.

Fig. 208 represents a side view of this combination, and fig. 209 a cross section at right angles to the axis of the smaller screw. It has been shown by Mr. Willis, that if each section of both screws be made by a plane perpendicular to the axis of the large screw or wheel, the outlines of the threads of the larger and smaller screw should be those of the teeth of a wheel and rack respectively: $B_1 B_1$, in fig. 208, for example, being the pitch circle of the wheel, and $B_2 B_2$ the pitch line of the rack.

The periphery and teeth of the wheel are usually hollowed to fit the screw, as shown at T, fig. 209.

To make the teeth or threads of a pair of screws fit correctly and work smoothly, a hardened steel screw is made of the figure of the smaller screw, with its thread or threads notched so as to form a cutting tool; the larger screw, or wheel, is cast approximately of the required figure; the larger screw and the steel screw are fitted up in their proper relative position, and made to rotate in contact with each other by turning the steel screw, which cuts the threads of the larger screw to their true figure.

476. **The Relative Sliding of a Pair of Screws** at their point of contact is found thus:—Let r_1, r_2, be the radii of their pitch cylinders, and i_1, i_2, the obliquities of their threads to their pitch circles, one of which is to be considered as negative if the screws are contrary-handed. Let u be the common component of the velocities of a pair of points of contact along a line touching the pitch surfaces and perpendicular to the threads, at the pitch point, and v the velocity of sliding of the threads over each other. Then

so that

$$\left.\begin{array}{l} u = a_1 r_1 \cdot \sin i_1 = a_2 r_2 \cdot \sin i_2 ; \\ a_1 = \dfrac{u}{r_1 \cdot \sin i_1} ; \ a_2 = \dfrac{u}{r_2 \cdot \sin i_2} ; \end{array}\right\} \dots\dots\dots\dots\dots (1.)$$

and

$$v = a_1 r_1 \cdot \cos i_1 + a_2 r_2 \cdot \cos i_2 = u\,(\text{cotan}\, i_1 + \text{cotan}\, i_2) \dots\dots (2.)$$

When the screws are contrary-handed, the difference instead of the sum of the terms in equation 2 is to be taken.

477. **Oldham's Coupling.**—A *coupling* is a mode of connecting a pair of shafts so that they shall rotate in the same direction, with the same mean angular velocity. If the axes of the shafts are in the same straight line, the coupling consists in so connecting their contiguous ends that they shall rotate as one piece; but if the axes are not in the same straight line, combinations of mechanism are required. A coupling for parallel shafts which acts by *sliding contact* was invented by Oldham, and is represented in fig. 210.

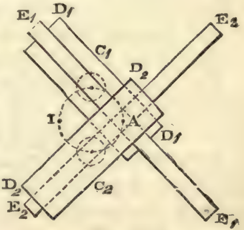

Fig. 210.

C_1, C_2, are the axes of the two parallel shafts; D_1, D_2, two cross-heads, facing each other, fixed on the ends of the two shafts respectively; E_1, E_1, a bar, sliding in a diametral groove in the face of

D_1; E_2, E_2, a bar, sliding in a diametral groove in the face of D_2; those bars are fixed together at A, so as to form a rigid cross. The angular velocities of the two shafts and of the cross are all equal at every instant. The middle point of the cross, at A, revolves in the dotted circle described upon the line of centres C_1 C_2, as a diameter, twice for each turn of the shafts and cross; the instantaneous axis of rotation of the cross, at any instant, is at I, the point in the circle C_1 C_2, diametrically opposite to A.

Oldham's coupling may be used with advantage where the axes of the shafts are intended to be as nearly in the same straight line as is possible, but where there is some doubt as to the practicability or permanency of their exact continuity.

SECTION 3.—*Connection by Bands.*

478. **Bands Classed.**—Bands, or wrapping connectors, for communicating motion between pulleys or drums rotating about fixed axes, or between rotating pulleys and drums and shifting pieces, may be thus classed :—

I. *Belts*, which are made of leather or of gutta percha, are flat and thin, and require nearly cylindrical pulleys. A belt tends to move towards that part of a pulley whose radius is greatest; pulleys for belts, therefore, are slightly swelled in the middle, in order that the belt may remain on the pulley unless forcibly shifted. A belt when in motion is shifted off a pulley, or from one pulley on to another of equal size alongside of it, by pressing against that part of the belt which is moving *towards* the pulley.

II. *Cords*, made of catgut, hempen or other fibres, or wire, are nearly cylindrical in section, and require either drums with ledges, or grooved pulleys.

III. *Chains*, which are composed of links or bars jointed together, require pulleys or drums, grooved, notched, and toothed, so as to fit the links of the chains.

Bands for communicating continuous motion are *endless*.

Bands for communicating reciprocating motion have usually their ends made fast to the pulleys or drums which they connect, and which in this case may be sectors.

479. **Principle of Connection by Bands.**—The *line of connection* of a pair of pulleys or drums connected by means of a band, is the central line or axis of that part of the band whose tension transmits the motion. The principle of Article 433 being applied to this case, leads to the following consequences :—

I. *For a pair of rotating pieces*, let r_1, r_2, be the perpendiculars let fall from their axes on the centre line of the band, a_1, a_2, their angular velocities, and i_1, i_2, the angles which the centre line of the band makes with the two axes respectively. Then the longitudinal velocity of the band, that is, its component velocity in the direction of its own centre line, is

$$u = r_1\, a_1 \sin i_1 = r_2\, a_2 \sin i_2 \,; \ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

whence the angular velocity-ratio is

$$\frac{a_2}{a_1} = \frac{r_1 \sin i_1}{r_2 \sin i_2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.)$$

When the axes are parallel (which is almost always the case), $i_1 = i_2$, and

$$\frac{a_2}{a_1} = \frac{r_1}{r^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.)$$

The same equation holds when both axes, whether parallel or not, are perpendicular in direction to that part of the band which transmits the motion; for then $\sin i_1 = \sin i_2 = 1$.

II. *For a rotating piece and a sliding piece*, let r be the perpendicular from the axis of the rotating piece on the centre line of the band, a the angular velocity, i the angle between the directions of the band and axis, u the longitudinal velocity of the band, j the angle between the direction of the centre line of the band and that of the motion of the sliding piece, and v the velocity of the sliding piece; then

$$u = r\, a \sin i = v \cos j\,; \text{ and} \ldots\ldots\ldots\ldots\ldots\ldots (4.)$$

$$v = \frac{r\, a \sin i}{\cos j} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5.)$$

When the centre line of the band is parallel to the direction of motion of the sliding piece, and perpendicular to the direction of the axis of the rotating piece, $\sin i = \cos j = 1$, and

$$v = u = r\, a \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.)$$

480. The **Pitch Surface of a Pulley or Drum** is a surface to which the line of connection is always a tangent; that is to say, it is a surface parallel to the acting surface of the pulley or drum, and distant from it by half the thickness of the band.

481. **Circular Pulleys and Drums** are used to communicate a constant velocity-ratio. In each of them, the length denoted by r

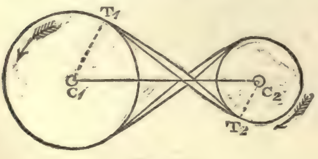

Fig. 211.

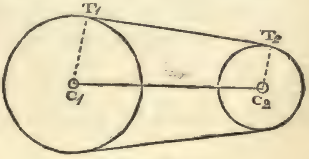

Fig. 212.

in the equations of Article 479 is constant, and is called the *effective radius*, being equal to the real radius of the pulley or drum added to half the thickness of the band.

A *crossed belt* connecting a pair of circular pulleys, as in fig. 211, reverses the direction of rotation; an *open belt*, as in fig. 212, preserves that direction.

482. **The Length of an Endless Belt,** connecting a pair of pulleys whose effective radii are $\overline{\mathrm{C}_1\mathrm{T}_1} = r_1$, $\overline{\mathrm{C}_2\mathrm{T}_2} = r_2$, with parallel axes whose distance apart is $\overline{\mathrm{C}_1\mathrm{C}_2} = c$, is given by formulæ founded on equation 1 of Article 402, viz.,—$\mathrm{L} = \Sigma \cdot s + \Sigma \cdot r\, i$. Each of the two equal straight parts of the belt is evidently of the length

$$\left.\begin{array}{l} s = \sqrt{c^2 - (r_1 + r_2)^2} \text{ for a crossed belt;} \\ s = \sqrt{c^2 - (r_1 - r_2)^2} \text{ for an open belt;} \end{array}\right\} \dots\dots\dots (1.)$$

r_1 being the greater radius, and r_2 the less. Let i_1 be the arc to radius unity of the greater pulley, and i_2 that of the less pulley, with which the belt is in contact; then for a crossed belt

$$\left.\begin{array}{l} i_1 = i_2 = \left(\pi + 2 \text{ arc} \cdot \sin \dfrac{r_1 + r_2}{c}\right); \\ \text{and for an open belt,} \\ i_1 = \left(\pi + 2 \text{ arc} . \sin \dfrac{r_1 - r_2}{c}\right);\ i_2 = \left(\pi - 2 \text{ arc} \cdot \sin \dfrac{r_1 - r_2}{c}\right); \end{array}\right\} (2.)$$

and the introduction of those values into equation 1 of Article 402 gives the following results:—

For a crossed belt,

$$\left.\begin{array}{l} \mathrm{L} = 2\sqrt{c^2 - (r_1 + r_2)^2} + (r_1 + r_2) \cdot \left(\pi + 2 \text{ arc} \cdot \sin \cdot \dfrac{r_1 + r_2}{c}\right); \\ \text{and for an open belt,} \\ \mathrm{L} = 2\sqrt{c^2 - (r_1 - r_2)^2} + \pi\,(r_1 + r_2) + 2\,(r_1 - r_2) \cdot \text{arc} \cdot \sin \cdot \dfrac{r_1 - r_2}{c}. \end{array}\right\} (3.)$$

As the last of these equations would be troublesome to employ in a practical application to be mentioned in the next Article, an approximation to it, sufficiently close for practical purposes, is obtained by considering, that if $r - r_2$ is small compared with c, $\sqrt{c^2 - (r_1 - r_2)^2} = c - \dfrac{(r_1 - r_2)^2}{2\,c}$ nearly, and $\text{arc} \cdot \sin \cdot \dfrac{r_1 - r_2}{c} = \dfrac{r_1 - r_2}{c}$ nearly; whence, for an open belt,

$$\mathrm{L} \text{ nearly} = 2\,c + \pi\,(r_1 + r_2) + \frac{(r_1 - r_2)^2}{c} \dots\dots\dots (3\ \text{A.})$$

483. Speed-Cones (figs. 213, 214, 215, 216) are a contrivance for varying and adjusting the velocity-ratio communicated between a pair of parallel shafts by means of a belt, and may be either continuous cones or conoids, as in figs. 213, 214, whose velocity-ratio can be varied gradually while they are in motion by shifting the belt; or sets of pulleys whose radii vary by steps, as in figs. 215, 216, in which case the velocity-ratio can be changed by shifting the belt from one pair of pulleys to another.

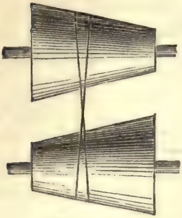
Fig. 213.

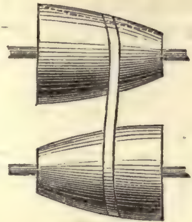
Fig. 214.

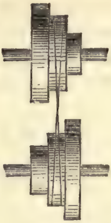
Fig. 215.

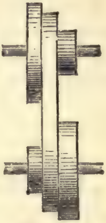
Fig. 216.

In order that the belt may be equally tight in every possible position on a pair of speed-cones, the quantity L in the equations of Article 482 must be constant.

For a *crossed* belt, as in figs. 213 and 215, L depends solely on c and on $r_1 + r_2$. Now c is constant, because the axes are parallel, therefore the *sum of the radii* of the pitch circles connected in every position of the belt is to be constant. That condition is fulfilled by a pair of continuous cones generated by the revolution of two straight lines inclined opposite ways to their respective axes at equal angles, and by a set of pairs of pulleys in which the sum of the radii is the same for each pair.

For an *open* belt, the following practical rule is deduced from the approximate equation 3 A of Article 482 :—

Let the speed-cones be equal and similar conoids, as in fig. 214, but with their large and small ends turned opposite ways. Let r_1 be the radius of the large end of each, r_2 that of the small end, r_0 that of the middle; and let y be the *sagitta*, measured perpendicular to the axis, of the arc by whose revolution each of the conoids is generated, or, in other words, the *bulging* of the conoids in the middle of their length; then

$$y = r_0 - \frac{r_1 + r_2}{2} = \frac{(r_1 - r_2)^2}{2\pi c} \dots\dots\dots\dots\dots\dots (1.)$$

$2\pi = 6{\cdot}2832$; but 6 may be used in most practical cases without sensible error.

The radii at the middle and ends being thus determined, make the generating curve an arc either of a circle or of a parabola.

For a pair of stepped cones, as in fig. 216, let a series of *differences* of the radii, or values of $r_1 - r_2$, be assumed; then for each pair of pulleys, the sum of the radii is to be computed from the difference by the formula

$$r_1 + r_2 = 2\, r_0 - \frac{(r_1 - r_2)^2}{\pi\, c}; \text{..................} (2.)$$

$2\, r_0$ being that sum when the radii are equal.

SECTION 4.—*Linkwork.*

484. **Definitions.**—The pieces which are connected by linkwork, if they rotate or oscillate, are usually called *cranks, beams,* and *levers.* The *link* by which they are connected is a rigid bar, which may be straight or of any other figure; the straight figure being the most favourable to strength, is used when there is no special reason to the contrary. The link is known by various names under various circumstances, such as *coupling rod, connecting rod, crank rod, eccentric rod,* &c. It is attached to the pieces which it connects by two pins, about which it is free to turn. The effect of the link is to maintain the distance between the centres of those pins invariable; hence the line joining the centres of the pins is *the line of connection;* and those centres may be called the *connected points.* In a turning piece, the perpendicular let fall from its connected point upon its axis of rotation is the *arm* or *crank arm.*

485. **Principles of Connection.**—The whole of the equations already given in Article 479 for bands, are applicable to linkwork. The axes of rotation of a pair of turning pieces connected by a link are almost always parallel, and perpendicular to the line of connection; in which case the angular velocity-ratio at any instant is the reciprocal of the ratio of the common perpendiculars let fall from the line of connection upon the respective axes of rotation (Article 479, equation 3).

486. **Dead Points.**—If at any instant the direction of one of the crank arms coincides with the line of connection, the common perpendicular of the line of connection and the axis of that crank arm vanishes, and the directional relation of the motions becomes indeterminate. The position of the connected point of the crank arm in question at such an instant is called a *dead point.* The velocity of the other connected point at such an instant is null, unless it also reaches a dead point at the same instant, so that the line of connection is in the plane of the two axes of rotation, in which case the velocity-ratio is indeterminate.

487. **Coupling of Parallel Axes.**—The only case in which an uniform angular velocity-ratio (being that of equality) is communicated by linkwork, is that in which two or more parallel shafts (such as those of the driving wheels of a locomotive engine) are made to rotate with constantly equal angular velocities, by having equal cranks, which are maintained parallel by a coupling rod of such a length that the line of connection is equal to the distance between the axes. The cranks pass their dead points simultaneously. To obviate the unsteadiness of motion which this tends to cause, the shafts are provided with a second set of cranks at right angles to the first, connected by means of a similar coupling rod, so that one set of cranks pass their dead points at the instant when the other set are farthest from theirs.

488. **The Comparative Motion of the Connected Points** in a piece of linkwork at a given instant is capable of determination by the method explained in Article 384; that is, by finding the instantaneous axis of the link; for the two connected points move in the same manner with two points in the link, considered as a rigid body.

If a connected point belongs to a turning piece, the direction of its motion at a given instant is perpendicular to the plane containing the axis and crank arm of the piece. If a connected point belongs to a shifting piece, the direction of its motion at any instant is given, and a plane can be drawn perpendicular to that direction.

The line of intersection of the planes perpendicular to the paths of the two connected points at a given instant, is the *instantaneous axis of the link* at that instant; and the *velocities of the connected points are directly as their distances from that axis.*

In drawing on a plane surface, the two planes perpendicular to the paths of the connected points are represented by two lines (being their sections by a plane normal to them), and the instantaneous axis by a point; and should the length of the two lines render it impracticable to produce them until they actually intersect, the velocity-ratio of the connected points may be found by the principle, that it is equal to the ratio of the segments which a line parallel to the line of connection cuts off from any two lines drawn from a given point, perpendicular respectively to the paths of the connected points.

Example I. *Two Rotating Pieces with Parallel Axes* (fig. 217).—Let C_1, C_2, be the parallel axes of the pieces; T_1, T_2, their connected points; $\overline{C_1 T_1}$, $\overline{C_2 T_2}$, their crank arms; $\overline{T_1 T_2}$, the link. At a given instant, let v_1 be the velocity of T_1; v_2 that of T_2.

To find the ratio of those velocities, produce $C_1 T_1$, $C_2 T_2$, till they intersect in K; K is the instantaneous axis of the link or connecting rod, and the velocity-ratio is

$$v_1 : v_2 :: \overline{\mathrm{K\,T_1}} : \overline{\mathrm{K\,T_2}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

Should K be inconveniently far off, draw any triangle with its sides respectively parallel to C_1T_1, C_2T_2, and T_1T_2; the ratio of the two sides first mentioned will be the velocity-ratio required. For example, draw C_2A parallel to C_1T_1, cutting T_1T_2 in A; then

$$v_1 : v_2 :: \overline{\mathrm{C_2\,A}} : \overline{\mathrm{C_2\,T_2}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.)$$

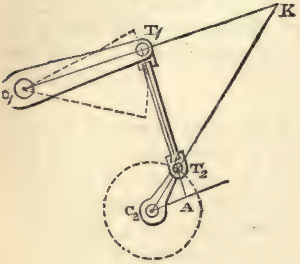

Fig. 217.

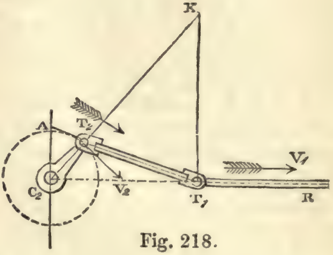

Fig. 218.

Example II. *Rotating Piece and Sliding Piece* (fig. 218).—Let C_2 be the axis of a rotating piece, and T_1R the straight line along which a sliding piece moves. Let T_1, T_2, be the connected points, $\overline{C_2T_2}$ the crank arm of the rotating piece, and $\overline{T_1T_2}$ the link or connecting rod. The points T_1, T_2, and the line T_1R, are supposed to be in one plane, perpendicular to the axis C. Draw T_1K perpendicular to T_1R, intersecting C_2T_2 in K; K is the instantaneous axis of the link; and the rest of the solution is the same as in Example I.

489. An **Eccentric** (fig. 219) being a circular disc keyed on a shaft, with whose axis its centre does not coincide, and used to give a reciprocating motion to a rod, is equivalent to a crank whose connected point is T, the centre of the eccentric disc, and whose crank arm is $\overline{CT}$, the distance of that point from the axis of the shaft, called the *eccentricity*.

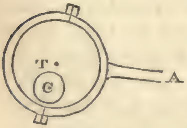

Fig. 219.

An eccentric may be made capable of having its eccentricity altered by means of an adjusting screw, so as to vary the extent of the reciprocating motion which it communicates, and which is called the *throw*, or *travel*, or *length of stroke.*

490. **The Length of Stroke** of a point in a reciprocating piece is the distance between the two ends of the path in which that point moves. When it is connected by a link with a point in a con-

tinuously rotating piece, the ends of the stroke of the reciprocating point correspond with the *dead points* of the continuously revolving piece (Article 486).

Let S be the length of stroke of the reciprocating piece, L the length of the line of connection, and R the crank arm of the continuously turning piece. Then if the two ends of the stroke be in one straight line with the axis of the crank,

$$S = 2\,R\,; \qquad \text{(1.)}$$

and if their ends be not in one straight line with that axis, then S, L − R, and L + R, are the three sides of a triangle, having the angle opposite S at that axis; so that if θ be the supplement of the arc between the dead points,

$$\left.\begin{aligned} S^2 &= 2\,(L^2 + R^2) - 2\,(L^2 - R^2)\cos\theta\,; \\ \cos\theta &= \frac{2\,L^2 + 2\,R^2 - S^2}{2\,(L^2 - R^2)}. \end{aligned}\right\} \qquad \text{(2.)}$$

491. **Hooke's Universal Joint** (fig. 220) is a contrivance for coupling shafts whose axes intersect each other in a point.

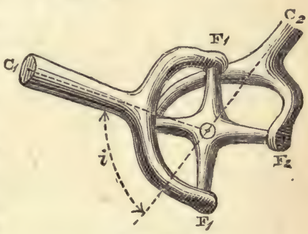

Fig. 220.

Let O be the point of intersection of the axes O C_1, O C_2, and i their angle of inclination to each other. The pair of shafts C_1, C_2, terminate in a pair of forks F_1, F_2, in bearings at the extremities of which turn the gudgeons at the ends of the arms of a rectangular cross having its centre at O. This cross is the link; the connected points are the centres of the bearings F_1, F_2. At each instant each of those points moves at right angles to the central plane of its shaft and fork, therefore the line of intersection of the central planes of the two forks, at any instant, is the instantaneous axis of the cross, and the *velocity-ratio* of the points F_1, F_2 (which, as the forks are equal, is also the *angular velocity-ratio* of the shafts), is equal to the ratio of the distances of those points from that instantaneous axis. The *mean* value of that velocity-ratio is that of equality; for each successive *quarter turn* is made by both shafts in the same time; but its actual value fluctuates between the limits,

$$\left.\begin{aligned} &\frac{a^2}{a_1} = \frac{1}{\cos i} \text{ when } F_1 \text{ is in the plane of the axes;} \\ &\frac{a_2}{a_1} = \cos i \text{ when } F_2 \text{ is in that plane.} \end{aligned}\right\} \quad \text{...(1.)}$$

Its value at intermediate instants, as well as the relation between the positions of the shafts, are given by the following equations :—Let φ_1, φ_2, be the angles respectively made by the central planes of the forks and shafts with the plane of the two axes at a given instant; then

$$\left.\begin{array}{l} \tan\varphi_1 \cdot \tan\varphi_2 = \cos i; \\ \dfrac{a_2}{a_1} = -\dfrac{d\,\varphi_2}{d\,\varphi_1} = \dfrac{\tan\varphi_1 + \operatorname{cotan}\varphi_1}{\tan\varphi_2 + \operatorname{cotan}\varphi_2}. \end{array}\right\} \dots\dots\dots\dots (2.)$$

492. The **Double Hooke's Joint** (fig. 221) is used to obviate the vibratory and unsteady motion caused by the fluctuation of the velocity-ratio indicated in the equations of Article 491. Between the two shafts to be connected, C_1, C_3, there is introduced a short intermediate shaft C_2, making equal angles with C_1 and C_3, connected with each of them by a Hooke's joint, and having both its own forks in the same plane.

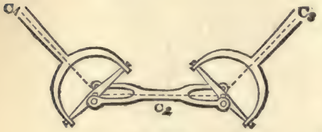

Fig. 221.

Let i be the angle of inclination of C_1 and C_2, and also that of C_2 and C_3. Let φ_1, φ_2, φ_3, be the angles made at a given instant by the planes of the forks of the three shafts with the plane of their axes, and let a_1, a_2, a_3, be their angular velocities. Then

$$\tan\varphi_2 \cdot \tan\varphi_3 = \cos i = \tan\varphi_1 \cdot \tan\varphi_2;$$

whence $$\tan\varphi_3 = \tan\varphi_1; \text{ and } a_3 = a_1;$$

so that the angular velocities of the first and third shafts are equal to each other at every instant.

493. **A Click,** being a reciprocating bar, acting upon a ratchet wheel or rack, which it pushes or pulls through a certain arc at each forward stroke, and leaves at rest at each backward stroke, is an example of intermittent linkwork. During the forward stroke, the action of the click is governed by the principles of linkwork; during the backward stroke, that action ceases. A *catch* or *pall*, turning on a fixed axis, prevents the ratchet wheel or rack from reversing its motion.

SECTION 5.—*Reduplication of Cords.*

494. **Definitions.**—The combination of pieces connected by the several plies of a cord or rope consists of a pair of cases or frames called *blocks*, each containing one or more pulleys called *sheaves.* One of the blocks called the *fall-block,* B_1, is fixed; the other, or

running-block, B_2, is moveable to or from the fall-block, with which it is connected by means of a rope of which one end is attached either to the fall-block or to the running-block, while the other end, T_1, called the *fall*, or *tackle-fall*, is free; while the intermediate portion of the rope passes alternately round the pulleys in the fall-block and running-block. The whole combination is called a *tackle* or *purchase*.

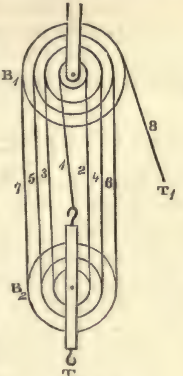

Fig. 222.

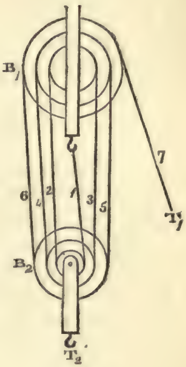

Fig. 223.

495. The **Velocity-Ratio** chiefly considered in a tackle is that between the velocities of the running-block, u, and of the tackle-fall, v. That ratio is given by equation 6 of Article 402 (which see), viz. :—

$$v = n\, u; \qquad (1.)$$

where n is the *number of plies* of rope by which the running-block is connected with the fall-block. Thus, in fig. 222, $n = 7$; and in fig. 223, $n = 6$.

496. The **Velocity of Any Ply** of the rope is found in the following manner :—

I. For a ply on the side of the fall-block next the tackle-fall, such as 2, 4, 6, fig. 222, and 3, 5, fig. 223, it is to be considered what would be the velocity of that ply if it were itself the tackle-fall. Let that velocity be denoted by v', and let n' be the number of plies *between* the ply in question and the point of attachment by which the first ply (marked 1 in the figures) is fixed to one or other block. Then

$$v' = n'\, u \qquad (1.)$$

II. For a ply on the side of the fall-block farthest from the tackle-fall, the velocity is equal and contrary to that of the next succeeding ply, with which it is directly connected over one of the sheaves of the fall-block.

III. If the first ply, as in fig. 223, is attached to the fall-block, its velocity is nothing; if to the running-block, its velocity is equal to that of the block.

497. **White's Tackle.**—The sheaves in a block are usually made all of the same diameter, and turn on a fixed pin; and they have, consequently, different angular velocities. But by making the

diameter of each sheave proportional to the velocity, *relatively to the block*, of the ply of rope which it is to carry, the angular velocities of the sheaves in one block may be rendered equal, so that the sheaves may be made all in one piece, and may have journals turning in fixed bearings. This is called *White's Tackle*, from the inventor, and is represented in figs. 222 and 223.

SECTION 6.—*Hydraulic Connection.*

498. The **General Principle** of the communication of motion between two pistons by means of an intervening fluid of constant density has already been stated in Article 411, viz., that the velocities of the pistons are inversely as their areas, measured on planes normal to their directions of motion.

Should the density of the fluid vary, the problem is no longer one of pure mechanism; because in that case, besides the communication of motion from one piston to the other, there is an additional motion of one or other, or both pistons, due to the change of volume of the fluid.

499. **Valves** are used to regulate the communication of motion through a fluid, by opening and shutting passages through which the fluid flows; for example, a cylinder may be provided with valves which shall cause the fluid to flow in through one passage, and out through another. Of this use of valves, two cases may be distinguished.

I. *When the piston moves the fluid*, the valves may be what is called *self-acting;* that is, moved by the fluid. If there be two passages into the cylinder, one provided with a valve opening inwards, and the other with a valve opening outwards; then during the outward stroke of the piston the former valve is opened and the latter shut by the inward pressure of the fluid, which flows in through the former passage; and during the inward stroke of the piston, the former valve is shut and the latter opened by the outward pressure of the fluid, which flows out through the latter passage. This combination of cylinder, piston, and valves, constitutes a *pump*.

II. *When the fluid moves the piston*, the valves must be opened and shut by mechanism, or by hand. In this case the cylinder is a *working cylinder*.

500. In the **Hydraulic Press,** the rapid motion of a small piston in a pump causes the slow motion of a large piston in a working cylinder. The pump draws water from a reservoir, and forces it into the working cylinder; during the outward stroke of the pump piston, the piston of the working cylinder stands still; during the inward stroke of the pump piston, the piston of the working

cylinder moves outward with a velocity as much less than that of the pump piston as its area is greater. When the piston of the working cylinder has finished its outward stroke, which may be of any length, it is permitted to be moved inwards again by opening a valve by hand and allowing the water to escape.

501. In the **Hydraulic Hoist,** the slow inward motion of a large piston drives water from a large cylinder into a smaller cylinder, and causes a more rapid outward motion of the piston of the smaller cylinder. When the latter piston is to be moved inward, a valve between the two cylinders is closed, and the valve of an outlet from the smaller cylinder opened, by hand, so as to allow the water to escape from the smaller cylinder. The larger cylinder is filled and its piston moved outward, when required, by means of a pump, in a manner resembling the action of a hydraulic press.

SECTION 7.—*Trains of Mechanism.*

502. **Trains of Elementary Combinations** have been defined in Article 435, and illustrated in the case of wheelwork, in Article 449, and in the case of a double Hooke's joint, in Article 492. The general principle of their action is that the comparative motion of the first driver and last follower is expressed by a ratio, which is found by multiplying together the several velocity-ratios of the series of elementary combinations of which the train consists, each with the sign denoting the directional relation.

Two or more trains of mechanism may *converge* into one; as when the two pistons of a pair of steam engines, each through its own connecting rod, act upon one crank shaft. One train of mechanism may *diverge* into two or more; as when a single shaft, driven by a prime mover, carries several pulleys, each of which drives a different machine. The principles of comparative motion in such converging and diverging trains are the same as in simple trains.

CHAPTER III.

ON AGGREGATE COMBINATIONS.

503. The **General Principles** of aggregate combinations have already been given in Part III., Chapter II., Section 3. The problems to which those principles are to be applied may be divided into two classes.

I. Where a secondary moving piece is connected at three, or at two points, as the case may be, with three or with two other pieces whose motions are given; so that the problem is, *from the motions of three or of two points in the secondary piece, to find its motion as a whole, and the motion of any point in it.* The solution of this problem is given in Articles 383 and 384.

II. Where a secondary piece, C, is carried by another piece, B; and denoting the frame of the machine by A, there are given two out of the three motions of A, B, and C, relatively to each other, and the third is required. The motion of C relatively to A is the resultant of the motion of C relatively to B, and of B relatively to A; and the problem is solved by the methods already explained in Articles 385 to 395, inclusive.

Mr. Willis distinguishes the effects of aggregate combinations into *aggregate velocities,* whether linear or angular, produced in secondary pieces by the combined action of different drivers, and *aggregate paths,* being the curves, such as cycloids and trochoids, epicycloids and epitrochoids, described by given points in such secondary pieces.

The following Articles give examples of the more ordinary and useful aggregate combinations.

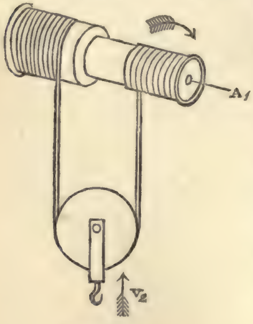

Fig. 224.

504. **Differential Windlass.**—In fig. 224, the axis A_1 carries two barrels of different radii, r_1 being the greater, and r_2 the less. A running block containing a single pulley is hung by a rope which passes below the pulley, and has one end wound round the larger barrel, and the other wound the contrary way round the smaller barrel. When the two barrels rotate together with the

common angular velocity a, the division of the rope which hangs from the larger barrel moves with the velocity $a\, r_1$, and the division which hangs from the smaller barrel moves in the contrary direction with the velocity $-a\, r_2$ (whose direction is denoted by the negative sign). These are also the velocities of the two points at opposite extremities of a diameter of the pulley, where it is touched by the two vertical divisions of the rope. The velocity of the centre of the pulley is a mean between those two velocities; that is, their half-difference, because their signs are opposite; or denoting it by v,

$$v = \frac{a\,(r_1 - r_2)}{2} \text{..........................} (1.)$$

The *instantaneous axis* of the pulley may be found by the method of Article 384, as follows:—In fig. 184 *c*, let A and B be the two ends of the horizontal diameter of the pulley, and let $\overline{\mathrm{A\,V}_a} = a\, r_1$, and $\overline{\mathrm{B\,V}_b} = a\, r_2$ represent their velocities; join $\overline{\mathrm{V}_a\,\mathrm{V}_b}$ cutting A B in O; this is the instantaneous axis, and its distance from the centre or moving axis of the pulley is obviously

$$\overline{\mathrm{A\,B}} \cdot \frac{r_1 - r_2}{2\,(r_1 + r_2)} \text{........................} (2.)$$

The motion of the centre of the pulley is the same with that of a point in a rope wound on a barrel of the radius $\frac{r_1 - r_2}{2}$. The use of the contrivance is to obtain a slow motion of the pulley without using a small, and therefore a weak, barrel.

505. **Compound Screws.**—(Fig. 225.) On the same axis let there be two screws $S_1\, S_1$, and $S_2\, S_2$, of the respective pitches

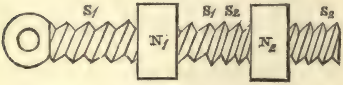

Fig. 225.

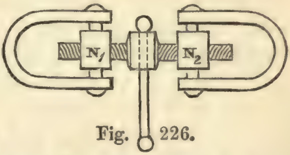

Fig. 226.

p_1 and p_2, p_1 being the greater, and let the screws in the first instance be both right-handed or both left-handed. Let N_1 and N_2 be two nuts, fitted on the two screws respectively. When the compound screw rotates with the angular velocity a, the nuts approach towards or recede from each other with the relative velocity,

$$v = \frac{a\,(p_1 - p_2)}{2\,\pi}; \text{........................} (1.)$$

being that due to a screw whose pitch is the *difference* of the two pitches of the compound screw. (See Article 382, equation 1.) The object of this contrivance is to obtain the slow advance due to a fine pitch, together with the strength of large threads.

Fig. 226 represents a compound screw in which the two screws are contrary-handed, and the relative velocity of the nuts N_1, N_2, is that due to the *sum* of the two pitches; or, as these are usually equal, to *double* the pitch of each screw. This combination is used in coupling railway carriages.

506. **Link Motion.**—Let C be the axis of the shaft of a steam engine, C T the crank, f the *connected point* (see Article 489) of the *forward eccentric* (which is suited to move the slide valve when the engine moves forwards), b the connected point of the *backward eccentric* (which is suited to move the slide valve when the engine is reversed), f F the forward and b B the backward eccentric rods, F B a piece called the *link*, jointed to those two rods at F and B, S a slider, which is capable of being slid to and fixed at different positions in the link, and to which the slide valve rod is jointed. Let the arrow represent the direction of *forward* rotation of the shaft, and at the instant represented in the figure, let the piston be at one end of its stroke. Let L L be a line showing the position in which the crank arm of an eccentric should stand, in order that the middle of the stroke of the slide valve should be at the same instant with the extremity of the stroke of the piston. The angle $\angle$ L C f is the *angular lead* or *advance* of the forward eccentric, and the angle $\angle$ L C b (usually equal to the former) the *angular lead* or *advance* of the backward eccentric.

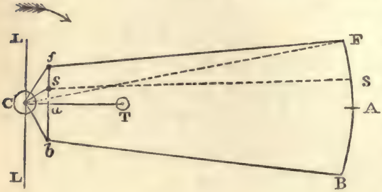

Fig. 227.

When S is at F, the engine is in *full forward gear*, the motion of the slide valve being governed by the forward eccentric alone. The stroke or *throw* of the slide valve is $2\,\overline{Cf}$, and its lead corresponds to the angle $\angle$ L C f.

When S is at B, the engine is in *full backward gear*, the motion of the slide valve being governed by the backward eccentric alone. The stroke or throw of the slide valve is $2\,\overline{Cb}$ (usually $= 2\,\overline{Cf}$), and its lead corresponds to the angle $\angle$ L C b (usually $= \angle$ L C f).

When S is at A, the engine is in *mid gear*, the velocity of the valve rod at each instant being a mean between those which it would receive from either eccentric separately.

The lead corresponds to 90°, or a quarter of a revolution. The throw is nearly, though not exactly, $= 2\,\overline{\mathrm{C}\,a}$, a being the middle of the straight line fb.

To find *exactly* the motions of the slide valve for different positions of the slider S, it is best to draw a diagram to a scale, representing the positions of the eccentrics, rods, and link, for a series of angular positions of the crank (usually dividing a revolution into 24 equal angles); and the corresponding series of positions of S when fixed at various points in the link. Several examples of this process are given in Mr. D. K. Clark's treatise on Railway Machinery.

A useful *approximation* to the motions of the valve, when the rods are long compared with the link, is got by dividing the line fb at s in the same proportion in which S divides F B, and considering the motion of the valve as produced by the crank C s; so that the throw is approximately $2\,\overline{\mathrm{C}\,s}$, and the lead approximately $\angle\,\mathrm{L\,C}\,s$.

507. **Parallel Motions** are jointed combinations of linkwork, designed to guide the motion of a reciprocating piece, such as the piston rod of a steam engine, either exactly or approximately in a straight line, in order to avoid the friction which attends the use of straight guides. Four kinds of parallel motion will now be described:—

I. An **Exact Parallel Motion,** believed to have been first proposed by Mr. Scott Russell, is represented in fig. 228. The same parts of the mechanism are marked with the same letters, and different successive positions are indicated by numerals affixed. The lever C T turns about the fixed centre C, and carries, jointed to its other end, the bar or link P T Q, in which $\overline{\mathrm{PT}} = \overline{\mathrm{TQ}} = \overline{\mathrm{CT}}$. The point Q is jointed to a slider which slides in guides along the straight line C Q. From Q draw Q D ⊥ C Q, cutting C T produced in D; then by Article 488, D is the instantaneous axis of the link; and because D P ∥ C Q, the motion of P, which is ⊥ D P, is always ⊥ C Q; that is to say, the point P moves in the straight line $\mathrm{P}_1\,\mathrm{C}\,\mathrm{P}_3$, ⊥ C Q. In a steam engine, a pair of the combinations here shown are used, one at each side of the cylinder; and the pair of bars P Q are jointed at their extremities P to the head of the piston rod. The distance through which Q slides at each single stroke of the piston, of the length $\overline{\mathrm{P}_1\,\mathrm{P}_3} = \mathrm{S}$, is given by the equation

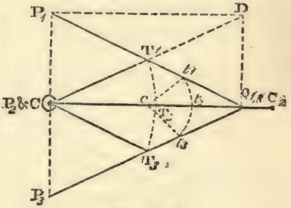

Fig. 228.

$$2\,\overline{Q_1\,Q_2} = 2\left\{\overline{P\,Q} - \sqrt{\overline{P\,Q}^2 - \frac{S^2}{4},}\right\} \ldots\ldots\ldots\ldots (1)$$

and is small compared with the length of stroke of the piston.

II. An **Approximate Parallel Motion,** somewhat resembling the preceding, is obtained by guiding the link P Q entirely by means of oscillating levers, instead of by a lever and a slide. To find the length and the position of the axis of one of those levers, *c t*, select any convenient point, *t*, in the link P Q, and lay down on a drawing the extreme and middle positions, t_1, t_2, t_3, of that point, corresponding to the extreme and middle positions of the link P Q. The centre *c* of a circle traversing those three points will be the required axis of the lever, and *c t* will be its length; and if the link P Q is guided by two such levers, the extreme and middle positions of P will be in one straight line, and the other positions of that point very nearly in one straight line.

III. **Watt's Approximate Parallel Motion.**—In fig. 229, let C T, *c t*, be a pair of levers, connected by a link T *t*, and oscillating about

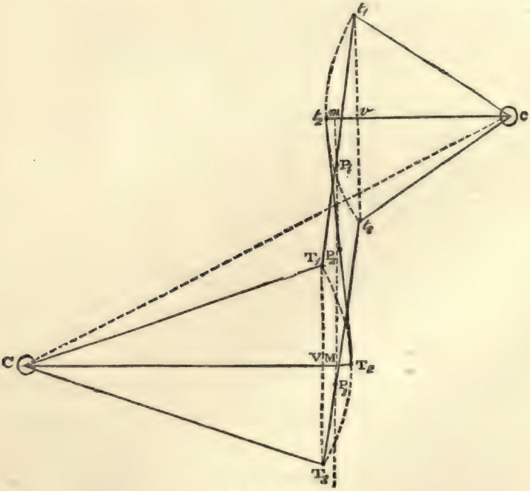

Fig. 229.

the axes C, *c*, between the positions marked 1 and 3. Let the middle positions of the levers, C T_2, *c* t_2, be parallel to each other. It is required to find a point P in the link T *t*, such, that its middle position P_2, and its extreme positions P_1, P_3, shall be in the same straight line perpendicular to $C T_2$, $c t_2$, and so to place the axes C, *c*, on the lines $C T_2$, $c t_2$, that the path of P, between the positions P_1, P_2, P_3, shall be as near as possible to a straight line.

The axes C, *c*, are to be so placed, that the middle M of the versed sine $V T_2$, and the middle *m* of the versed sine $v t_2$, of the respective arcs whose equal chords $\overline{T_1 T_3} = \overline{t_1 t_3}$ represent the stroke, may each be in the line of stroke M *m*. Then T_1 and T_3 will be as far to one side of that line as T_2 is to the other, and t_1 and t_3 will be as far to the latter side of the same line as t_2 is to the former; consequently, the two extreme positions of the link, $T_1 t_1$, $T_3 t_3$, are parallel to each other, and inclined to M *m* at the same angle in one direction that the middle position of the link $T_2 t_2$ is inclined to that line in the other direction; and the three intersections $P_1 P_2 P_3$, are at the same point on the link.

The position of the point P on the link is found by the following proportional equation :—

$$\left.\begin{array}{r} \overline{Tt} : \overline{PT} : \overline{Pt} \\ :: \overline{TV} + \overline{tv} : \overline{TV} : \overline{tv} \\ :: \overline{CM} + \overline{cm} : \overline{cm} : \overline{CM} \end{array}\right\} \text{..............(2.)}$$

The positions of the point P in the link, intermediate between its middle and extreme positions, are near enough to a straight line for practical purposes. When there are given, the axes C, *c*, the line of stroke $P_1 P_2 P_3$, the length of stroke $\overline{P_1 P_3} = S$, and the perpendicular distance $\overline{Mm}$ between the middle positions of the two levers, the following equations serve to compute the lengths of the levers and link :—

$$\left.\begin{array}{ll} \text{Versed sines,} & \overline{TV} = \dfrac{S^2}{8\,\overline{CM}};\ \overline{tv} = \dfrac{S^2}{8\,\overline{cm}}; \\ \text{Levers,} & \overline{CT} = \overline{CM} + \dfrac{\overline{TV}}{2};\ \overline{ct} = \overline{cm} + \dfrac{\overline{tv}}{2}; \\ \text{Link,} & \overline{Tt} = \sqrt{\left\{\overline{Mm}^2 + \dfrac{(\overline{TV} + \overline{tv})^2}{4}\right\}}. \end{array}\right\} \text{.......(3.)}$$

IV. **Watt's Parallel Motion Modified** by having the guided point P in the prolongation of the link T *t* beyond its connected points, instead of between those points, is represented by fig. 230. In this case, the centres of the two levers are at the same side of the link, instead of at opposite sides, the shorter lever being the farther from the guided point P; and the equations 2 and 3 are modified as follows :—

$$\left.\begin{array}{ll}\text{Segments of the link,} & \overline{\mathrm{T}t} : \overline{\mathrm{PT}} : \mathrm{P}t \\ & :: \overline{tv} - \overline{\mathrm{TV}} : \overline{\mathrm{TV}} : \overline{tv} \\ & :: \overline{\mathrm{CM}} - \overline{cm} : \overline{cm} : \overline{\mathrm{CM}}.\end{array}\right\} \dots\dots\dots\dots(4.)$$

$$\left.\begin{array}{ll}\text{Versed sines,} & \overline{\mathrm{TV}} = \dfrac{\mathrm{S}^2}{8\,\overline{\mathrm{CM}}};\ \overline{tv} = \dfrac{\mathrm{S}^2}{8\,\overline{cm}}; \\ \text{Levers,} & \overline{\mathrm{CT}} = \overline{\mathrm{CM}} + \dfrac{\overline{\mathrm{TV}}}{2};\ \overline{ct} = \overline{cm} + \dfrac{\overline{tv}}{2}; \\ \text{Link,} & \overline{\mathrm{T}t} = \sqrt{\left\{\overline{\mathrm{M}m^2} + \dfrac{(tv - \mathrm{TV})^2}{4}\right\}}.\end{array}\right\} \dots\dots\dots(5.)$$

This parallel motion is used in some marine engines, in a position inverted with respect to that in the figure, P being the upper, and *t* the lower end of the link.

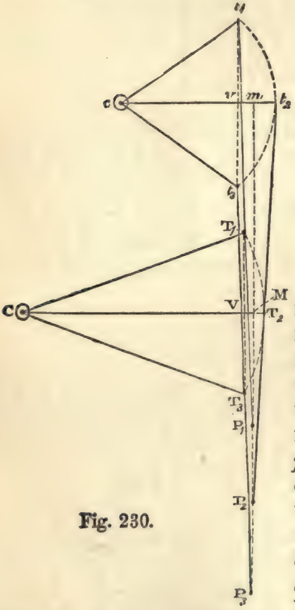

Fig. 230.

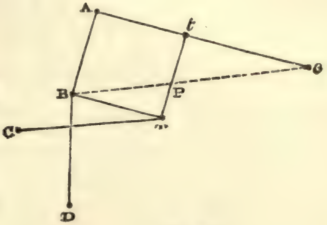

Fig. 231.

When Watt's parallel motion (III.) is applied to steam engines with beams, it is more usual to guide the air pump rod than the piston rod directly by means of the point P. The head of the piston rod is guided by being connected with that point by means of a *parallelogram* of bars, shown in fig. 231. *c* is the axis of motion of the beam of the engine, *c t* A one arm of that beam, C T a lever called the *radius bar* or *bridle rod*, T *t* a link called the *back link*. C T, *c t*, and T *t*, form the combination already described (III.), and shown in fig. 229; and the point P, found as already shown, is guided in a vertical line, almost exactly straight. The total length of the beam arm, $\overline{c\mathrm{A}}$, is fixed by the proportion

$$\overline{\mathrm{P}\,t} : \overline{\mathrm{T}\,t} : : \overline{\mathrm{C}\,t} : \overline{\mathrm{C\,A}};\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6.)$$

that is, t A is very nearly a third proportional to $\overline{\mathrm{C\,T}}$ and $\overline{c\,t}$. Draw A B || T t, and c P B intersecting it; then from the proportion 6 it follows that $\overline{\mathrm{A\,B}} = \overline{\mathrm{T}\,t}$. $\overline{\mathrm{A\,B}}$ is the *main link*, by the lower end of which, B, the head of the piston rod is guided. $\overline{\mathrm{B\,T}}$ = and || $\overline{t\,\mathrm{A}}$ is the *parallel bar*, by which the main and back links are connected. P moves sensibly in a straight line; $\frac{\overline{c\,\mathrm{B}}}{\overline{c\,\mathrm{P}}} = \frac{\overline{c\,\mathrm{A}}}{\overline{c\,t}}$ is a constant ratio; therefore B moves sensibly in a straight line parallel to that in which P moves.

A *parallelogram* analogous to A B T t may also be combined with the parallel motion IV.

508. **Epicyclic Trains.**—The term *epicyclic train* is used by Mr. Willis to denote a train of wheels carried by an arm, and having certain rotations relatively to that arm, which itself rotates. The arm may either be driven by the wheels, or assist in driving them. The comparative motions of the wheels and of the arm relatively to each other and to the frame, and the *aggregate paths* traced by points in the wheels, are determined by the principles of the composition of rotations, already explained in Articles 385 to 395.

PART V.

PRINCIPLES OF DYNAMICS.

509. **Division of the Subject.**—The science of Dynamics, which treats of the relations between the motions of bodies and the forces acting amongst them, may be divided into two primary divisions, according as it has reference to balanced forces and uniform motions, or to unbalanced forces and varying motions. A secondary mode of dividing the subject is founded on the distinction between questions respecting the motions of masses which are either insensibly small, or which, being of sensible magnitude, have motions of translation only,—questions respecting the motions of rigid bodies and rigidly connected systems which rotate,—and questions respecting the motions of pliable bodies and of fluids. The dynamics of fluids has received the special name of *hydrodynamics*. It is a branch of mechanics so extensive in its applications, and depending so much in its details upon special experiments, as to require a separate work for its full exposition; nevertheless, in the present treatise its fundamental principles will be set forth in their proper place.

The dynamical principles of the motions of rotating rigid bodies, of pliable bodies, and of fluids, are deduced from those of the motions of rigid bodies having motions of simple translation, by conceiving the bodies under consideration to be divided into indefinitely small molecules or particles, so that the laws of the motion of each molecule shall differ from those of a body having a motion of simple translation to an extent less than any given difference. It is to such indefinitely small molecules that the term *physical point*, already mentioned in Article 7, is applied.

Hence it appears that the laws of the relations between the motions of a so-called *physical point*, and the forces acting on it, are the foundation of the science of dynamics; and the same laws are applicable to a rigid body in which every point moves in the same manner at the same instant; that is to say, which has a motion of translation, as defined in Article 369.

The subjects to which the principles of dynamics relate will therefore be classed in the following manner:—

I. Uniform Motion.
II. Varied Translation of Points and Rigid Bodies.
III. Rotations of Rigid Bodies.
IV. Motions of Pliable Bodies.
V. Motions of Fluids.

CHAPTER I.

ON UNIFORM MOTION UNDER BALANCED FORCES.

510. **First Law of Motion.**—*A body under the action of no force, or of balanced forces, is either at rest, or moves uniformly.* (Uniform motion has been defined in Article 354.)

Such is the first law of motion as usually stated; but in that statement is implied something more than the literal meaning of the words; for it is understood, that the *rest* or *motion of the body* to which the law refers, is its rest or motion *relatively to another body which is also under the action of no force, or of balanced forces.* Unless this implied condition be fulfilled, the law is not true. Therefore the complete and explicit statement of the first law of motion is as follows:—

If a pair of bodies be each under the action of no force, or of balanced forces, the motion of each of those bodies relatively to the other is either none or uniform.

The first law of motion has been learned by experience and observation: not directly, for the circumstances supposed in it never occur; but indirectly, from the fact that its consequences, when it is taken in conjunction with other laws, are in accordance with all the phenomena of the motions of bodies.

The first law of motion may be regarded as a consequence of the definitions of *force* and of *balance* (Articles 12, 13): at the same time it is to be observed, that the framing of those definitions has been guided by experimental knowledge.

511. **Effort; Resistance; Lateral Force.**—Let F denote a force applied to a moving point, and θ the angle made by the direction of that force with the direction of the motion of the point. Then, by the principles of Article 57, the force F may be resolved into two rectangular components, one along, and the other across, the direction of motion of the point, viz.:—

The *direct* force, $F \cos \theta$.
The *lateral* force, $F \sin \theta$.

A direct force is further distinguished, according as it acts *with* or *against* the motion of the point (that is, according as θ is acute or obtuse), by the name of *effort*, or of *resistance*, as the case may be. Hence each force applied to a moving point may be thus decomposed:—

$$\left.\begin{array}{l} \textit{Effort},\ P = F \cos \theta, \text{ if } \theta \text{ is acute;} \\ \textit{Resistance},\ R = F \cos (\pi - \theta) \text{ if } \theta \text{ is obtuse;} \\ \textit{Lateral force},\ Q = F \sin \theta. \end{array}\right\} \ldots\ldots (1.)$$

512. The **Conditions of Uniform Motion** of a pair of points are, that the forces applied to each of them shall balance each other; that is to say, *that the lateral forces applied to each point shall balance each other, and that the efforts applied to each point shall balance the resistances.*

The direction of a force being, as stated in Article 20, that of the motion which it tends to produce, it is evident that the balance of lateral forces is the condition of *uniformity of direction* of motion, that is, of motion in a straight line; and that the balance of efforts and resistances is the condition of *uniformity of velocity.*

513. **Work** consists in moving against resistance. The work is said to be *performed,* and the resistance *overcome.* Work is measured by the product of the resistance into the distance through which its point of application is moved. The *unit of work* commonly used in Britain is a resistance of one pound overcome through a distance of one foot, and is called a *foot-pound.*

514. **Energy** means *capacity for performing work.* The *energy of an effort,* or *potential energy,* is measured by the product of the effort into the distance through which its point of application is *capable* of being moved. The unit of energy is the same with the unit of work.

When the point of application of an effort *has been moved* through a given distance, energy is said to have been *exerted* to an amount expressed by the product of the effort into the distance through which its point of application has been moved.

515. **Energy and Work of Varying Forces.**—If an effort has different magnitudes during different portions of the motion of its point of application through a given distance, let each different magnitude of the effort P be multiplied by the length Δs of the corresponding portion of the path of the point of application; the sum

$$\Sigma \cdot P \Delta s \ldots\ldots \ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

is the whole energy exerted. If the effort varies by insensible degrees, the energy exerted is the *integral* or limit towards which that sum approaches continually, as the divisions of the path are made smaller and more numerous, and is expressed by

$$\int \mathrm{P}\, d\, s \text{..............................(2.)}$$

Similar processes are applicable to the finding of the work performed in overcoming a varying resistance. As to integration in general, see Article 81.

516. A **Dynamometer or Indicator** is an instrument which measures and records the energy exerted by an effort. It usually consists essentially, *first*, of a piece of paper moving with a velocity proportional to that of the point of application of the effort, and having a straight line marked on it parallel to its direction of motion, called the zero line; and *secondly*, of a spring, acted upon and bent by the effort, and carrying a pencil whose perpendicular distance from the zero line, as regulated by the bending of the spring, is proportional to the effort. The pencil traces on the piece of paper a line like that in fig. 24 of Article 81, such that its *ordinate* $\overline{\mathrm{E}\,\mathrm{F}}$, perpendicular to the zero line O X at a given point, represents the effort P for the corresponding point in the path of the point of application of the effort; and the *area between two ordinates*, such as A C D B, represents the energy exerted, $\int \mathrm{P}\, d\, s$, for the corresponding portion, A B, of the path of the point of application of the effort.

517. The **Energy and Work of Fluid Pressure** may be expressed as follows:—Let A denote the projection *on a plane perpendicular to the direction of motion* of the moving body, of that portion of the body's surface to which the pressure is applied, p the intensity of the pressure in units of force per unit of area (Article 86), and Δs the distance through which the body is moved in a given interval of time; then during that interval, the energy exerted by, or work performed against, the fluid pressure, according as it acts with or against the motion, is given by the formula

$$\mathrm{P} \cdot \Delta s \ (\text{or } \mathrm{R} \cdot \Delta s) = p\, \mathrm{A} \cdot \Delta s = p \cdot \Delta \mathrm{V}; \text{..........(1.)}$$

where $\Delta \mathrm{V}$ is the *volume* of the space swept through by the portion of the body's surface which is pressed upon, during the given interval of time.

518. The **Conservation of Energy,** in the case of uniform motion, means the fact, that *the energy exerted is equal to the work performed;* and is a consequence of the first law of motion, as is shown by the consideration of the following cases:—

CASE 1. *For the forces acting on a single point,* the principle is self-evident; for as the effort applied to the point balances the resistance, the products of these forces into the distance traversed by the point in any interval must be equal; that is,

$$P \cdot \Delta s = R \cdot \Delta s \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

Case 2. *For the forces acting on any system of balanced points*, the principle must be true, because it is true for those acting on each single point of the system. This is expressed as follows :—

$$\Sigma \cdot P \Delta s = \Sigma \cdot R \Delta s \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.)$$

Case 3. When a system of points are *rigidly connected*, so that their relative positions do not alter, there is neither energy exerted nor work performed by the forces which act *amongst the points of the system themselves;* and therefore, from case 2 it follows, that the principle of the conservation of energy is true *of the forces acting between the points of the system and external bodies.*

Symbolically, let the efforts acting amongst the points of the system be denoted by P_1, the resistances by R_1; the efforts acting between the points of the system and external bodies by P_2, and the resistances by R_2. Then by case 2,

$$\Sigma (P_1 + P_2) \Delta s = \Sigma \cdot (R_1 + R_2) \Delta s;$$

but by the condition of rigidity,

$$\Sigma \cdot P_1 \Delta s = 0; \; \Sigma \cdot R_1 \Delta s = 0;$$

therefore,

$$\Sigma \cdot P_2 \Delta s = \Sigma \cdot R_2 \Delta s \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.)$$

Case 4. The same principle is demonstrable in the same manner, for the forces acting between external bodies and the points of a system so connected, that though not absolutely rigid, *they do not vary their relative positions in the directions in which the internal forces of the system act* Such is the ideal condition in which a train of mechanism would be, if no resistance arose from the mode of connection of the pieces.

519. The **Principle of Virtual Velocities** is the name given to the application of the principle of the conservation of energy to the determination of the conditions of equilibrium amongst the forces externally applied to any connected system of points. That application is effected in the following manner :—Let F be any one of the externally applied forces in question. The conditions of equilibrium are those of uniform motion. Conceive the points of the system to be moving with uniform velocities in any manner which is consistent with the absence of all exertion of energy and performance of work by their mutual or internal forces. Let v be the

velocity, or any number proportional to the velocity, of the point to which the external force F is applied, and θ the angle between the direction of that force and the direction of motion of its point of application. Then from cases 3 and 4 of the principle of the conservation of energy, it follows that the condition of equilibrium amongst the forces F is

$$\Sigma \cdot \mathrm{F}\, v \cos \theta = 0; \quad \text{(1.)}$$

attention being paid to the principle, that cos θ is $\left\{ \begin{matrix} \text{positive} \\ \text{negative} \end{matrix} \right\}$ when θ is $\left\{ \begin{matrix} \text{acute} \\ \text{obtuse} \end{matrix} \right\}$. The same principle may be otherwise expressed thus: let v be the virtual velocity of any point to which an effort P is applied, u the virtual velocity of any point to which a resistance R is applied; then

$$\Sigma \cdot \mathrm{P}\, v = \Sigma \cdot \mathrm{R}\, u. \quad \text{(2.)}$$

The principle thus expressed is called that of *virtual velocities*, because the velocities denoted by v are merely velocities which the points of the system *might* have.

As the *proportions* of the several velocities v are all that are required in using this principle, it enables the conditions of equilibrium of the forces applied to any body or machine to be found, so soon as the *comparative velocities* of the points of application of those forces have been determined by means of the principles of cinematics, and of the theory of mechanism; and every proposition which has been proved in Parts III. and IV. of this treatise, respecting the comparative velocities of points in a body or in a train of mechanism, can at once be converted into a proposition respecting the equilibrium of forces applied to those points in given directions.

520. **Energy of Component Forces and Motions.**—Let the motion Δs of a point in a given interval of time make angles, α, β, γ, with three rectangular axes; then

$$\Delta s \cdot \cos \alpha,\ \Delta s \cdot \cos \beta,\ \Delta s \cdot \cos \gamma,$$

are the three components of that motion. To that point let there be applied a force F, making with the same axes the angles α', β', γ', so that its rectangular components are

$$\mathrm{F} \cdot \cos \alpha',\ \mathrm{F} \cdot \cos \beta',\ \mathrm{F} \cdot \cos \gamma'.$$

Then multiplying each component of the motion by the component of the force in its own direction, there are found the three quantities of energy exerted,

$$\left.\begin{array}{l} \mathrm{F} \cdot \Delta s \cdot \cos \alpha \cos \alpha'; \\ \mathrm{F} \cdot \Delta s \cdot \cos \beta \cos \beta'; \\ \mathrm{F} \cdot \Delta s \cdot \cos \gamma \cos \gamma'; \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.)$$

and the sum of those three quantities of energy is the whole energy exerted. Now it is well known, that

$$\cos \alpha \cos \alpha' + \cos \beta \cos \beta' + \cos \gamma \cos \gamma' = \cos \theta,$$

θ being the angle between the directions of the force and of the motion; so that the addition of the three quantities of energy in the formulæ 1 gives for the whole energy exerted, simply

$$\mathrm{F} \cdot \Delta s \cdot \cos \theta,$$

as in former examples; and similar remarks apply to work performed.

CHAPTER II.

ON THE VARIED TRANSLATION OF POINTS AND RIGID BODIES.

SECTION 1.—*Definitions.*

521. **The Mass, or Inertia,** of a body, is a quantity proportional to the unbalanced force which is required in order to produce a given definite change in the motion of the body in a given interval of time.

It is known that the weight of a body, that is, the attraction between it and the earth, at a fixed locality on the earth's surface, acting unbalanced on the body for a fixed interval of time (*e. g.*, for a second), produces a change in the body's motion, which is the same for all bodies whatsoever. Hence it follows, that the *masses of all bodies are proportional to their weights at a given locality on the earth's surface.*

This fact has been learned by experiment; but it can also be shown that it is necessary to the permanent existence of the universe; for if the gravity of all bodies whatsoever were not proportional to their respective masses, it would not produce similar and equal changes of motion in all bodies which arrive at similar positions with respect to other bodies, and the different parts which make up stars and systems would not accompany each other in their motions, never departing beyond certain limits, but would be dispersed and reduced to chaos. Neither an imponderable body, nor a body whose gravity, as compared with its mass, differs in the slightest conceivable degree from that of other bodies, can belong to the system of the universe.*

522. **The Centre of Mass** of a body is its centre of gravity, found in the manner explained in Part I., Chapter V., Section 1.

523. **The Momentum** of a body means, the product of its mass into its velocity relatively to some point assumed as fixed. The momentum of a body, like its velocity, can be resolved into components, rectangular or otherwise, in the manner already explained for motions in Part III., Chapter I.

524. **The Resultant Momentum** of a system of bodies is the resultant of their separate momenta, compounded as if they were motions or statical couples.

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* See the Rev. Dr. Whewell's demonstration "that all matter gravitates."

THEOREM. *The momentum of a system of bodies is the same as if all their masses were concentrated at the centre of gravity of the system.* Conceive the velocity of each of the bodies to be resolved into three rectangular components. Consider all the component velocities parallel to one of the rectangular directions. These are the rates of variation of the perpendicular distances of the bodies from a certain plane. If the mass of each of the bodies be multiplied by its distance from a certain plane, the products added, and the sum divided by the sum of the masses, the result is the distance of the centre of gravity of the whole system from that plane; therefore, if the component velocity of each of the bodies in a direction perpendicular to that plane be multiplied by the mass of the body, the sum of such products for all the bodies of the system will be the product of the entire mass of the system into the velocity of its centre of gravity in a direction perpendicular to the plane in question; so that this product is one of the three rectangular components of the resultant momentum of the system of bodies; and the same may be proved for the other rectangular components. Expressed symbolically, let u, v, w, be the three rectangular components of the velocity of any mass, m, belonging to a system of bodies, and u_0, v_0, w_0, the rectangular components of the velocity of the centre of gravity of that system of bodies; then

$$\left.\begin{aligned} u_0 \cdot \Sigma m &= \Sigma \cdot m u; \\ v_0 \cdot \Sigma m &= \Sigma \cdot m v; \\ w_0 \cdot \Sigma m &= \Sigma \cdot m w. \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

COROLLARY. *The resultant momentum of a system of bodies relatively to their common centre of gravity is nothing;* that is to say,

$$\left.\begin{gathered} \Sigma m\,(u-u_0)=0;\ \Sigma m\,(v-v_0)=0; \\ \Sigma m\,(w-w_0)=0. \end{gathered}\right\} \ldots\ldots\ldots\ldots (2.)$$

525. **Variations and Deviations of Momentum** are the products of the mass of a body into the rates of variation of its velocity and deviation of its direction, found as explained in Part III., Chapter I., Section 3.

526. **Impulse** is the product of an unbalanced force into the *time* during which it acts unbalanced, and can be resolved and compounded exactly like force. If F be a force, and dt an interval of time during which it acts unbalanced, $\mathrm{F}\,dt$ is the impulse exerted by the force during that time. The impulse of an unbalanced force in an unit of time is the magnitude of the force itself.

527. **Impulse, Accelerating, Retarding, Deflecting.**—Corresponding to the resolution of a force applied to a moving body into effort or resistance, as the case may be, and lateral stress, as explained in

Article 511, there is a resolution of impulse into accelerating or retarding impulse, which acts with or against the body's motion, and deflecting impulse, which acts across the direction of the body's motion. Thus if θ, as before, be the angle which the unbalanced force F makes with the body's path during an indefinitely short interval, dt,

$$\left.\begin{array}{l} \mathrm{P}\,dt = \mathrm{F}\cos\theta\cdot dt \text{ is accelerating impulse if } \theta \text{ is acute ;} \\ \mathrm{R}\,dt = \mathrm{F}\cos(\pi-\theta)\cdot dt \text{ is retarding impulse if } \theta \text{ is obtuse ;} \\ \mathrm{Q}\,dt = \mathrm{F}\sin\theta\, dt \text{ is deflecting impulse.} \end{array}\right\} (1.)$$

528. **Relations between Impulse, Energy, and Work.**—If v be the mean velocity of a moving body during the interval dt of the action of the unbalanced force F, then $ds = v\,dt$ is the distance described by that body; and according as θ is acute or obtuse, there is either *energy exerted on the body by the accelerating impulse* to the amount

$$\mathrm{P}\,ds = \mathrm{F}\,v\cos\theta\cdot dt; \quad\text{.....................(1.)}$$

or *work performed by the body against the retarding impulse* to the amount

$$\mathrm{R}\,ds = \mathrm{F}\,v\cos(\pi-\theta)\cdot dt \quad\text{...................(2.)}$$

SECTION 2.—*Law of Varied Translation.*

529. **Second Law of Motion.**—*Change of momentum is proportional to the impulse producing it.* In this statement, as in that of the first law of motion, Article 510, it is implied that the motion of the moving body under consideration is referred to a fixed point or body whose motion is uniform. In questions of applied mechanics, the motion of any part of the earth's surface may be treated as uniform without sensible error in practice. The units of mass and of force may be so adapted to each other as to make *change of momentum equal to the impulse producing it.* (See Articles 531, 532.)

530. **General Equations of Dynamics.**—To express the second law of motion algebraically, two methods may be followed: the first method being to resolve the change of momentum into direct variation and deviation, and the impulse into direct and deflecting impulse; and the second method being to resolve both the change of momentum and the impulse into components parallel to three rectangular axes.

First method. m being the mass of the body, v its velocity, and r the radius of curvature of its path, it follows from Articles 361 and 362 that the *rate of direct variation* of its momentum is

$$m\frac{dv}{dt} = m\cdot\frac{d^2s}{dt^2};$$

and from Articles 363 and 364, that the rate of deviation of its momentum is

$$m\frac{v^2}{r}.$$

Equating these respectively to the direct and lateral impulse per unit of time, exerted by an unbalanced force F, making an angle θ with the direction of the body's motion, we find the two following equations:—

$$\text{P or } -\text{R} = \text{F}\cos\theta = m\cdot\frac{d\,v}{d\,t} = m\frac{d^2 s}{d\,t^2};\ldots\ldots\ldots\ldots(1.)$$

$$\text{Q} = \text{F}\sin\theta = \frac{m\,v^2}{r}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.)$$

The radius of curvature r is in the direction of the deviating force Q.

Second method. As in Article 366, let the velocity of the body be resolved into three rectangular components, $\frac{d\,x}{d\,t}$, $\frac{d\,y}{d\,t}$, $\frac{d\,z}{d\,t}$; so that the three component rates of variation of its momentum are

$$m\frac{d^2x}{d\,t^2},\ m\frac{d^2y}{d\,t^2},\ m\frac{d^2z}{d\,t^2}.$$

Also let the unbalanced force F, making the angles α, β, γ, with the axes of co-ordinates, and its impulse per unit of time, be resolved into three components, F_x, F_y, F_z. Then we obtain

$$\left.\begin{aligned} \text{F}_x &= \text{F}\cos\alpha = m\cdot\frac{d^2x}{d\,t^2};\\ \text{F}_y &= \text{F}\cos\beta = m\,\frac{d^2y}{d\,t^2};\\ \text{F}_z &= \text{F}\cos\gamma = m\,\frac{d^2z}{d\,t^2}; \end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(3.)$$

three equations, which are substantially identical with the equations 1 and 2.

531. **Mass in Terms of Weight.**—A body's own weight, acting unbalanced on the body, produces velocity towards the earth, increasing at a rate per second denoted by the symbol g, whose numerical value is as follows:—Let λ denote the latitude of the place, h its elevation above the mean level of the sea,

$$g_1 = 32{\cdot}1695 \text{ feet, or } 9{\cdot}8051 \text{ mètres, per second};$$

being the value of g for $\lambda = 45^\circ$ and $h = 0$, and

$$\text{R} = 20900000 \text{ feet, or } 6370000 \text{ mètres, nearly,}$$

being the earth's mean radius; then

$$g = g_1 \cdot (1 - 0{\cdot}00284 \cos 2\lambda) \cdot \left(1 - \frac{2h}{\mathrm{R}}\right) \ldots\ldots\ldots\ldots (1.)$$

For latitudes exceeding 45°, it is to be borne in mind that cos 2 λ is negative, and the terms containing it as a factor have their signs reversed.

For practical purposes connected with ordinary machines, it is sufficiently accurate to assume

$$g = 32{\cdot}2 \text{ feet, or } 9{\cdot}81 \text{ mètres, per second nearly} \ldots\ldots (2.)$$

If, then, a body of the weight W be acted upon by an unbalanced force F, the change of velocity in the direction of F produced in a second will be

$$\frac{\mathrm{F}}{m} = \frac{\mathrm{F}\,g}{\mathrm{W}};$$

whence

gm = W

$$m = \frac{\mathrm{W}}{g} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.)$$

is the expression for the *mass* of a body in terms of its weight, suited to make a change of momentum *equal* to the impulse producing it. m being absolutely constant for the same body, g and W vary in the same proportion at different elevations and in different latitudes.

532. An **Absolute Unit of Force** is the force which, acting during an unit of time on an arbitrary unit of mass, produces an unit of velocity. In Britain, the unit of time being a second (as it is elsewhere), and the unit of velocity one foot per second, the unit of mass employed is the mass whose weight in vacuo at London and at the level of the sea is a standard avoirdupois pound.

The *weight* of an unit of mass, in any given locality, has for its value, in absolute units of force, the co-efficient g. When the *unit of weight* is employed as the unit of force, instead of the *absolute unit*, the corresponding unit of mass becomes g times the unit just mentioned: that is to say, in British measures, the mass of 32·2 lbs.; or in French measures, the mass of 9·81 kilogrammes.

533. The **Motion of a Falling Body,** under the unbalanced action of its own weight, a sensibly uniform force, is a case of the uniformly varied velocity described in Article 361. In the equations of that Article, for the rate of variation of velocity a, is to be substituted the co-efficient g, mentioned in the last Article. Then if v_0 be the velocity of the body at the beginning of an interval of time t, its velocity at the end of that time is

$$v = v_0 + g\,t, \quad \text{(1.)}$$

the mean velocity during that time is

$$\frac{v_0 + v}{2} = v_0 + \frac{g\,t}{2}, \quad \text{(2.)}$$

and the vertical height fallen through is

$$h = v_0\,t + \frac{g\,t^2}{2} \quad \text{(3.)}$$

The preceding equations give the final velocity of the body, and the height fallen through, each in terms of the initial velocity and the time. To obtain the height in terms of the initial and final velocities, or *vice versa,* equation 2 is to be multiplied by $v - v_0 = g\,t$, the acceleration, and compared with equation 3; giving the following results :—

$$\left.\begin{array}{c} \dfrac{v^2 - v_0^2}{2} = v_0\,g\,t + \dfrac{g^2\,t^2}{2} = g\,h; \\ h = \dfrac{v^2 - v_0^2}{2\,g}. \end{array}\right\} \quad \text{(4.)}$$

When the body falls from a state of rest, v_0 is to be made $= 0$; so that the following equations are obtained :—

$$v = g\,t;\ h = \frac{g\,t^2}{2} = \frac{v^2}{2\,g} \quad \text{(5.)}$$

The height h in the last equation is called *the height or fall due to the velocity v;* and that velocity is called *the velocity due to the height or fall h.*

Should the body be at first projected vertically upwards, the initial velocity v_0 is to be made negative. To find the height to which it will rise before reversing its motion and beginning to fall, v is to be made $= 0$ in the last of the equations 4; then

$$h = -\frac{v_0^2}{2\,g}, \quad \text{(6.)}$$

being a rise equal to the fall due to the initial velocity v_0.

534. An **Unresisted Projectile,** or a projectile to whose motion there is no sensible resistance, has a motion compounded of the vertical motion of a falling body, and of the horizontal motion due to the horizontal component of its velocity of projection. In fig. 232, let O represent the point from which the projectile is originally

projected in the direction O A, making the angle X O A = θ with a horizontal line O X in the same vertical plane with O A. Let horizontal distances parallel to O X be denoted by x, and vertical ordinates parallel to O Z by z, positive upwards, and negative downwards. In the equations of vertical motion, the symbol h of the equations of Article 533 is to be replaced by $-z$, because of h and z being measured in opposite directions.

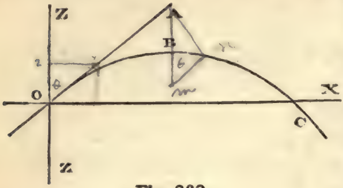

Fig. 232.

Let v_0 be the velocity of projection. Then at the instant of projection, the components of that velocity are,

$$\text{horizontal, } \frac{d\,x}{d\,t} = v_0 \cos\theta; \text{ vertical, } \frac{d\,z}{d\,t} = v_0 \sin\theta;$$

and after the lapse of a given time t, those components have become

$$\left.\begin{aligned} \frac{d\,x}{d\,t} &= v_0 \cos\theta = \text{constant}; \\ \frac{d\,z}{d\,t} &= v_0 \sin\theta - g\,t. \end{aligned}\right\} \text{..................} (1.)$$

Hence the co-ordinates of the body at the end of the time t are

$$\left.\begin{aligned} &\text{horizontal,} & x &= v_0 \cos\theta \cdot t; \\ &\text{vertical,} & z &= v_0 \sin\theta \cdot t - \frac{g\,t^2}{2}; \end{aligned}\right\} \text{..................} (2.)$$

and because $t = \dfrac{x}{v_0 \cos\theta}$, those co-ordinates are thus related,

$$z = x \cdot \tan\theta - \frac{g}{2\,v_0^2 \cos^2\theta} \cdot x^2; \text{..............} (3.)$$

an equation which shows the path O B C of the projectile to be a parabola with a vertical axis, touching O A in O.

The total velocity of the projectile at a given instant, being the resultant of the components given by equation 1, has for the value of its square

Sub. val. of $v_0 \sin\theta\, t$ from Eq 2.

$$v^2 = \frac{d\,x^2}{d\,t^2} + \frac{d\,z^2}{d\,t^2} = v_0^2 - 2\,v_0 \sin\theta \cdot g\,t + g^2\,t^2 = v_0^2 - 2\,g\,z; \text{...} (4.)$$

from the last form of which is obtained the equation

$$z = \frac{v_0^2 - v^2}{2g}; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5.)$$

which, being compared with equation 4 of Article 533, shows that *the relation between the variation of vertical elevation, and the variation of the square of the resultant velocity, is the same, whether the velocity is in a vertical, inclined, or horizontal direction.* This is a particular case of a more general principle, to be explained in the sequel.

The resistance of the air prevents any actual projectile near the earth's surface from moving exactly as an unresisted projectile. The approximation of the motion of an actual projectile to that of an unresisted projectile is the closer, the slower is the motion, and the heavier the body, because of the resistance of the air increasing with the velocity, and because of its proportion to the body's weight being dependent upon that of the body's surface to its weight.

535. The **Motion of a Body Along an Inclined Path,** under the force of gravity alone, is regulated by the principle, that the variation of momentum in each interval of time is equal to the impulse exerted in that interval, by that component of the body's weight which acts along the direction of motion. If the path is straight, the other rectangular component of the body's weight is balanced by the resistance of the surface or other guiding body which causes the inclined path to be described; if the path is curved, the difference between those two forces which act across it is employed in deviating the direction of motion of the body.

Let v be the velocity of the body at any instant, $\frac{dv}{dt}$, as before, the rate of variation of that velocity, θ the inclination of the body's path to the horizon, positive upwards, and negative downwards. Then the body is acted upon in a direction *along* its path by a force equal to its weight multiplied by sin θ, which is a *resistance* if θ is positive, and an *effort* if θ is negative; therefore

$$\frac{dv}{dt} = -g \sin \theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

When the inclination of the path is uniform, this rate of variation of velocity is constant, and the body moves in the same manner with an unresisted body moving vertically, except that each change of velocity occupies an interval of time longer in the ratio of 1 : sin θ for the inclined path than for the vertical path.

The motion of a body in any path on an INCLINED PLANE being resolved into two rectangular components, one horizontal, and the other in the direction of steepest declivity,—the horizontal component (in the absence of friction) is uniform, and the inclined component takes place according to the law expressed by equation 1 of this Article. Consequently, the resultant motion of the body is that of an unresisted projectile, as described in Article 534, except that $g \cdot \sin \theta$ is to be substituted for g.

The motions of bodies on inclined planes being slower, and therefore more easily observed than their vertical motions, were used by Galileo to ascertain the laws of dynamics, which he discovered.

For a body sliding on an inclined plane without friction, the equation connecting the velocity directly with the position of the body is the following :— See Eq. 5 Art. 11.

$$v_0^2 - v^2 = 2\,g \sin \theta \cdot z';$$

where v_0 is the velocity at the origin of the motion, and v the velocity which the body has when it reaches a position whose *inclined co-ordinate* relatively to the origin of the motion is z', positive upwards. But $z' \sin \theta = z$, the *difference of vertical elevation* of the two positions of the body; so that the variation of the square of the velocity bears the same relation to the difference of vertical elevation in the present case as in the case of an unresisted projectile, or a free body moving vertically.

536. An **Uniform Effort or Resistance,** unbalanced, causes the velocity of a body to vary according to the law expressed by this equation,

$$\frac{d\,v}{d\,t} = f\,g; \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

where f is the constant ratio which the unbalanced force bears to the weight of the moving body, positive or negative according to the direction of the force; so that by substituting $f\,g$ for g in the equations of Article 533, those equations are transformed into the equations of motion of the body in question, h being taken to represent the distance traversed by it in a positive direction.

In the apparatus known by the name of its inventor, Atwood, for illustrating the effect of uniform moving forces, this principle is applied in order to produce motions following the same law with those of falling bodies, but slower, by a method less liable to errors caused by friction than that of Galileo. Two weights, P and R, of which P is the greater, are hung to the opposite ends of a cord passing over a finely constructed pulley. Considering the masses of the cord and pulley to be insensible, the weight of the mass to be moved is P + R, and the moving force P − R, being less than the weight in the ratio,

$$f = \frac{\mathrm{P} - \mathrm{R}}{\mathrm{P} + \mathrm{R}}.$$

See Fig 232.

$\frac{1}{\sqrt{dx^2+dz^2}/dx} = \frac{1}{\sqrt{1+\frac{dz^2}{dx^2}}}$ ∴ $Q = W \times \frac{1}{\sqrt{1+\frac{dz^2}{dx^2}}} = W \div \sqrt{1+\frac{dz^2}{dx^2}}$

Consequently the two weights move according to the same law with a falling body, but slower in the ratio of *f* to 1.

537. A **Deviating Force,** which acts unbalanced in a direction perpendicular to that of a body's motion, and changes that direction without changing the velocity of the body, is equal to the rate of deviation of the body's momentum per unit of time, as the following equation expresses:—

$$Q = \frac{W v^2}{g r}; \ldots\ldots\ldots\ldots(1.)$$

Q being the deviating force, W the weight of the body, W ÷ g its mass, v^2 its velocity, and r the radius of curvature of its path.

In the case of an unresisted projectile, already mentioned in Article 534, the deviating force at any instant is that component of the body's weight which acts perpendicular to its direction of motion; that is to say See above.

$v_0 \cos\theta$ = hor. comp. z × Eq.
v = × 0 $W \sin\theta = W \times$

$$Q = W \div \sqrt{\left(1 + \frac{d z^2}{d x^2}\right)} = \frac{W v_0 \cdot \cos\theta}{v} \ldots\ldots\ldots(2.)$$

The well known expression for the radius of curvature of any curve whose co-ordinates are x and z is

$$r = \left(1 + \frac{d z^2}{d x^2}\right)^{\frac{3}{2}} \div \frac{d^2 z}{d x^2} = \left(\frac{v}{v_0 \cdot \cos\theta}\right)^3 \cdot \frac{v_0^2 \cdot \cos^2\theta}{g} \ldots\ldots(3.)$$

Consequently $Q\, r = \frac{W v^2}{g}$, which agrees with equation 1.

In the case of projectiles, just described, and of the heavenly bodies, deviating force is supplied by that component of the mutual attraction of two masses which acts perpendicular to the direction of their relative motion. In machines, deviating force is supplied by the strength or rigidity of some body, which *guides* the deviating mass, making it move in a curve.

A pair of free bodies attracting each other have both deviated motions, the attraction of each guiding the other; and their deviations of momentum are equal in equal times; that is, their deviations of motion are inversely as their masses.

In a machine, each revolving body tends to press or draw the body which guides it away from its position, in a direction from the centre of curvature of the path of the revolving body; and that tendency is resisted by the strength and stiffness of the guiding body, and of the frame with which it is connected.

538. **Centrifugal Force** is the force with which a revolving body reacts on the body that guides it, and is equal and opposite to the

deviating force with which the guiding body acts on the revolving body.

In fact, as has been stated in Article 12, every force is an action between two bodies; and *deviating force* and *centrifugal force* are but two different names for the same force, applied to it according as its action on the revolving body or on the guiding body is under consideration at the time.

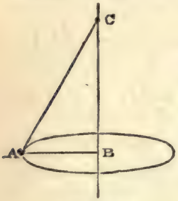

Fig. 233.

539. A **Revolving Simple Pendulum** consists of a small mass A, suspended from a point C by a rod or cord C A of insensibly small weight as compared with the mass A, and revolving in a circle about a vertical axis C B. The tension of the rod is the resultant of the weight of the mass A, acting vertically, and of its centrifugal force, acting horizontally; and therefore the rod will assume such an inclination that

$$\frac{\text{height } \overline{\mathrm{B\,C}}}{\text{radius } \overline{\mathrm{A\,B}}} = \frac{\text{weight}}{\text{centrifugal force}} = \frac{g\,r}{v^2} \ldots\ldots\ldots\ldots(1.)$$

where $r = \mathrm{A\,B}$. Let n be the *number of turns per second* of the pendulum; then

$$v = 2\,\pi\,n\,r;$$

and therefore, making $\overline{\mathrm{B\,C}} = h$,

$$h = \frac{g\,r^2}{v^2} = \frac{g}{4\,\pi^2\,n^2}$$

$$= \text{(in the latitude of London)}\,\frac{0{\cdot}8154 \text{ foot}}{n^2} = \frac{9{\cdot}7848 \text{ inches}}{n^2}\ldots\ldots(2.)$$

When the speed of revolution varies, the inclination of the pendulum varies, so as to adjust the height to the varying speed.

540. **Deviating Force in Terms of Angular Velocity.**—If the radius of curvature of the path of a revolving body be regarded as a sort of *arm* of constant or variable length at the end of which the body is carried, the angular velocity of that arm is given by the expression,

$$a = \frac{v}{r}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.)$$

Let $a\,r$ be substituted for v in the value of deviating force of Article 537, and that value becomes

$$\mathrm{Q} = \frac{\mathrm{W}\,a^2\,r}{g}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.)$$

In the case of a body revolving with uniform velocity in a circle, like the bob A of the revolving pendulum of Article 539, $a = 2\pi n$, where n is the number of revolutions per second, so that

$$Q = \frac{4\pi^2 W n^2 r}{g}; \dots\dots\dots\dots\dots\dots (3.)$$

from which equation the height of a revolving pendulum might be deduced with the same result as in the last Article.

541. **Rectangular Components of Deviating Force.**—*First Demonstration.* Let O in fig. 234 be the centre of the circular path E F G H of a body revolving in a circle with an uniform velocity, through which centre draw rectangular axes, O X and O Y, in the plane of revolution. Let the angle $\angle$ X O A, which at any instant the radius vector of the revolving body makes with the axis of x, be denoted by θ. Let

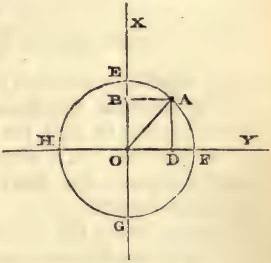

Fig. 234.

$$\left.\begin{array}{l}\overline{AD} = x = r\cdot\cos\theta,\text{ and}\\ \overline{AB} = y = r\cdot\sin\theta,\end{array}\right\}(1.)$$

be the rectangular co-ordinates of the revolving body at any instant. Let Q_x, Q_y, be the components of the deviating force, parallel to O X and O Y respectively. Then from the obvious proportion between the magnitudes of those components,

$$Q : Q_x : Q_y :: r : x : y, \dots\dots\dots\dots\dots\dots (2.)$$

combined with the equation 2 of Article 540, follow the values of those components,

$$Q_x = -\frac{W a^2 x}{g};\ Q_y = -\frac{W a^2 y}{g} \dots\dots\dots\dots (3.)$$

Those two components have the negative sign affixed, because they represent forces tending to diminish the co-ordinates x and y, to which they are proportional.

Second Demonstration.—The same result may be obtained, though less simply, by the second method described in Article 530, as follows:—Let intervals of time, t, be reckoned from an instant when the revolving body is at E. Then $\theta = a\,t$, and the values of the co-ordinates x and y, in terms of the time, are

$$x = r\cos a\,t;\ y = r\sin a\,t \dots\dots\dots\dots\dots\dots (4.)$$

The components of the velocity of the body are,

$$\frac{dx}{dt} = -ar \sin at;\ \frac{dy}{dt} = ar \cos at, \ldots\ldots\ldots\ldots (5.)$$

the velocity parallel to each co-ordinate being proportional to the other. The components of the variation of motion are

$x = r \cos at$

$$\left.\begin{aligned} \frac{d^2 x}{dt^2} &= -a^2 r \cos at = -a^2 x; \\ \frac{d^2 y}{dt^2} &= -a^2 r \sin at = -a^2 y; \end{aligned}\right\} \ldots\ldots\ldots\ldots (6.)$$

which being multiplied by the mass $\frac{W}{g}$, reproduce the components of the deviating force as before given in equation 3.

542. **Straight Oscillation** is the motion performed by a body which moves to and fro in a straight line, alternately to one side and to the other of a central point; and in order that this motion may take place, the body must be urged at each instant towards the central point.

In most cases, the force so acting on the oscillating body is either exactly or very nearly proportional to its *displacement*, or distance from the central point of equilibrium; that is to say, that force follows the law of one of the rectangular components of the deviating force of a body revolving uniformly in a circle once for each double oscillation of the oscillating body.

In fig. 234, let a body B, equal in weight to the body A, start at the same instant from E, and oscillate to and fro along the diameter E G, while A revolves in the circle E F G H. Then if B is urged towards the centre O with a force at each instant proportional to its distance from that point, and given by the equation

$$Q_x = -\frac{W a^2 x}{g}, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

being equal to the parallel component of the deviating force of A, B will *accompany* A in its motion parallel to O X; both those bodies being at each instant in the same straight line B A || O Y at the distance

$$x = r \cos at = r \cos \theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.)$$

from O: the velocity of B being at each instant equal to the parallel component of the velocity of A; that is to say,

$$\frac{dx}{dt} = -ar \sin at = -ar \sin \theta; \ldots\ldots\ldots\ldots (3.)$$

and each *double oscillation* of B, from E to G and back again to E, being performed in the same time with a revolution of A; that is in the time

from Eq 3 Art. 540.

$$\frac{1}{n} = \frac{2\pi}{a} = 2\pi\sqrt{\frac{r\,\mathrm{W}}{g\,\mathrm{Q}}};\dots\dots\dots\dots\dots\dots(4.)$$

where r is the *semi-amplitude* of the oscillation, $\overline{\mathrm{O\,E}} = \overline{\mathrm{O\,G}}$, Q is the corresponding greatest magnitude of the force urging the body towards O, being the same with the entire deviating force of A, and n is the number of double oscillations in a second. (The angle $\theta = a\,t$ is called the PHASE of the oscillation.)

The greatest value Q of the force which must act on B to produce n double oscillations of the semi-amplitude r in a second, is given by the equation

$$\mathrm{Q} = \frac{\mathrm{W}\,a^2\,r}{g} = \frac{4\,\pi^2\,\mathrm{W}\,n^2\,r}{g},\dots\dots\dots\dots\dots\dots(5.)$$

being similar to equation 3 of Article 540.

Revolution in a circle may be regarded as compounded of two oscillations of equal amplitude, in directions at right angles to each other.

543. **Elliptical Oscillations or Revolutions** compounded of two straight oscillations of equal periods, but unequal amplitudes, may be performed by a body urged towards a central point by a force proportional to its distance from that point. In fig. 235, let A be the position of the body at any instant; let $\mathrm{O\,A} = \varrho$, and let the force urging the body towards O be

$$\mathrm{F} = \frac{\mathrm{W}\,b^2\,\varrho}{g},\dots\dots\dots\dots\dots(1.)$$

b being a constant. Then the rectangular components of that force are

$$\mathrm{F}_x = -\frac{\mathrm{W}\,b^2\,x}{g};\ \mathrm{F}_y = -\frac{\mathrm{W}\,b^2\,y}{g};\dots(2.)$$

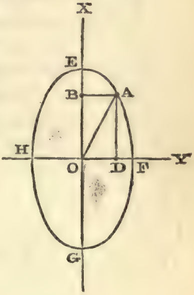

Fig. 235.

the former force being suited to maintain a straight oscillation parallel to O X, and the latter, a straight oscillation parallel to O Y, the period of a double oscillation in either case being the same, viz.:—

$$\frac{1}{n} = \frac{2\pi}{b},\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(3.)$$

according to equation 4 of Article 542. Hence let $x_1 = \overline{\mathrm{O\,E}} = \overline{\mathrm{O\,G}}$ be the semi-amplitude of the former straight oscillation, and $y_1 =$

O F = O H that of the latter; then at any instant the co-ordinates of the body will be

$$x = x_1 \cos b\,t;\ y = y_1 \sin b\,t; \ldots\ldots\ldots\ldots\ldots (4.)$$

which equations being respectively divided by x_1 and y_1, the results squared, and the squares added together, give

$$\frac{x^2}{x_1^2} + \frac{y^2}{y_1^2} = 1; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5.)$$

the well known equation of an ellipse described about O as a centre with the semi-axes x_1, y_1. The components of the velocity of the body at any instant are

Eq. 4

$$\left.\begin{aligned} \frac{d\,x}{d\,t} &= -b\,x_1 \sin b\,t = -b\frac{x_1}{y_1}y; \\ \frac{d\,y}{d\,t} &= b\,y_1 \cos b\,t = b\frac{y_1}{x_1}x. \end{aligned}\right\} \ldots\ldots\ldots\ldots (6.)$$

544. A **Simple Oscillating Pendulum** consists of an indefinitely small weight A, fig. 236, hung by a cord or rod of insensible weight A C from a point C, and swinging in a vertical plane to and fro on either side of a central point D vertically below C. The path of the weight or *bob* is a circular arc, A D E.

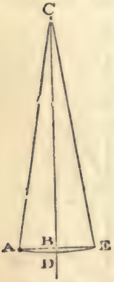

Fig. 236.

The weight W of the bob, acting vertically, may be resolved at any instant into two components, viz.:—

$$\mathrm{W} \cdot \cos \angle \mathrm{D\,C\,A} = \mathrm{W} \cdot \frac{\overline{\mathrm{B\,C}}}{\overline{\mathrm{C\,A}}},$$

acting along C A, and balanced by the tension of the rod or cord, and

$$\mathrm{W} \cdot \sin \angle \mathrm{D\,C\,A} = \mathrm{W} \cdot \frac{\overline{\mathrm{A\,B}}}{\overline{\mathrm{C\,A}}},$$

acting in the direction of a tangent to the arc, towards D, and unbalanced. The motion of A depends on the latter force.

When the arc A D E is small compared with the length of the pendulum A C, it very nearly coincides with the chord A B E; and the horizontal distance A B, to which the moving force is proportional, is very nearly equal to the distance of the bob from D, the central point of its oscillations. Hence the bob is very nearly in the condition of straight oscillation described in Article 542; and the time which it occupies in making a *double oscillation* is therefore found approximately by means of equation 4 of that Article, viz.:—

$$\frac{1}{n} = 2\,\pi \sqrt{\frac{r\,\mathrm{W}}{g\,\mathrm{Q}}},$$

where r denotes the semi-amplitude, and Q the maximum value of $\mathrm{W} \cdot \frac{\overline{\mathrm{A\,B}}}{\overline{\mathrm{C\,A}}}$. But if the length of the pendulum, $\overline{\mathrm{C\,A}}$, be made $= l$, we have

Q = Dev, F. Q [illegible]

$$\frac{\mathrm{Q}}{\mathrm{W}} = \text{max. } \frac{\overline{\mathrm{A\,B}}}{\overline{\mathrm{C\,A}}} = \frac{r}{l}, \text{ nearly};$$

whence, approximately, for small arcs of oscillation,

$$\left. \begin{array}{l} \dfrac{1}{n} = 2\,\pi \sqrt{\dfrac{l}{g}}; \text{ and} \\ l = \dfrac{g}{4\,\pi^2\,n^2}; \end{array} \right\} \text{..................(1.)}$$

which being compared with equation 2 of Article 539, shows, *that the length of a simple oscillating pendulum, making a given number of small double oscillations in a second, is sensibly equal to the height of a revolving pendulum, making the same number of revolutions in a second.*

When the amplitude of oscillation becomes of considerable magnitude, the period of oscillation is no longer sensibly independent of the length of the arc, but becomes longer for greater amplitudes, according to a law which can be expressed by an elliptic function, but which it is unnecessary to explain in this treatise. (See Legendre, *Traité des Fonctions elliptiques*, vol. i., chap. viii.)

545. **Cycloidal Pendulum.**—In order that the oscillations of a simple pendulum may be exactly *isochronous* (or of equal duration) for all amplitudes, the bob must oscillate in a curve, the lengths of whose arcs, measured from its lowest point, are proportional to the sines of their angles of declivity at their upper ends, to which sines the moving forces at those upper ends are proportional. That this may be the case, the radius of curvature at each point of the curve must be proportional to the cosine of the declivity: the greatest radius of curvature, at the lowest point of the curve, being equal to l, as given by equation 1 of Article 544; and from Article 390, case 3, equation 6, it appears that such a curve is a cycloid, traced by a rolling circle whose radius is

$$r_0 = \frac{l}{4} \text{..............................(1.)}$$

It is well known that a cycloid is the involute of an equal and similar cycloid. Hence, in fig. 237, let C F, C G, be a pair of cycloidal *cheeks*, described by rolling a circle of the radius r_0 on a horizontal line traversing C; let C A be a flexible line, fixed at C, and having a bob at A, its length being $l = 4 r_0 = \overline{CD}$ = the length of each of the semicycloids C F, C G. Then as the pendulum C A swings between the cycloidal cheeks, the bob oscillates in an arc of the cycloid F D G; its double oscillations, for all amplitudes, have exactly the periodic time given by equation 1 of Article 544, being that of a revolution of a revolving pendulum of the height $\overline{CD}$; and the motion of the bob in its cycloidal path follows the law of straight oscillations described in Article 542.

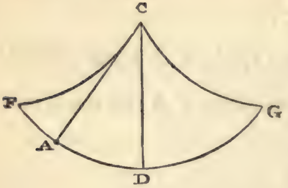

Fig. 237.

546. **Residual Forces.**—If two bodies be acted upon at every instant by unbalanced forces which are parallel in direction, and proportional to the masses of the bodies in magnitude, the variations of the motions of those two bodies, relatively to a fixed body, whether by change of velocity or by deviation, are simultaneous and equal; so that their motion, relatively to each other, is the same with that of a pair of bodies acted upon by no force or by balanced forces; that is, according to the first law of motion, Article 510, that motion is none or uniform.

If two bodies, A and B, be acted upon by any unbalanced forces whatsoever, and if from the force acting on B there be *taken away* a force parallel to that acting on A, and proportional to the mass of B (in other words, if with the actual force acting on B there be combined a force equal and opposite to that which would make the motion of B change in the same manner with that of A), then the resultant or *residual* unbalanced force acting on B is that corresponding to *the variations of the motion of* B *relatively to* A.

This is the exact statement of the case of a body near the earth's surface. From the *total attraction* between the body and the earth is to be taken away the *deviating force* necessary to make the body accompany the earth's surface in its motion, by revolving in a circle round the earth's axis once in a sidereal day (Article 352). The *residual force* is the weight of the body, $W = g\ m$, which regulates its motions *relatively to the earth's surface.* Thus the variations of the co-efficient g in different localities of the earth's surface, at different elevations, expressed by the formulæ of Article 531, are due partly to variations of attraction, and partly to variations of deviating force.

When bodies are carried in a ship or vehicle, and are free to move with respect to it, then when the ship or vehicle varies its motion, the bodies in question perform motions relatively to the ship or vehicle, such as would, in the case of the uniform motion of the ship or vehicle, be produced by the application to the bodies of forces equal and contrary to those which would make them accompany the ship or vehicle in the variations of its motion.

SECTION 3.— *Transformation of Energy.*

547. The **Actual Energy** of a moving body relatively to a fixed point is the product of the *mass* of the body into *one-half* of the *square of its velocity*, or, as Article 533 shows, the product of the *weight* of the body into the *height* due to its velocity; that is to say, it is represented by

$$\frac{m v^2}{2} = \frac{W v^2}{2 g} \ldots\ldots\ldots\ldots\ldots\ldots(1.)$$

The product $m v^2$, the double of the actual energy of a body, was formerly called its *vis-viva*. Actual energy, being the product of a *weight* into a *height*, is expressed, like potential energy and work, in *foot-pounds* (Article 513, 514).

548. **Components of Actual Energy.**—The actual energy of a body (unlike its momentum) is essentially positive, and irrespective of direction. Let the velocity v be resolved into three components, $\frac{d x}{d t}$, $\frac{d y}{d t}$, $\frac{d z}{d t}$, parallel to three rectangular axes; then the quantities of actual energy due to those three components respectively are

$$\frac{W}{2 g} \cdot \frac{d x^2}{d t^2}; \frac{W}{2 g} \cdot \frac{d y^2}{d t^2}; \frac{W}{2 g} \cdot \frac{d z^2}{d t^2}.$$

But the square of the resultant velocity is the sum of the squares of its three components, or

$$v^2 = \frac{d x^2}{d t^2} + \frac{d y^2}{d t^2} + \frac{d z^2}{d t^2};$$

therefore the actual energy of the body is simply the *sum* of the actual energies due to the rectangular components of its velocity.

549. **Energy of Varied Motion.**—THEOREM I. *A deviating force produces no change in a body's actual energy,* because such force produces change of direction only, and not of velocity; and actual energy is irrespective of direction, and depends on velocity only.

THEOREM II. *The increase of actual energy produced by an unbalanced effort is equal to the potential energy exerted.* This theorem is a consequence of the second law of motion, deduced as follows :—

Let $m = W \div g$, be the mass of a moving body acted upon by an effort P, and a resistance R, the effort being the greater, so that there is an *unbalanced effort* $P - R$; and in the first place let that unbalanced effort be constant. Then the body is uniformly accelerated; and if its velocity at the beginning of a given interval of time Δt is v_1, and its velocity at the end of that interval v_2, the increase of the body's momentum is

ch. of M.

$$\frac{W}{g}(v_2 - v_1) = (P - R)\,\Delta t \text{.................}(1.)$$

Because of the uniformity of the acceleration of the body, its mean velocity is $\frac{v_1 + v_2}{2}$, and the distance traversed by it is

$$\Delta s = \frac{v_1 + v_2}{2} \cdot \Delta t.$$

$\frac{v_1 + v_2}{2} = \frac{\Delta s}{\Delta t}$

Let both sides of equation 1 be multiplied by that mean velocity; the following equation is obtained :—

$$\frac{W(v_2^2 - v_1^2)}{2g} = (P - R)\,\Delta s\,;\text{..................}(2.)$$

now the first side of this equation is the *increase of the body's actual energy*, and the second is the *potential energy exerted* by the unbalanced effort; and those two quantities are equal.—Q. E. D.

When the unbalanced effort varies, let $d\,s$ be taken to denote a distance in which it varies less than in any given proportion, and $d \cdot v^2$ the change in the square of the velocity in that distance; then

Differentiate

$$\frac{W\,d \cdot v^2}{2g} = \frac{W\,v\,d\,v}{g} = (P - R)\,d\,s;\text{.............}(3.)$$

or if s_1, s_2, denote the two extremities of a finite portion of the body's path,

$$\frac{W \cdot (v_2^2 - v_1^2)}{2g} = \int_{s_1}^{s_2} (P - R)\,d\,s\text{............}(3\ \text{A.})$$

Theorem III. *The diminution of actual energy produced by an unbalanced resistance is equal to the work performed in moving against the resistance.* This is a consequence of the second law of motion, demonstrated by considering R to be greater than P in the equations of the preceding theorem; so that equation 1 becomes

$$\frac{W}{g}(v_1 - v_2) = (R - P)\,\Delta t;\text{.................}(4.)$$

equation 2 becomes

$$\frac{W\,(v_1^2 - v_2^2)}{2\,g} = (R - P)\,\Delta\,s;\ldots\ldots\ldots\ldots\ldots\ldots(5.)$$

and equation 3 and 3 A become

$$-\frac{W\,d\cdot v^2}{2\,g} = -\frac{W\,v\,d\,v}{g} = (R - P)\,d\,s\ldots\ldots\ldots\ldots(6.)$$

$$\frac{W\,(v_1^2 - v_2^2)}{2\,g} = \int_{s_1}^{s_2}(R - P)\,d\,s\ldots\ldots\ldots\ldots(6\ \text{A.})$$

550. **Energy Stored and Restored.**—A body alternately accelerated and retarded, so as to be brought back to its original speed, performs work by means of its retardation exactly equal in amount to the potential energy exerted in producing its acceleration; and that amount of energy may be considered as *stored* during the acceleration, and *restored* during the retardation.

551. The **Transformation of Energy** is a term applied to such processes as the expenditure of potential energy in the production of an equal amount of actual energy, and *vice versa.*

552. The **Conservation of Energy in Varied Motion** is a fact or principle expressed by combining the Theorems II. and III. of Article 549 with the definition of stored and restored energy of Article 550, and may be stated as follows:—*in any interval of time during a body's motion, the potential energy exerted, added to the energy restored, is equal to the energy stored added to the work performed.* This principle, expressed in the form of a differential equation, is as follows:— Ch. in A. Energy

$$P\,d\,s - \frac{W\,v\,d\,v}{g} - R\,d\,s = 0;\ldots\ldots\ldots\ldots\ldots\ldots(1.)$$

B[illegible] of stat[illegible]

which includes equations 3 and 6 of Article 549; and in the form of an integral equation,

$$\int P\,d\,s - \frac{W\,(v_2^2 - v_1^2)}{2\,g} - \int R\,d\,s = 0\ldots\ldots\ldots\ldots(2.)$$

553. **Periodical Motion.**—If a body moves in such a manner that it periodically returns to its original velocity, then at the end of each period, the entire variation of its actual energy is nothing; and in each such period the whole potential energy exerted is equal to the whole work performed, exactly as in the case of a body moving uniformly (Article 518).

554. **Measures of Unbalanced Force.**—From Articles 530 and 531, and from Article 549, it appears that the magnitude of an unbalanced force may be computed in two ways,—either from the change of momentum which it produces by acting for a given time,

or by the change of energy which it produces by acting along a given distance. Both those ways of computing are expressed in the following equation:—

$$P = \frac{W}{g} \cdot \frac{dv}{dt} = \frac{W}{g} \cdot \frac{v\,dv}{ds},$$

and each is a necessary consequence of the other; yet in former times a fallacy prevailed that they were inconsistent and contradictory, and a bitter controversy long raged between their respective partizans.

555. **Energy due to Oblique Force.**—It has already been stated in Chapter I. of this Part, and especially in Article 520, that if an unbalanced force F acts on a body while it moves through the distance ds, making the angle θ with the direction of the force, the product

$$F \cos \theta \cdot ds$$

represents the energy exerted, if θ is acute, or the work performed, if θ is obtuse, during that motion. Now that product may be treated mathematically in two ways: either as the product of $F \cos \theta = P$ (or, as the case may be, $F \cos (\pi - \theta) = R$), the component of the force along the direction of motion, into ds, the motion; or as the product of F, the entire force, into $\cos \theta \cdot ds$, the component of the motion in the direction of the force. The former method is that pursued in the preceding Articles; but occasionally the latter may be the more convenient. For example, when the force F is either directed towards or from a central point, or is always perpendicular to a given surface; let z denote the distance of the body at any instant from the central point, or its normal distance from the given surface, as the case may be; then

$$dz = \cos \theta \cdot ds \text{.............................(1.)}$$

is the component of the motion of the body in the direction of z.

The force F is to be treated as positive or negative according as it tends to increase or diminish z. Then if v_1, v_2, be the velocities of the body, and z_1, z_2, its distances from the given point or surface at the beginning and end of a given interval, the change of its actual energy in that interval is

$$\frac{W(v_2^2 - v_1^2)}{2g} = \int_{s_1}^{s_2} F \cos \theta \cdot ds = \int_{z_1}^{z_2} F\,dz\text{;.........(2.)}$$

and if F is either constant, or a function of z only, the velocity of v varies with z alone.

This principle, as applied to the force of gravity near the earth's surface, has already been illustrated in Articles 533, 534, and 535;

for in that case, z denotes the elevation of the body above a given level, $F = -W$ (because it tends to diminish z), and therefore

Eq. 5. p. 489.

$$\frac{v_2^2 - v_1^2}{2g} = z_1 - z_2, \text{..........................} (3.)$$

as was formerly proved by another process. p. 489

556. A **Reciprocating Force** is a force which acts alternately as an effort and as an equal and opposite resistance, according to the direction of motion of the body. Such a force is the weight of a body which alternately rises and falls; or the attraction of a body towards a point from which its distance periodically changes. Such a force is the force F in the last Article, when it is constant, or a function of z only; and such is the elasticity of a perfectly elastic body. The work which a body performs in moving against a reciprocating force is employed in increasing its own potential energy, and is not lost by the body.

557. The **Total Energy** of a body is the sum of its potential and actual energies. It is evident, that if at each point of the course of a moving body its total energy, or capacity for performing work, be added to the work which it has already performed, the sum must be a constant quantity, and equal to the INITIAL ENERGY which the body possessed before beginning to perform work. If a body performs no work, its total energy is constant; and the same is the case if its work consists only in *moving itself to a place where its potential energy is greater*, that is, moving against a reciprocating force; and the increase of potential energy so obtained being taken into account, balances the work performed in obtaining it.

Example 1. If a body whose weight is W be at a height z_1 above the ground, and be moving with the velocity v_1 in any direction, its initial total energy relatively to the ground is

$$W\left(z_1 + \frac{v_1^2}{2g}\right); \text{..........................} (1.)$$

of which $W z_1$ is potential and $W\frac{v_1^2}{2g}$ actual. Supposing the body to have moved without any resistance except such as may arise from a component of its own weight, which is a reciprocating force, to a different height z_2 above the ground, its total energy relatively to the ground is now

$$W\left(z_2 + \frac{v_2^2}{2g}\right); \text{..........................} (2.)$$

being the same in amount as before, but differently divided between the actual and potential forms.

Example II. Should the motion of the body be opposed by a resistance such as friction, which is not a reciprocating force, then the total energy in the second position of the body is diminished to

$$\mathrm{W}\left(z_2+\frac{v_2^2}{2\,g}\right)=\mathrm{W}\left(z_1+\frac{v_1^2}{2\,g}\right)-\int \mathrm{R}\,d\,s\,\ldots\ldots\ldots(3.)$$

Example III. Let a body oscillate (as in Article 542) in a straight line traversing a central point towards which the body is urged by a force varying as the distance from the point; let x_1 be the semi-amplitude of oscillation, x the displacement at any instant, $-\mathrm{Q}_1$ the greatest value of the moving force, so that $\frac{-\mathrm{Q}_1 x}{x_1}$ is the value for the displacement x. Then when the body is at its extreme displacement, its actual energy is nothing; and its total energy, which is all potential, is

$$\frac{\mathrm{Q}_1}{x_1}\int_0^{x_1} x\,d\,x=\frac{\mathrm{Q}_1\,x_1}{2}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4.)$$

When the body is in the act of passing the central point, its potential energy is nothing, and its total energy, which is now all actual, is in amount the same as before, viz.:—

$$\frac{\mathrm{W}\,v_0^2}{2\,g}=\frac{\mathrm{Q}_1\,x_1}{2};\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5.)$$

v_0 being the maximum velocity. At any intermediate point, the total energy, partly actual and partly potential, is still the same, being

$$\frac{\mathrm{W}\,v^2}{2\,g}+\frac{\mathrm{Q}\,x^2}{2\,x_1}=\frac{\mathrm{W}\,v_0^2}{2\,g}\cdot\sin^2 at+\frac{\mathrm{Q}_1\,x_1}{2}\cos^2 at=\frac{\mathrm{Q}_1\,x_1}{2};\ldots\ldots(6.)$$

where, as before, $a=2\,\pi\,n$; n being the number of double oscillations in a second. For the elliptic oscillations of Article 543, the total energy of the body is at each instant the sum of the quantities of energy due to the two straight oscillations of which the elliptic oscillation is compounded; and for a body revolving in a circle, and urged towards the centre by a deviating force proportional to the radius vector, the total energy relatively to the centre is one-half actual and one-half potential, viz.:—

$$\frac{\mathrm{W}\,v^2}{2\,g}+\frac{\mathrm{Q}\,r}{2}=\mathrm{Q}\,r\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7.)$$

SECTION 4.—*Varied Translation of a System of Bodies.*

558. **Conservation of Momentum.**—THEOREM. *The mutual actions of a system of bodies cannot change their resultant momentum.* (Resultant momentum has been defined in Article 524.) Every force is a pair of equal and opposite actions between a pair of bodies; in any given interval of time it constitutes a pair of equal and opposite impulses on those bodies, and produces equal and opposite momenta. Therefore the momenta produced in a system of bodies by their mutual actions neutralize each other, and have no resultant, and cannot change the resultant momentum of the system.

559. **Motion of Centre of Gravity.**—COROLLARY. *The variations of the motion of the centre of gravity of a system of bodies are wholly produced by forces exerted by bodies external to the system;* for the motion of the centre of gravity is that which, being multiplied by the total mass of the system, gives the resultant momentum, and this can be varied by external forces only.

It follows that in all dynamical questions in which the mutual actions of a certain system of bodies are alone considered, the centre of gravity of that system of bodies may be correctly treated as a point whose motion is none or uniform; because its motion cannot be changed by the forces under consideration.

79 560. The **Angular Momentum,** relatively to a fixed point, of a body having a motion of translation, is the product of the momentum of the body into the perpendicular distance of the fixed point from the line of direction of the motion of the body's centre of gravity at the instant in question; and is obviously equal to the product of the mass of the body into double the area swept by the radius vector drawn from the given point to its centre of gravity in an unit of time. Let m be the mass of the body, v its velocity, l the length of the before-mentioned perpendicular; then

$$m v l = \frac{\mathrm{W} v l}{g}$$

is the angular momentum relatively to the given point.

Angular momenta are compounded and resolved like forces, each angular momentum being represented by a line whose length is proportional to the magnitude of the angular momentum, and whose direction is perpendicular to the plane of the motion of the body and of the fixed point, and such, that when the motion of the body is viewed from the extremity of the line, the radius vector of the body seems to have right-handed rotation. The direction of such a line is called the *axis* of the angular momentum which it represents. The *resultant angular momentum* of a system of bodies is the resultant of all their angular momenta relatively to their common centre of gravity; and the axis of that resultant angular momentum is called the *axis of angular momentum* of the system. The term *angular momentum* was introduced by Mr. Hayward.

561. **Angular Impulse** is the product of the moment of a couple of forces (Article 29) into the time during which it acts. Let F be the force of a couple, l its leverage, and $d\,t$ the time during which it acts, then

$$\mathrm{F}\,l\,d\,t$$

is angular impulse. Angular impulses are compounded and resolved like the moments of couples.

562. **Relations of Angular Impulse and Angular Momentum.**—THEOREM. *The variation, in a given time, of the angular momentum of a body, is equal to the angular impulse producing that variation, and has the same axis.* This is a consequence which is deduced from the second law of motion in the following manner:—Conceive an unbalanced force F to be applied to a body m, and an equal, opposite, and parallel force, to a fixed point, during the interval $d\,t$; and let l be the perpendicular distance from the fixed point to the line of action of the first force. Then the couple in question exerts the angular impulse

$$\mathrm{F}\,l\,d\,t.$$

At the same time, the body m acquires a variation of momentum in the direction of the force applied to it, of the amount

$$m\,d\,v = \mathrm{F}\,d\,t;$$

so that relatively to the fixed point, the variation of the body's angular momentum is

$$m\,l\,d\,v = \mathrm{F}\,l\,d\,t; \qquad (1.)$$

being equal to the angular impulse, and having the same axis.—Q. E. D.

563. **Conservation of Angular Momentum.**—THEOREM. *The resultant angular momentum of a system of bodies cannot be changed in magnitude, nor in the direction of its axis, by the mutual actions of the bodies.*

Considering the common centre of gravity of the system of bodies as a fixed point, conceive that for each force with which one of the bodies of the system is urged in virtue of the combined action of all the other bodies upon it, there is an equal, opposite, and parallel force applied to the common centre of gravity, so as to form a couple. The forces with which the bodies act on each other are equal and opposite in pairs, and their resultant is nothing; therefore, the resultant of the ideal forces conceived to act at the common centre of gravity is nothing, and the supposition of these forces does not effect the equilibrium or motion of the system. Also, the resultant of all the couples thus formed is nothing; therefore, the resultant of their angular impulses is nothing; therefore, the resultant of the several variations of angular momentum produced by those angular impulses is nothing; therefore, the resultant angular momentum of the system is invariable in amount and in the direction of its axis.—Q. E. D.

This theorem is sometimes called the *principle of the conservation of areas.* When applied to a system consisting of two bodies only, it forms one of the laws discovered by Kepler, by observation of the motions of the planets.

In considering the relative motions of a system of bodies as depending on their mutual actions only, the axis of angular momentum may be treated as a *fixed direction,* as already stated in Article 348. A plane perpendicular to the axis of angular momentum is called by some writers the *invariable plane.* The nearest approach to an absolutely fixed direction yet known is the invariable axis of the discovered bodies of the solar system.

564. **Actual Energy of a System of Bodies.**—THEOREM. *The actual energy of a system of bodies relatively to a point external to the system, is the sum of the actual energies of the bodies relatively to their common centre of gravity, added to the actual energy due to the motion of the mass of the whole system with a velocity equal to that which its centre of gravity has relatively to the external point.*

Let the motion of each of the bodies, and of their common centre of gravity, relatively to the external point, be resolved into three rectangular components. Let m be any one of the masses, and u, v, w, the components of its velocity relatively to the external point; let $\Sigma \cdot m$ be the mass of the whole system, and u_0, v_0, w_0, the components of the velocity of its centre of gravity relatively to the external point.

Conceive the motion of each of the bodies to be resolved into two parts; that which it has *in common with the centre of gravity* relatively to the external point, and that which it has *relatively to the centre of gravity.* The component velocities of the first part are

$$u_0, v_0, w_0;$$

and those of the second part

$$u - u_0 = u'; \; v - v_0 = v'; \; w - w_0 = w';$$

so that the components of the whole motion of the body may be represented by

$$u = u_0 + u'; \; v = v_0 + v'; \; w = w_0 + w'.$$

Then the actual energy of the system relatively to the external point is

$$\tfrac{1}{2} \Sigma \cdot m \left\{(u_0 + u')^2 + (v_0 + v')^2 + (w_0 + w')^2\right\};$$

which being developed, and common factors removed outside the sign of summation, gives

$$\begin{gathered}\tfrac{1}{2}(u_0^2 + v_0^2 + w_0^2) \cdot \Sigma\, m \\ + u_0 \cdot \Sigma \cdot m\, u' + v_0 \cdot \Sigma \cdot m\, v' + w_0 \cdot \Sigma \cdot m\, w' \\ + \tfrac{1}{2}\, \Sigma \cdot m\,(u'^2 + v'^2 + w'^2).\end{gathered}$$

But in Article 524 it has been shown, that the resultant momentum of a system of bodies relatively to their common centre of gravity is nothing; that is to say,

$$\Sigma \cdot m\, u' = 0;\ \Sigma \cdot m\, v' = 0;\ \Sigma \cdot m\, w' = 0;$$

so that the above expression for the actual energy of the system becomes simply

$$\tfrac{1}{2}(u_0^2 + v_0^2 + w_0^2) \cdot \Sigma\, m + \tfrac{1}{2}\Sigma \cdot m\,(u'^2 + v'^2 + w'^2); \ldots\ldots (1.)$$

of which the first term is *the actual energy of the whole mass of the system due to the motion of the centre of gravity relatively to the external point*, and the second term is *the sum of the actual energies of the bodies relatively to their common centre of gravity.*—Q. E. D.

Those two parts of the actual energy of a system may be distinguished as the *external* and *internal* actual energy.

COROLLARY. *The mutual actions of a system of bodies change their internal actual energy alone.*

565. **Conservation of Internal Energy.**—LAW. *The total internal energy, actual and potential, of a system of bodies, cannot be changed by their mutual actions.* This is a proposition made known partly by reasoning and partly by experiment. The total internal energy of a system is the sum of the total energies of the bodies of which it consists relatively to their common centre of gravity. It has been shown in Articles 549 to 557, that the total energy of a single body can be diminished only by performing work against a resistance which is not a reciprocating force; in other words, against an *irreversible* or *passive* resistance.

Now it has been proved by experiment, that all work performed against passive resistances is accompanied by the production of an equal amount of energy in a different form (as when friction produces heat); therefore the total internal energy of a system of bodies cannot be changed by their mutual actions.—Q. E. D.

Although this law has become known in the first instance by experiment and observation, it can be shown to be necessary to the permanent existence of the universe as actually constituted.

566. **Collision** is a pressure of inappreciably short duration between two bodies. The most usual problem in cases of collision is, when two bodies whose masses are given move before the collision in one straight line with given velocities, and it is required to find

their velocities after the collision. The two bodies form a system whose resultant momentum and internal energy are each unaltered by the collision; but a certain fraction of the internal energy disappears as visible motion, and appears as vibration and heat. If the bodies are equal, similar, and perfectly elastic, that fraction is nothing.

Let m_1, m_2, be the masses of the two bodies, and u_1, u_2, their velocities before the collision, whose directions should be indicated by their signs. Then the velocity of their common centre of gravity is

$$u_0 = \frac{u_1 m_1 + u_2 m_2}{m_1 + m_2}; \qquad \ldots\ldots\ldots\ldots\ldots\ldots(1.)$$

and this is not altered by the collision; neither is the *external energy*, whose amount is

$$(m_1 + m_2) \frac{u_0^2}{2} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(2.)$$

The *internal energy* of the system of two bodies is

$$\frac{m_1 (u_1 - u_0)^2}{2} + \frac{m_2 (u_2 - u_0)^2}{2} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(3.)$$

When the bodies strike together, this actual internal energy is expended in altering the figures of the bodies at and near their surface of contact, in opposition to their elastic force. So soon as the relative motion of the bodies has been thus stopped, the elastic force begins to restore their figures, and drive them asunder; and if they were equal, similar, and perfectly elastic, it would reproduce all the energy of relative motion given by the formula 3, so that the bodies would separate with velocities relatively to their common centre of gravity, equal and opposite to their original velocities relatively to that point; that is to say, with the velocities

$$u_0 - u_1,\ u_0 - u_2,$$

relatively to the common centre of gravity, and the velocities

$$v_1 = 2\,u_0 - u_1,\ v_2 = 2\,u_0 - u_2, \qquad \ldots\ldots\ldots\ldots\ldots\ldots(4.)$$

relatively to the earth. But as a certain proportion, which may be denoted by $1 - k^2$, of the internal actual energy takes the forms of internal vibration and of heat, the internal actual energy due to visible motion after the collision is

$$\frac{k^2 m_1 (u_1 - u_0)^2}{2} + \frac{k^2 m_2 (u_2 - u_0)^2}{2}; \qquad \ldots\ldots\ldots\ldots\ldots(5.)$$

the velocities of the bodies, relatively to their common centre of gravity, after the collision, are

$$k\,(u_0-u_1),\ k\,(u_0-u_2)\,;$$

and their velocities relatively to the earth are

$$v_1=(1+k)\,u_0-k\,u_1\,;\ v_2=(1+k)\,u_0-k\,u_2 \ldots\ldots\ldots\ldots (6.)$$

Should the bodies be *perfectly soft*, or *inelastic*, $k=0$; in which case

$$v_1=v_2=u_0\,; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7.)$$

that is, the bodies do not fly asunder, but proceed together with the velocity of their common centre of gravity. (See Addendum, p. 512.)

567. The **Action of Unbalanced External Forces** on a system of bodies, considered as a whole, is to vary the resultant momentum and the resultant angular momentum. It has been shown in Article 60, that every system of forces can be reduced to a single force and a couple. The system of forces applied to a system of bodies is to be reduced to a single force acting through the centre of gravity of the system, and a couple, as shown in equations 5, 6, 7, 8, of Article 60; then in a given interval of time, the variation of resultant momentum of the system is equal to and in the direction of the impulse of the single resultant force, and the variation of angular momentum is equal to the angular impulse, and about the axis, of the resultant couple.

To express this by general equations, let the components of the momentum of any mass m belonging to the system, whose rectangular co-ordinates are x, y, z, be $m\frac{dx}{dt}$, $m\frac{dy}{dt}$, $m\frac{dz}{dt}$. Then the rates of variation of these components are

$$m\frac{d^2x}{dt^2},\ m\frac{d^2y}{dt^2},\ m\frac{d^2z}{dt^2} \ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

Also, the rectangular components of the angular momentum of that mass are

$$\text{about } x,\ m\left(z\frac{dy}{dt}-y\frac{dz}{dt}\right);\ \text{about } y,\ m\left(x\frac{dz}{dt}-z\frac{dx}{dt}\right);$$

$$\text{about } z,\ m\left(y\frac{dx}{dt}-x\frac{dy}{dt}\right); \ldots\ldots\ldots\ldots\ldots (2.)$$

whose rates of variation are

$$\left.\begin{matrix} m\left(z\frac{d^2y}{dt^2}-y\frac{d^2z}{dt^2}\right);\ m\left(x\frac{d^2z}{dt^2}-z\frac{d^2x}{dt^2}\right); \\ m\left(y\frac{d^2x}{dt^2}-x\frac{d^2y}{dt^2}\right). \end{matrix}\right\} \ldots\ldots (3.)$$

Let F_x, F_y, F_z, be the components of the force externally applied to a point whose co-ordinates are x, y, z. Then by the equality of the resultant impulse to the variation of resultant momentum,

$$\left.\begin{aligned} \Sigma\left\{F_x - m\frac{d^2x}{dt^2}\right\} = 0;\ \Sigma\left\{F_y - m\frac{d^2y}{dt^2}\right\} = 0; \\ \Sigma\left\{F_z - m\frac{d^2z}{dt^2}\right\} = 0; \end{aligned}\right\} \ldots(4.)$$

Sec. [illegible]

and by the equality of the resultant angular impulse to the variation of the resultant angular momentum,

$$\left.\begin{aligned} \Sigma\left\{z\left(F_y - m\frac{d^2y}{dt^2}\right) - y\left(F_z - m\frac{d^2z}{dt^2}\right)\right\} = 0; \\ \Sigma\left\{x\left(F_z - m\frac{d^2z}{dt^2}\right) - z\left(F_x - m\frac{d^2x}{dt^2}\right)\right\} = 0; \\ \Sigma\left\{y\left(F_x - m\frac{d^2x}{dt^2}\right) - x\left(F_y - m\frac{d^2y}{dt^2}\right)\right\} = 0; \end{aligned}\right\} \ldots\ldots(5.)$$

The use of those equations is to determine the effect of a given system of external forces on a system of bodies when the relations amongst the motions of those bodies are known, without taking into consideration the internal forces acting between the bodies, which latter forces it is sometimes difficult or impossible to determine until the effects of the external forces have first been found.

568. **Determination of the Internal Forces.**—When the relations which exist between the motion of the system as a whole,—that is, its resultant momentum and angular momentum,—and the motions of the several bodies of which it consists, are fixed by cinematical principles, then the motion of each body can be determined when the externally applied forces are known. *Then if, from the force externally applied to each body at each instant, there is taken away the force required to produce the change of motion of the body which takes place at that instant, the remainder must be balanced by, and equal and opposite to, the internal force acting on the body in question;* and this, which is the PRINCIPLE OF D'ALEMBERT, serves to determine the internal forces. Using the notation of the last Article, the components of the internal force applied to a given body of the system are

$$m\frac{d^2x}{dt^2} - F_x;\ m\frac{d^2y}{dt^2} - F_y;\ m\frac{d^2z}{dt^2} - F_z.$$

569. **Residual External Forces.**—If the resultant external force acting through the centre of gravity of a system of bodies be sup-

posed to be divided into parallel components, each applied to one of the bodies and proportional to the mass of the body to which it is applied, such will be the system of external forces required to make all the bodies of the system have equal and parallel motions at each instant in common with their centre of gravity. Then if the forces so determined be taken away from the forces actually applied to the several bodies, the residual external forces, being combined with the internal forces, will constitute those forces which regulate the motions of the bodies relatively to their common centre of gravity considered as a fixed point.

ADDENDUM TO ARTICLE 566, PAGE 510.

Collision.—It was formerly supposed that the disappearance of energy after collision was wholly due to imperfect elasticity, and that any two perfectly elastic bodies would fly asunder after collision with a relative velocity equal to their relative velocity of approach before collision. But M. de St. Venant showed that, except when the bodies are similar and equal, a certain quantity of energy disappears, even in perfectly elastic bodies, in producing internal vibrations of each body. The value of the co-efficient k, being the ratio of the relative velocity of the recoil to that of the approach, in the case of a pair of perfectly elastic prismatic bars, striking each other endwise, is given as follows: let a_1 and a_2 be the lengths of the bars; p_1 and p_2 their weights per unit of length; s_1 and s_2 the velocities of the transmission of sound (that is, of longitudinal vibrations) along them; let $\frac{a_1}{s_1} < \frac{a_2}{s_2}$; and also let $s_1 p_1 < s_2 p_2$; in other words, let $\frac{s_2}{s_1} < \frac{a_2}{a_1}$ and $> \frac{p_1}{p_2}$; then

$$k = 2 \cdot \frac{a_1 p_1 + a_2 p_2}{a_2 p_2} \cdot \frac{p_2 s_2}{p_1 s_1 + p_2 s_2} - 1 \ldots\ldots\ldots (8.)$$

As to the velocity of sound, see Article 615, page 563. The paper of M. de St. Venant is published in full in the *Journal des Mathematiques pures et appliquées*, 1867; and an abstract in English of the more simple of its results in *The Engineer* for the 15th February, 1867.

CHAPTER III.

ROTATIONS OF RIGID BODIES.

570. **The Motion of a Rigid Body,** or of a body which sensibly preserves the same figure, has already been shown in Part III., Chapter II., to be always capable of being resolved at each instant into a translation and a rotation; and by the aid of the principles explained in Section 3 of that chapter, the component rotation can always be conceived to take place about an axis traversing the centre of gravity of the body, and to be combined, if necessary, with a translation of the whole body in a curved or straight path along with its centre of gravity. The variations of the *momentum* of the translation, whether in amount or in direction, are due to the resultant force acting through the centre of gravity of the body, and are exactly the same with those of the momentum of the entire mass if it were concentrated at that centre; the variations of the *angular momentum* of the rotation are due to the resultant couple which is combined with that resultant force. The variations of *actual energy* are due to both causes.

When the translation of the centre of gravity of a rotating body, and its rotation about an axis traversing that centre, are known, the motion of every point in the body is determined by cinematical principles, which have been explained in Part III., Chapter II., Section 3; so that by the aid of D'Alembert's principle (Article 568) the internal forces acting amongst the parts of the body can be completely determined.

In the investigations of questions respecting the motions of rigid bodies, there are certain quantities, lines, and points, depending on the figures of the bodies, the mode of distribution of their masses, and the way in which their motions are guided, whose use facilitates the understanding of the subject and the computation of results, and which are related to each other by geometrical principles. These are, *moments of inertia*, *radii of gyration*, *moments of deviation*, and *centres of percussion*. Their geometrical relations are considered in the following section.

SECTION 1.—*On Moments of Inertia, Radii of Gyration, Moments of Deviation, and Centres of Percussion.*

571. The **Moment of Inertia** of an indefinitely small body, or "physical point," relatively to a given axis, is the product of the mass of the body, or of some quantity proportional to the mass, such as the weight, into the square of its perpendicular distance from the axis: thus in the following equation:—

$$\frac{I}{g} = m\,r^2 = \frac{W\,r^2}{g}, \dots\dots\dots\dots (1.)$$

r is the perpendicular distance of the mass m, whose weight is W, from a given axis; and the moment of inertia, according to the unit employed, is either I, or $I \div g$; the former, when the unit is the moment of inertia of an unit of *weight* at the end of an arm whose length is unity; and the latter, when the unit is the moment of inertia of an unit of *mass* at the end of the same arm. For the purposes of applied mechanics, the former is the more convenient unit, and will be employed in this treatise.

By an extension of the term "moment of inertia," it is applied to the product of any quantity, such as a volume, or an area, into the square of the distance of the point to which that quantity relates from a given axis, as has already been exemplified in Article 95, and in the theory of resistance to bending; but in the remainder of this treatise the term will be used in its strict sense, and according to the unit of measure already specified; that is, in British measures, moment of inertia will be expressed by the product of a certain number of *pounds avoirdupois* into the square of a certain number of *feet*.

The geometrical relations amongst moments of inertia, to which the present section refers, are independent of the unit of measure.

572. The **Moment of Inertia of a System of Physical Points**, relatively to a given axis, is the sum of the moments of inertia of the several points; that is,

$$I = \Sigma \cdot W\,r^2 \dots\dots\dots\dots (1.)$$

573. The **Moment of Inertia of a Rigid Body** is the sum of the moments of inertia of all its parts, and is found by integration; that is, by conceiving the body to be divided into small parts of regular figure, multiplying the mass of each of those parts into the square of the distance of its centre of gravity from the axis, adding the products together, and finding the value towards which their sum converges when the size of the small parts is indefinitely diminished. For example, let the body be conceived to be built up of rectangular

molecules, whose dimensions are $d\,x$, $d\,y$, and $d\,z$, the volume of each $d\,x\,d\,y\,d\,z$, and the mass of unity of volume w. Then

$$\mathrm{I} = \int\int\int r^2 w \cdot d\,x\,d\,y\,d\,z \ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

Hence follows the general principle which will afterwards be illustrated in special cases, that propositions relative to the geometrical relations amongst the moments of inertia of systems of points are made applicable to continuous bodies by substituting integration for ordinary summation; that is, for example, by putting $\int\int\int$ for Σ, and $w \cdot d\,x\,d\,y\,d\,z$ for W.

574. The **Radius of Gyration** of a body about a given axis is that length whose square is the *mean of all the squares* of the distances of the indefinitely small equal particles of the body from the axis, and is found by dividing the moment of inertia by the mass, thus,

$$\varrho^2 = \frac{\mathrm{I}}{\Sigma \cdot \mathrm{W}} = \frac{\Sigma \cdot \mathrm{W}\, r^2}{\Sigma \cdot \mathrm{W}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

When symbols of integration are used, this becomes

$$\varrho^2 = \frac{\int\int\int r^2 w \cdot d\,x\,d\,y\,d\,z}{\int\int\int w \cdot d\,x\,d\,y\,d\,z} \ldots\ldots\ldots\ldots\ldots\ldots (2.)$$

575. **Components of Moment of Inertia.**—Let the positions of the particles of a body be referred to three rectangular axes, one of which, O X, is that about which the moment of inertia is to be taken. Then the square of the radius vector of any particle is

$$r^2 = y^2 + z^2;$$

so that the moment of inertia round the axis of x is

$$\mathrm{I}_x = \Sigma \cdot \mathrm{W}\, y^2 + \Sigma \cdot \mathrm{W}\, z^2; \ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

that is to say, *the moment of inertia of a body round a given axis may be found by adding together the sum of the products of the masses of the particles, each multiplied by the square of each of its distances from a pair of planes cutting each other at right angles in the given axis.*

In the same manner it may be shown that the moments of inertia of the same body round the other two axes are given by the equations

$$\mathrm{I}_y = \Sigma \cdot \mathrm{W}\, z^2 + \Sigma \cdot \mathrm{W}\, x^2;\ \mathrm{I}_z = \Sigma \cdot \mathrm{W}\, x^2 + \Sigma \cdot \mathrm{W}\, y^2 \ldots (2.)$$

576. **Moments of Inertia Round Parallel Axes Compared.**—THEOREM. *The moment of inertia of a body about any given axis is equal to its moment of inertia about an axis traversing its centre of gravity parallel to the given axis, added to the moment of inertia about the given axis due to the whole mass of the body concentrated at its centre of gravity.*

Take the given axis for the axis of x, and any two planes traversing it at right angles to each other as the planes of $x\,y$ and $z\,x$; then, as in the preceding Article,

$$\mathrm{I}_x = \Sigma \cdot \mathrm{W}\, y^2 + \Sigma \cdot \mathrm{W}\, z^2.$$

Let y_0, z_0, be the perpendicular distances of the centre of gravity of the body from the two co-ordinate planes before mentioned; conceive a new axis to traverse that centre of gravity, parallel to the given axis; let two co-ordinate planes parallel to the original co-ordinate planes traverse that new axis; and let y', z', be the perpendicular distances of a given particle from those new co-ordinate planes. Then

$$y = y_0 + y'\,;\ z = z_0 + z'\,;$$

and introducing those values of the original co-ordinates into the value of I_x, we find

$$\begin{aligned} \mathrm{I}_x &= \Sigma \cdot \mathrm{W}\,(y_0 + y')^2 + \Sigma \cdot \mathrm{W}\,(z_0 + z')^2 \\ &= (y_0^2 + z_0^2)\,\Sigma\,\mathrm{W} \\ &\quad + 2\,y_0\,\Sigma \cdot \mathrm{W}\,y' + 2\,z_0\,\Sigma \cdot \mathrm{W}\,z' + \Sigma \cdot \mathrm{W}\,(y'^2 + z'^2); \end{aligned}$$

but because y' and z' are the distances of a particle from planes traversing the centre of gravity of the body,

$$\Sigma \cdot \mathrm{W}\,y' = 0\,;\ \Sigma \cdot \mathrm{W}\,z' = 0\,;$$

Statics

and the preceding equation is reduced to the following:—

$$\mathrm{I}_x = (y_0^2 + z_0^2)\,\Sigma\,\mathrm{W} + \Sigma \cdot \mathrm{W}\,(y'^2 + z'^2) \ldots\ldots\ldots\ldots (1.)$$

which expresses the theorem to be proved.

This theorem may be more briefly expressed as follows:—Let I_0 be the moment of inertia of a body about an axis traversing its centre of gravity in any given direction, and I the moment of inertia of the same body about an axis parallel to the former at the perpendicular distance r_0; then

$$\mathrm{I} = r_0^2 \cdot \Sigma\,\mathrm{W} + \mathrm{I}_0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.)$$

An analogous proposition for surfaces has been demonstrated in Article 95, Theorem V.

COROLLARY I. The radius of gyration (ϱ) of a body about any axis is equal to the hypotenuse of a right-angled triangle, of which the two legs are respectively equal to the radius of gyration of the body about an axis traversing the centre of gravity parallel to the given axis (ϱ_0), and to the perpendicular distance between these two axes (r_0). That is to say,

$$\varrho^2 = r_0^2 + \varrho_0^2 \text{.......................} (3.)$$

COROLLARY II. The moment of inertia of a body about an axis traversing its centre of gravity in a given direction, is less than the moment of inertia of the same body about any other axis parallel to the first.

COROLLARY III. The moments of inertia of a body about all axes parallel to each other, which lie at equal distances from its centre of gravity, are equal.

577. **Combined Moments of Inertia.**—THEOREM. *The combined moment of inertia of a rigidly connected system of bodies about a given axis, is equal to the combined moment of inertia which the system would have about the given axis, if each body were concentrated at its own centre of gravity, added to the sum of the several moments of inertia of the bodies, about axes traversing their respective centres of gravity, parallel to the given axis.*

Let W now denote the mass of *one of the bodies*, I_0 its moment of inertia about an axis traversing its own centre of gravity parallel to the given common axis, and r_0 the distance of its centre of gravity from that common axis. Then the moment of inertia of that body about the common axis, according to Article 576, equation 2, is

$$I = W r_0^2 + I_0.$$

Consequently, the combined moment of inertia of the system of bodies is

$$\Sigma I = \Sigma \cdot W r_0^2 + \Sigma I_0; \text{...................} (1.)$$

—Q. E. D.

578. **Examples of Moments of Inertia and Radii of Gyration** of homogeneous bodies of some of the more simple and ordinary figures, are given in the following tables. In each case, the axis is supposed to traverse the *centre of gravity* of the body; for the principles of Article 576 enable any other case to be easily solved. The axes are also supposed, in each case, to be *axes of symmetry* of the figure of the body. In subsequent Articles, it will be shown what relations exist between the moments of inertia of the same body about axes traversing it in different directions.

The column headed W gives the mass of the body; that headed I_0 gives the moment of inertia; that headed ϱ_0^2, the *square* of the radius of gyration. The mass of an unit of volume is in each case denoted by w.

BODY.	AXIS.	W	I_0	ϱ_0^2
I. Sphere of radius r,........	Diameter	$\frac{4\pi w r^3}{3}$	$\frac{8\pi w r^5}{15}$	$\frac{2r^2}{5}$
II. Spheroid of revolution—polar semi-axis a, equatorial radius r,...........	Polar axis	$\frac{4\pi w a r^2}{3}$	$\frac{8\pi w a r^4}{15}$	$\frac{2r^2}{5}$
III. Ellipsoid — semi-axes, a, b, c,......................	Axis, $2a$	$\frac{4\pi wabc}{3}$	$\frac{4\pi wabc(b^2+c^2)}{15}$	$\frac{b^2+c^2}{5}$
IV. Spherical shell—external radius r, internal r',....	Diameter	$\frac{4\pi w(r^3-r'^3)}{3}$	$\frac{8\pi w(r^5-r'^5)}{15}$	$\frac{2(r^5-r'^5)}{5(r^3-r'^3)}$
V. Spherical shell, insensibly thin — radius r, thickness dr, ...	Diameter	$4\pi w r^2 dr$	$\frac{8\pi w r^4 dr}{3}$	$\frac{2r^2}{3}$
VI. Circular cylinder—length $2a$, radius r, ...	Longitudinal axis, $2a$	$2\pi w a r^2$	$\pi w a r^4$	$\frac{r^2}{2}$
VII. Elliptic cylinder—length $2a$, transverse semi-axes b, c,......................	Longitudinal axis, $2a$	$2\pi wabc$	$\frac{\pi wabc(b^2+c^2)}{2}$	$\frac{b^2+c^2}{4}$
VIII. Hollow circular cylinder—length $2a$, external radius r, internal r', ...	Longitudinal axis, $2a$	$2\pi wa(r^2-r'^2)$	$\pi wa(r^4-r'^4)$	$\frac{r^2+r'^2}{2}$
IX. Hollow circular cylinder, insensibly thin — length $2a$, radius r, thickness dr,	Longitudinal axis, $2a$	$4\pi w a r dr$	$4\pi w a r^3 dr$	r^2
X. Circular cylinder—length $2a$, radius r, ...	Transverse diameter	$2\pi w a r^2$	$\frac{\pi w a r^2(3r^2+4a^2)}{6}$	$\frac{r^2}{4}+\frac{a^2}{3}$
XI. Elliptic cylinder—length $2a$, transverse semi-axes b, c,......................	Transverse axis, $2b$	$2\pi wabc$	$\frac{\pi wabc(3c^2+4a^2)}{6}$	$\frac{c^2}{4}+\frac{a^2}{3}$
XII. Hollow circular cylinder—length $2a$, external radius r, internal r', ...	Transverse diameter	$2\pi wa(r^2-r'^2)$	$\frac{\pi wa}{6}\left\{3(r^4-r'^4)+4a^2(r^2-r'^2)\right\}$	$\frac{r^2+r'^2}{4}+\frac{a^2}{3}$
XIII. Hollow circular cylinder, insensibly thin — radius r, thickness dr,...........	Transverse diameter	$4\pi w a r dr$	$\pi wa(2r^3+\frac{4}{3}a^2r)dr$	$\frac{r^2}{2}+\frac{a^2}{3}$
XIV. Rectangular prism — dimensions $2a$, $2b$, $2c$,.....	Axis, $2a$	$8wabc$	$\frac{8wabc(b^2+c^2)}{3}$	$\frac{b^2+c^2}{3}$
XV. Rhombic prism — length $2a$, diagonals $2b$, $2c$,....	Axis, $2a$	$4wabc$	$\frac{2wabc(b^2+c^2)}{3}$	$\frac{b^2+c^2}{6}$
XVI. Rhombic prism, as above,	Diagonal, $2b$	$4wabc$	$\frac{2wabc(c^2+2a^2)}{3}$	$\frac{c^2}{6}+\frac{a^2}{3}$

579. **Moments of Inertia found by Division and Subtraction.**—Each of the solids mentioned in the table of the preceding Article can be divided into two equal and symmetrical halves by a plane perpendicular to the axis. The radius of gyration of each of those halves is the same with that of the original solid. Each of the solids can also be divided into four equal and symmetrical wedges or sectors by planes traversing the axis; and those which are solids of revolution can be divided into an unlimited number of such wedges or sectors. The radius of gyration of each such sector about the original axis, which forms its edge, is the same with that of the original solid.

To find the radius of gyration of any such sector about an axis parallel to its edge, the original axis, and traversing the centre of gravity of the sector, let r_0 be the distance of that centre of gravity from the original axis, ϱ_0 the radius of gyration of the original solid, and ϱ'_0 the radius of gyration of the sector about the new axis in question; then from Article 576, equation 3, it follows that

$$\varrho'^2_0 = \varrho^2_0 - r^2_0 \text{..............................} (1.)$$

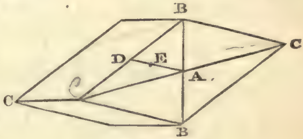

Fig. 238.

Example. In case 15 of Article 578, the square of the radius of gyration of a rhombic prism about its longitudinal axis is found to be $\frac{b^2+c^2}{6}$, b and c being the two semi-diagonals. Let fig. 238 represent such a prism, and let A be one end of its longitudinal axis, and $\overline{\text{B A B}} = 2\,b$, $\overline{\text{C A C}} = 2\,c$, its two diagonals. Divide the prism into four equal right-angled triangular prisms by two planes traversing the diagonals and the longitudinal axis; the radius of gyration of each of those prisms about that axis is the same with that of the original prism. Bisect $\overline{\text{B C}}$ in D, and join $\overline{\text{A D}}$, in which take $r_0 = \overline{\text{A E}} = \frac{2}{3}\,\overline{\text{A D}} = \frac{1}{3}\,\overline{\text{B C}} = \frac{\sqrt{b^2+c^2}}{3}$; then E is the extremity of a longitudinal axis traversing the centre of gravity of the triangular prism A B C, and the radius of gyration of that prism about that new axis is given by the equation

$$\varrho'^2_0 = \varrho^2_0 - r^2_0 = \frac{b^2+c^2}{6} - \frac{b^2+c^2}{9} = \frac{b^2+c^2}{18} \text{.............} (2.)$$

580. **Moments of Inertia found by Transformation.**—The moment of inertia and radius of gyration of a body about a given axis are not changed by any transformation of its figure which can be effected by shifting its particles parallel to the given axis; and the

radius of gyration is not altered by altering the dimensions of the body parallel to the axis in a constant ratio; for example, in cases 1 and 2 of Article 578, the radius of gyration of a spheroid about its polar axis is the same with that of a sphere of the same equatorial radius.

If the dimensions of a body in all directions transverse to the axis are altered in a constant ratio, the radius of gyration is altered in the same ratio.

If the dimensions of a body transverse to its axis, in two directions perpendicular to each other, are altered in different ratios; for example, if the dimensions denoted by y are altered in the ratio m, and the dimensions denoted by z in the ratio n, then the radius of gyration ϱ of the original body is to be conceived as the hypotenuse of a right-angled triangle whose sides are, η parallel to y, and ζ parallel to z, and are given by the equations

$$\overset{\text{Eta}}{\eta^2} = \frac{\Sigma \cdot W y^2}{\Sigma W}; \ \overset{\text{Zeta}}{\zeta^2} = \frac{\Sigma \cdot W z^2}{\Sigma W}; \ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

and the radius of gyration ϱ' of the transformed body will be the hypotenuse of a new right-angled triangle whose sides are $m\,\eta$ and $n\,\zeta$; that is to say,

$$\varrho'^2 = m^2\,\eta^2 + n^2\,\zeta^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.)$$

This method may be exemplified by deducing the radius of gyration of an ellipsoid about any one of its axes (Article 578, case 3) from that of a sphere (*ib.*, case 1).

581. The **Centre of Percussion** of a body, for a given axis, is a point so situated, that if part of the mass of the body were concentrated at that point, and the remainder at the point directly opposite in the given axis, the statical moment of the weight so distributed (Article 42), and its moment of inertia about the given axis, would be the same as those of the actual body in every position of the body.

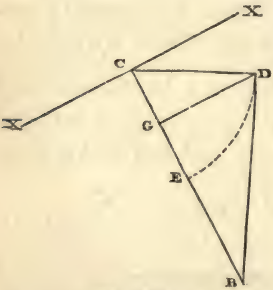

Fig. 239.

In fig. 239 let X X be the given axis, and G the centre of gravity of the body. It is evident, in the first place, that the centre of percussion must be somewhere in the perpendicular C G B let fall from the centre of gravity on the given axis. Secondly, in order that the statical moment of the whole mass, concentrated partly at C, and partly at the centre of percussion B (still unknown), may be the same with that of the actual body, the centre of gravity must be unaltered by that concentration of mass; that is to say, the masses concentrated at B and C must be inversely as the distances of those points from G. Hence denoting the weights of those masses by the letters B and C respectively, and the weight of the whole body by W, we have the proportion

$$\mathrm{W} : \mathrm{C} : \mathrm{B} : : \overline{\mathrm{BC}} : \overline{\mathrm{GB}} : \overline{\mathrm{GC}} \ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

Lastly, in order that the moment of inertia of the mass as supposed to be concentrated at B and C, about the axis X X, may be the same with that of the actual body, we must have

$$\mathrm{B} \cdot \overline{\mathrm{BC}}^2 = \mathrm{W} \varrho^2 = \mathrm{W}\,(\varrho_0^2 + r_0^2) \ldots\ldots\ldots\ldots\ldots\ldots (2.)$$

where $r_0 = \overline{\mathrm{GC}}$, and ϱ_0 is the radius of gyration of the body about an axis parallel to X X and traversing G; and substituting for B its value from equation 1, viz., $\mathrm{B} = \mathrm{W} r_0 \div \overline{\mathrm{BC}}$, we find, for the distance of the centre of percussion from the axis,

$$\overline{\mathrm{BC}} = \frac{\varrho^2}{r_0} = \frac{\varrho_0^2}{r_0} + r_0 ; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.)$$

and for its distance from the centre of gravity,

$$\overline{\mathrm{GB}} = \overline{\mathrm{BC}} - r_0 = \frac{\varrho_0^2}{r_0} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4.)$$

The last equation may also be expressed in the form

$$\overline{\mathrm{GB}} \cdot \overline{\mathrm{GC}} = \varrho_0^2 ; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5.)$$

which preserves the same value when $\overline{\mathrm{GB}}$ and $\overline{\mathrm{GC}}$ are interchanged; thus showing, that if a new axis parallel to the original axis XX be made to traverse the original centre of percussion, the new centre of percussion is the point C in the original axis.

The proportion in which the mass of the body is to be considered as distributed between B and C takes the following form, when each of the last three terms of the proportion 1 is multiplied by $r_0 = \overline{\mathrm{GC}}$:—

Same as Eq. 1

$$\mathrm{W} : \mathrm{C} : \mathrm{B} : : \varrho_0^2 + r_0^2 : \varrho_0^2 : r_0^2 \ldots\ldots\ldots\ldots\ldots\ldots (6.)$$

The preceding solution is represented by the following geometrical construction:—Draw $\overline{\mathrm{GD}} \perp \overline{\mathrm{CG}}$ and $= \varrho_0$; join $\overline{\mathrm{CD}}$, perpendicular to which draw $\overline{\mathrm{DB}}$ cutting $\overline{\mathrm{CG}}$ produced in B; this point is the centre of percussion.

Also, $\overline{\mathrm{CD}} = \varrho$, the radius of gyration about X X; and $\overline{\mathrm{DB}}$ is the radius of gyration about an axis traversing B parallel to XX.

If $\overline{\mathrm{CE}}$ be taken = $\overline{\mathrm{CD}}$, E is sometimes called the CENTRE OF GYRATION of the body for the axis XX.*

582. **No Centre of Percussion** exists when the axis traverses the centre of gravity of the body. In that case, the statical moment of the body is nothing; and an equal mass, concentrated and uniformly distributed round the circle BBB, whose radius is ϱ_0, the radius of gyration, or at a set of symmetrically arranged points in that circle, has the same moment of inertia with the actual body.

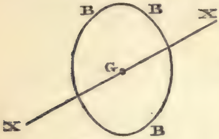

Fig. 240.

583. **Moments of Inertia about Inclined Axes.**—The object of the present Article and the remaining Articles of this section is to show the relations which exist amongst the moments of inertia of a body about axes traversing a fixed point in it in different directions. The mathematical processes which it is necessary to employ for that purpose, though not very abstruse, are somewhat complex; and the reader who wishes to study the more simple parts of the subject only, may take the conclusions for granted.

It has already been shown in Article 575 that the moment of inertia of a body about a given axis denoted by x, is given by the equation

$$\mathrm{I}_x = \mathrm{S}\,y^2 + \mathrm{S}\,z^2; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

in which, for the sake of brevity, $\Sigma \cdot \mathrm{W}$ has been replaced by the single symbol S. The fixed point being the origin of co-ordinates, let $\mathrm{S}\,\mathrm{R}^2$ be the sum of the products of the weight of each particle into the square of its distance from that point; a sum which is independent of the directions of the axis. Then because $\mathrm{R}^2 = x^2 + y^2 + z^2$, the moments of inertia of the body relatively to three rectangular axes may be expressed as follows:—

$$\mathrm{I}_x = \mathrm{S}\,\mathrm{R}^2 - \mathrm{S}\,x^2;\ \mathrm{I}_y = \mathrm{S}\,\mathrm{R}^2 - \mathrm{S}\,y^2;\ \mathrm{I}_z = \mathrm{S}\,\mathrm{R}^2 - \mathrm{S}\,z^2 \ldots\ldots\ldots (2.)$$

Further, let the three sums of the weights of the particles of the body, each multiplied by the product of a pair of its co-ordinates, be thus expressed:—

$$\mathrm{S}\,yz;\ \mathrm{S}\,zx;\ \mathrm{S}\,xy \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.)$$

These will be called *moments of deviation.*

Now, let three new rectangular axes of co-ordinates, denoted by x', y', z', traverse the same fixed point in the body; let the angles which they make with the original axes be denoted by

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* As to the centres of percussion and gyration, and other remarkable points in a rigid body, see a memoir by M. Poinsot in *Liouville's Journal* for 1857.

$$\left.\begin{array}{l}\widehat{xx'},\ \widehat{xy'},\ \widehat{xz'},\\ \widehat{yx'},\ \widehat{yy'},\ \widehat{yz'},\\ \widehat{zx'},\ \widehat{zy'},\ \widehat{zz'};\end{array}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(4.)$$

Then for any given particle, the new co-ordinates are thus expressed in terms of the original co-ordinates :—

$$x' = x \cdot \cos \widehat{xx'} + y \cdot \cos \widehat{yx'} + z \cdot \cos \widehat{zx'}, \ldots\ldots\ldots\ldots(5.)$$

and analogous equations for y' and z'; and the original co-ordinates are thus expressed in terms of the new co-ordinates :—

$$x = x' \cdot \cos \widehat{xx'} + y' \cdot \cos \widehat{xy'} + z' \cdot \cos \widehat{xz'};\ \&c.\ldots\ldots(6.)$$

The nine angles of equation 4 are connected by the relations :—that the sum of the squares of the cosines of any three angles in one line, or in one column, is unity; for example,

$$\cos^2 \widehat{xx'} + \cos^2 \widehat{xy'} + \cos^2 \widehat{xz'} = 1; \ldots\ldots\ldots\ldots\ldots\ldots(7.)$$

and that the sum of the three products of the pairs of cosines of the angles in a pair of lines, or a pair of columns, is nothing; for example,

$$\cos \widehat{yx'} \cdot \cos \widehat{zx'} + \cos \widehat{yy'} \cdot \cos \widehat{zy'} + \cos \widehat{yz'} \cdot \cos \widehat{zz'} = 0 \ldots.(8.)$$

A relation deduced from the preceding is this, that the cosine of each angle is equal to the difference between the binary products of the cosines of the four angles, which are neither in the same line nor in the same column with the first, these binary products being taken diagonally; for example,

$$\cos \widehat{xx'} = \cos \widehat{yy'} \cdot \cos \widehat{zz'} - \cos \widehat{yz'} \cdot \cos \widehat{zy'}. \ldots\ldots\ldots(9.)$$

and similarly for the other cosines.

Now, if for the new co-ordinates x', y', z', in the six integrals,

$$\mathrm{S}\,x'^2,\ \mathrm{S}\,y'^2,\ \mathrm{S}\,z'^2,\ \mathrm{S}\,y'z',\ \mathrm{S}\,z'x',\ \mathrm{S}\,x'y',$$

there are substituted their values in terms of the original co-ordinates, as given by equation 5 for x', and analogous equations for y' and z', there are obtained the six expressions for those integrals relatively to the new axes, in terms of the integrals relatively to the original axes, and of the cosines of the nine angles between the new and the original axes; but it is unnecessary here to write those equations at length, for they are *precisely similar to the equations of transformation in Article* 106 (pages 92, 93), substituting only

$$\mathrm{S}\, x^2,\ \mathrm{S}\, y^2,\ \mathrm{S}\, z^2,\ \mathrm{S}\, y z,\ \mathrm{S}\, z x,\ \mathrm{S}\, x y,$$

$$\text{for} \quad p_{xx},\ p_{yy},\ p_{zz},\ p_{yz},\ p_{zx},\ p_{xy},$$

and making the like substitutions in the symbols referring to the new co-ordinates.

584. **Principal Axes of Inertia.**—THEOREM. *At each point in a body there is a system of three rectangular axes, for which the moments of deviation are each equal to nothing.*

Supposing such a set of axes to exist, let co-ordinates parallel to them be denoted by x_1, y_1, z_1. Then the property which they are required to have is expressed by the equations

$$\mathrm{S}\, y_1 z_1 = 0;\ \mathrm{S}\, z_1 x_1 = 0;\ \mathrm{S}\, x_1 y_1 = 0 \ldots\ldots\ldots\ldots\ldots (1.)$$

Co-ordinates parallel to a set of axes, for which the integrals $\mathrm{S}\, x^2$, &c., have been determined, being denoted by x, y, z, we have for each particle,

$$x = x_1 \cos \widehat{x x_1} + y_1 \cos \widehat{x y_1} + z_1 \cos \widehat{x z_1};$$

$$x_1 x = x_1^2 \cos \widehat{x x_1} + x_1 y_1 \cos \widehat{x y_1} + x_1 z_1 \cos \widehat{x z_1};$$

and consequently,

$$\mathrm{S}\, x_1 x = \cos \widehat{x x_1} \cdot \mathrm{S}\, x_1^2 + \cos \widehat{x y_1} \cdot \mathrm{S}\, x_1 y_1 + \cos \widehat{x z_1} \cdot \mathrm{S}\, z_1 x_1;$$

but because of the conditions expressed by the equations 1, this is reduced to

$$\mathrm{S}\, x_1 x = \cos \widehat{x x_1} \cdot \mathrm{S}\, x_1^2; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.)$$

and by similar reasoning it is shown that

$$\left.\begin{array}{l} \mathrm{S}\, x_1 y = \cos \widehat{y x_1} \cdot \mathrm{S}\, x_1^2; \\ \mathrm{S}\, x_1 z = \cos \widehat{z x_1} \cdot \mathrm{S}\, x_1^2. \end{array}\right\} \ldots\ldots\ldots\ldots\ldots (2\ \text{A.})$$

Now, from the equation

$$x_1 = x \cos \widehat{x x_1} + y \cos \widehat{y x_1} + z \cos \widehat{z x_1},$$

are deduced the following values of the integrals in the equations 2, 2 A:—

$$\begin{aligned} \mathrm{S}\, x_1 x &= \cos \widehat{x x_1} \,.\, \mathrm{S}\, x^2 + \cos \widehat{y x_1} \cdot \mathrm{S}\, x y + \cos \widehat{z x_1} \cdot \mathrm{S}\, z x; \\ \mathrm{S}\, x_1 y &= \cos \widehat{x x_1} \cdot \mathrm{S}\, x y + \cos \widehat{y x_1} \cdot \mathrm{S}\, y^2 + \cos \widehat{z x_1} \cdot \mathrm{S}\, y z; \\ \mathrm{S}\, x_1 z &= \cos \widehat{x x_1} \cdot \mathrm{S}\, z x + \cos \widehat{y x_1} \cdot \mathrm{S}\, y z + \cos \widehat{z x_1} \cdot \mathrm{S}\, z^2. \end{aligned}$$

Subtracting the equations 2 and 2 A from these, we find the following equations :—

$$\left.\begin{aligned} &\cos \widehat{x x_1} (\mathrm{S}\, x^2 - \mathrm{S}\, x_1^2) + \cos \widehat{y x_1} \cdot \mathrm{S}\, x y + \cos \widehat{z x_1} \cdot \mathrm{S}\, z x = 0; \\ &\cos \widehat{x x_1} \cdot \mathrm{S}\, x y + \cos \widehat{y x_1} \cdot (\mathrm{S}\, y^2 - \mathrm{S}\, x_1^2) + \cos \widehat{z x_1} \cdot \mathrm{S}\, y z = 0; \\ &\cos \widehat{x x_1} \cdot \mathrm{S}\, z x + \cos \widehat{y x_1} \cdot \mathrm{S}\, y z + \cos \widehat{z x_1} (\mathrm{S}\, z^2 - \mathrm{S}\, x_1^2) = 0. \end{aligned}\right\} (3.)$$

The elimination of the three cosines from these three equations leads to the following cubic equation :—

$$(\mathrm{S}\, x_1^2)^3 - \mathrm{A}\, (\mathrm{S}\, x_1^2)^2 + \mathrm{B} \cdot \mathrm{S}\, x_1^2 - \mathrm{C} = 0; \ldots\ldots\ldots\ldots (4.)$$

in which the co-efficients have the following values :—

$$\left.\begin{aligned} \mathrm{A} &= \mathrm{S}\, x^2 + \mathrm{S}\, y^2 + \mathrm{S}\, z^2 = \mathrm{S}\, \mathrm{R}^2; \\ \mathrm{B} &= \mathrm{S}\, y^2 \cdot \mathrm{S}\, z^2 + \mathrm{S}\, z^2 \cdot \mathrm{S}\, x^2 + \mathrm{S}\, x^2 \cdot \mathrm{S}\, y^2 \\ &\quad - (\mathrm{S}\, y z)^2 - (\mathrm{S}\, z x)^2 - (\mathrm{S}\, x y)^2; \\ \mathrm{C} &= \mathrm{S}\, x^2 \cdot \mathrm{S}\, y^2 \cdot \mathrm{S}\, z^2 + 2\, \mathrm{S}\, y z \cdot \mathrm{S}\, z x \cdot \mathrm{S}\, x y \\ &\quad - \mathrm{S}\, x^2 \cdot (\mathrm{S}\, y z)^2 - \mathrm{S}\, y^2 \cdot (\mathrm{S}\, z x)^2 - \mathrm{S}\, z^2\, (\mathrm{S}\, x y)^2. \end{aligned}\right\} (5.)$$

It is evident that A is always positive. By considering the terms of which B is composed, it can be shown that it is equivalent to

$$\mathrm{S}\, (y z' - z y')^2 + \mathrm{S}\, (z x' - x z')^2 + \mathrm{S}\, (x y' - y x')^2;$$

x, y, z, x', y', z', being the co-ordinates of *a pair of different particles*, and the particles being taken in pairs in every possible way; and by considering the terms of which C is made up, it can be shown to be equivalent to

$$\mathrm{S}\, (x y' z'' + x' y'' z + x'' y z' - x y'' z' - x'' y' z - x' y z'')^2;$$

in which the letters without accents, with one accent, and with two accents, denote the co-ordinates of a set of three different particles, and the particles are taken in triplets in every possible way. Hence B and C, being both sums of squares, are positive, as well as A; and the cubic equation 4 has *three real positive roots*, corresponding to the three rectangular axes which satisfy the conditions of equation 1. These roots are the values of $\mathrm{S}\, x_1^2$, $\mathrm{S}\, y_1^2$, $\mathrm{S}\, z_1^2$;

and their existence proves the existence of the three rectangular PRINCIPAL AXES OF INERTIA.—Q. E. D.

The angles which any one of the principal axes makes with the three original axes are given by the following equations, which are deduced from the equations 3 :—

$$\left.\begin{array}{c} \cos \widehat{x\,x_1} : \cos \widehat{y\,x_1} : \cos \widehat{z\,x_1} \\ :: \dfrac{1}{(\mathrm{S}x_1^2 - \mathrm{S}x^2)\mathrm{S}yz + \mathrm{S}zx \cdot \mathrm{S}xy} : \dfrac{1}{(\mathrm{S}x_1^2 - \mathrm{S}y^2)\mathrm{S}zx + \mathrm{S}xy \cdot \mathrm{S}yz} \\ : \dfrac{1}{(\mathrm{S}\,x_1^2 - \mathrm{S}\,z^2)\,\mathrm{S}\,x\,y + \mathrm{S}\,y\,z \cdot \mathrm{S}\,z\,x}. \end{array}\right\} \quad (6.)$$

Similar equations, substituting y_1 and z_1 successively for x_1, give the ratios of the other two sets of cosines.

From the properties of the roots of equations, it follows, that the co-efficients of the cubic equation 4 have the following values in terms of the integrals $\mathrm{S}\,x_1^2$, &c. :—

$$\left.\begin{array}{c} \mathrm{A} = \mathrm{S}\,x_1^2 + \mathrm{S}\,y_1^2 + \mathrm{S}\,z_1^2 = \mathrm{S}\,\mathrm{R}^2 \text{ as before;} \\ \mathrm{B} = \mathrm{S}\,y_1^2 \cdot \mathrm{S}\,z_1^2 + \mathrm{S}\,z_1^2 \cdot \mathrm{S}\,x_1^2 + \mathrm{S}\,x_1^2 \cdot \mathrm{S}\,y_1^2; \\ \mathrm{C} = \mathrm{S}\,x_1^2 \cdot \mathrm{S}\,y_1^2 \cdot \mathrm{S}\,z_1^2; \end{array}\right\} \quad \ldots\ldots(7.)$$

and hence it appears, that the functions of the six integrals $\mathrm{S}\,x^2$, &c., denoted by A, B, and C, in the equations 5, are *isotropic;* that is, are the same in magnitude for all directions of the rectangular axes of x, y, and z.

585. **Ellipsoid of Inertia.**—Let the principal axes of a body, traversing a given point, be now taken for axes of co-ordinates; and the moments of inertia about them, called the *principal moments of inertia*, being given, and denoted by I_1, I_2, I_3, let it be required to determine the moment of inertia, I, about any axis traversing the same point, and making with the principal axes the angle α, β, γ. Let co-ordinates along this new axis be denoted by x, and along the principal axes by x_1, y_1, z_1, as before.

It has already been shown that

$$\mathrm{S}\,x^2 = \cos^2\alpha \cdot \mathrm{S}\,x_1^2 + \cos^2\beta \cdot \mathrm{S}\,y_1^2 + \cos^2\gamma \cdot \mathrm{S}\,z_1^2, \ldots (1.)$$

and that

$$\mathrm{I} = \mathrm{S}\,\mathrm{R}^2 - \mathrm{S}\,x^2;\ \mathrm{I}_1 = \mathrm{S}\,\mathrm{R}^2 - \mathrm{S}\,x_1^2;\ \mathrm{I}_2 = \mathrm{S}\,\mathrm{R}^2 - \mathrm{S}\,y_1^2;$$
$$\mathrm{I}_3 = \mathrm{S}\,\mathrm{R}^2 - \mathrm{S}\,z_1^2; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.)$$

and from these equations the following is easily deduced :—

$$\mathrm{I} = \mathrm{I}_1 \cdot \cos^2\alpha + \mathrm{I}_2 \cdot \cos^2\beta + \mathrm{I}_3 \cdot \cos^2\gamma \ldots\ldots\ldots (3.)$$

Let a, b, c, be the three semi-axes of an ellipsoid, and s its semidiameter in any direction which makes the angles α, β, γ, with those semi-axes. Then it is well known that

$$\frac{1}{s^2} = \frac{\cos^2\alpha}{a^2} + \frac{\cos^2\beta}{b^2} + \frac{\cos^2\gamma}{c^2}; \ldots\ldots\ldots\ldots\ldots (4.)$$

and by comparing this with equation 3 it is made evident, that if an ellipsoid be constructed whose semi-axes are in direction the principal axes of the body at a given point, and represent in magnitude the reciprocals of the square roots of the moments of inertia about those axes respectively, as shown by the equations

$$a = \frac{1}{\sqrt{\mathrm{I}_1}};\ b = \frac{1}{\sqrt{\mathrm{I}_2}};\ c = \frac{1}{\sqrt{\mathrm{I}_3}}; \ldots\ldots\ldots\ldots (5.)$$

then will the reciprocal of the square of the semidiameter of that ellipsoid in any direction represent the moment of inertia about an axis traversing the origin in that direction, as expressed by the equation

$$\mathrm{I} = \frac{1}{s^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.)$$

Such an ellipsoid, when described about the centre of gravity of the body as a centre, is called by M. Poinsot the *central ellipsoid.*

If I_1, I_2, I_3, be ranged in their order of magnitude, it is evident that the greatest of them, I_1, is the greatest moment of inertia of the body about any axis traversing the fixed point; that the least, I_3, is the least moment of inertia about any such axis; and that the intermediate principal moment of inertia, I_2, is the least moment of inertia about any axis traversing the fixed point perpendicular to the axis of I_3, and the greatest moment of inertia about any axis traversing the fixed point perpendicular to the axis of I_1.

Should two of the principal moments of inertia be equal, as $\mathrm{I}_2 = \mathrm{I}_3$, the ellipsoid becomes a spheroid of revolution: all the moments of inertia about axes traversing the fixed point in the plane of the axes of I_2 and I_3 are equal; and the moments of inertia about all axes traversing the fixed point and equally inclined to the axis of I_1 are equal. In this case equation 3 becomes

$$\mathrm{I} = \mathrm{I}_1 \cos^2\alpha + \mathrm{I}_2 \sin^2\alpha \ldots\ldots\ldots\ldots\ldots\ldots (7.)$$

If all three principal moments of inertia are equal, the ellipsoid becomes a sphere, and the moments of inertia are equal about all axes traversing the fixed point.

Suppose the fixed point in the first place to be the centre of gravity of the body, whose weight is W, and that I_{01}, I_{02}, I_{03}, are the principal moments of inertia about rectangular axes traversing it. Let a new fixed point be taken whose distance from the centre of gravity is r_0, in a direction making the angles α, β, γ, with the principal axes at the centre of gravity. Then with respect to a set of rectangular axes traversing the new point parallel to the original axes, the new moments of inertia are

$$\left.\begin{aligned} I_x &= I_{01} + W\, r_0^2 \sin{}^2\alpha; \\ I_y &= I_{02} + W\, r_0^2 \sin{}^2\beta; \\ I_z &= I_{03} + W\, r_0^2 \sin{}^2\gamma; \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots(8.)$$

and there are at the same time moments of deviation represented by

$$\left.\begin{array}{c} S\,yz = W\, r_0^2 \cdot \cos\beta\cos\gamma;\ S\,zx = W\, r_0^2 \cdot \cos\gamma\cos\alpha; \\ S\,xy = W\, r_0^2 \cdot \cos\alpha\cos\beta; \end{array}\right\} (9.)$$

so that the principal axes at the new point are not parallel to those at the centre of gravity, unless two at least of the direction cosines of r_0 are null; that is to say, unless the new point is in one of the original principal axes, when all the moments of deviation vanish, and the new axes are parallel to the original axes.

586. The **Resultant Moment of Deviation** about a given axis is represented by the diagonal of a rectangular parallelogram of which the sides represent the moments of deviation relatively to two rectangular co-ordinate planes traversing the given axis.

Let the principal axes and moments of inertia at a given point be known, and let three new axes of moments, denoted by x, y, z, be taken in any three rectangular directions making angles with the original axes denoted as in the equations of Article 583. Then the moments of deviation in the new co-ordinate planes are

$$\begin{aligned} S\,y\,z = \cos \widehat{y\,x_1} \cdot \cos \widehat{z\,x_1}\, S\,x_1^2 + \cos \widehat{y\,y_1} \cdot \cos \widehat{z\,y_1}\, S\,y_1^2 & \\ + \cos \widehat{y\,z_1} \cdot \cos \widehat{z\,z_1}\, S\,z_1^2, & \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.) \end{aligned}$$

and similar equations for $S\,z\,x$, and $S\,x\,y$, *mutatis mutandis.* Substituting for $S\,x_1^2$, &c., their values, $S\,R^2 - I_1$, &c., and observing that

$$\cos \widehat{y\,x_1} \cdot \cos \widehat{z\,x_1} + \cos \widehat{y\,y_1} \cdot \cos \widehat{z\,y_1} + \cos \widehat{y\,z_1} \cos \widehat{z\,z_1} = 0,$$

those equations become

$$\begin{aligned} S\,yz = - I_1 \cdot \cos \widehat{y\,x_1} \cdot \cos \widehat{z\,x_1} - I_2 \cdot \cos \widehat{y\,y_1} \cdot \cos \widehat{z\,y_1} & \\ - I_3 \cdot \cos \widehat{y\,z_1} \cdot \cos \widehat{z\,z_1}, & \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.) \end{aligned}$$

and similar equations, *mutatis mutandis*, for $\mathrm{S}zx$, $\mathrm{S}xy$; from which, by the aid of relations amongst the direction cosines already stated in Article 583, the following value is found for the resultant moment of deviation about one of the new axes, such as x:—

$$\left.\begin{aligned} \mathrm{K}_x = &\sqrt{\left\{\mathrm{I}_1^2 \cos^2 \widehat{x x_1} + \mathrm{I}_2^2 \cos^2 \widehat{x y_1} + \mathrm{I}_3^2 \cos^2 \widehat{x z_1}\right.} \\ &\quad \overline{\left. - (\mathrm{I}_1 \cos^2 \widehat{x x_1} + \mathrm{I}_2 \cos^2 \widehat{x y_1} + \mathrm{I}_3 \cos^2 \widehat{x z_1})^2\right\}}\,; \\ = &\sqrt{\left\{\mathrm{I}_1^2 \cos^2 \widehat{x x_1} + \mathrm{I}_2^2 \cos^2 \widehat{x y_1} + \mathrm{I}_3^2 \cos^2 \widehat{x z_1} - \mathrm{I}_x^2\right\}} \end{aligned}\right\} \ldots(3.)$$

This equation, expressed in terms of the axes of the ellipsoid of inertia, becomes as follows:—

$$\mathrm{K}_x = \sqrt{\left\{\frac{\cos^2 \widehat{x x_1}}{a^4} + \frac{\cos^2 \widehat{x y_1}}{b^4} + \frac{\cos^2 \widehat{x z_1}}{c^4} - \frac{1}{s^4}\right\}} \ldots(4.)$$

but the positive part of this expression is well known to be the value of $\frac{1}{s^2 n^2}$, where n represents the *normal* let fall from the centre of the ellipsoid of inertia upon a plane which touches the ellipsoid at the point where it is cut by the new axis x. Hence

$$\mathrm{K}_x = \sqrt{\left(\frac{1}{s^2 n^2} - \frac{1}{s^4}\right)} = \frac{\sqrt{s^2 - n^2}}{s^2 n}\,; \ldots\ldots\ldots\ldots\ldots(5.)$$

in which it is to be observed, that $\sqrt{s^2 - n^2}$ represents the *length of the tangent* to the ellipsoid, from the point of contact to the foot of the normal. Also, let θ be the angle between the normal n and the semidiameter s; then $\sqrt{s^2 - n^2} : n = \tan \theta$, and

$$\mathrm{K}_x = \mathrm{I}_x \tan \theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.)$$

SECTION 2.—*On Uniform Rotation.*

587. The **Momentum** of a body rotating about its centre of gravity is nothing, according to the principle of Article 524. As every motion of a rigid body can be resolved into a translation, and a rotation about its centre of gravity, the rotation will be supposed to take place about the centre of gravity of the body throughout this section.

588. The **Angular Momentum** is found in the following manner:—Let x denote the axis of rotation, and y and z any two axes fixed in the body, perpendicular to it and to each other. Let a be the

angular velocity of rotation. Then the velocity of any particle W, whose radius vector is $r = \sqrt{y^2 + z^2}$, is

$$a\,r = a\sqrt{y^2 + z^2},$$

and the angular momentum of that particle, *relatively to the axis of rotation*, is

$$\frac{\mathrm{W}\,a\,r^2}{g} = \frac{\mathrm{W}\,a}{g}\,(z^2 + y^2);$$

being the *product of its moment of inertia into its angular velocity*, divided by g, because of the weights of the particles having been used in computing the moment of inertia. Now let a line, parallel to the radius vector of the particle, be drawn in the plane of y and z; the distance of that line from the particle is x, and the angular momentum of the particle *relatively to that line* is

$$\frac{\mathrm{W}}{g}\,a\,r\,x = \frac{\mathrm{W}}{g}\,a\,x\sqrt{y^2 + z^2};$$

and this may be resolved into two components; one *relatively to the axis* of y,

$$\frac{\mathrm{W}\,a\,z\,x}{g};$$

and the other relatively to the axis of z,

$$\frac{\mathrm{W}\,a\,x\,y}{g};$$

and these are equal respectively to the angular velocity divided by the acceleration produced by gravity in a second, multiplied by the *moments of deviation* of the particle in the co-ordinate planes of $z\,x$ and $x\,y$.

Hence it appears that the resultant angular momentum of the whole body consists of three components, viz. :—

$$\left.\begin{array}{l}\text{Relatively to the axis of rotation,}\\ \qquad\dfrac{a}{g}\cdot(\mathrm{S}\,y^2 + \mathrm{S}\,z^2) = \dfrac{a}{g}\,\mathrm{I}_x;\\ \text{and relatively to the transverse axes,}\\ \qquad\dfrac{a}{g}\cdot\mathrm{S}\,z\,x;\ \dfrac{a}{g}\cdot\mathrm{S}\,x\,y;\end{array}\right\}\dots\dots\dots\dots\dots(1.)$$

and if lines proportional to those three components be set off upon the three axes, the diagonal of the rectangle described upon them will represent in direction the axis, and in length the magnitude, of the resultant angular momentum.

It follows that *the axis of angular momentum of a rotating body does not coincide with the axis of rotation, unless that axis is an axis of inertia;* in which case the moments of deviation are each equal to nothing, and the resultant angular momentum is simply *the product of the moment of inertia about the axis of rotation into the angular velocity*, divided by g.

Now let the axes of inertia be taken for axes of co-ordinates, and let the axis of rotation make with them the angles α, β, γ. Resolve the angular velocity a about that axis into three components about the axes of inertia

$$a \cos \alpha; \ a \cos \beta; \ a \cos \gamma;$$

then the angular momenta due to those three components are respectively

$$\frac{a}{g} \mathrm{I}_1 \cos \alpha; \ \frac{a}{g} \mathrm{I}_2 \cos \beta; \ \frac{a}{g} \mathrm{I}_3 \cos \gamma;$$

the resultant angular momentum is

$$\mathrm{A} = \frac{a}{g} \cdot \sqrt{\left\{\mathrm{I}_1^2 \cos^2 \alpha + \mathrm{I}_2^2 \cos^2 \beta + \mathrm{I}_3^2 \cos^2 \gamma\right\}}; \ldots\ldots\ldots\ldots (2.)$$

and the axis of angular momentum makes with the axes of inertia the angles whose cosines are

$$\frac{a \ \mathrm{I}_1 \cos \alpha}{g \ \mathrm{A}}; \ \frac{a \ \mathrm{I}_2 \cos \beta}{g \ \mathrm{A}}; \ \frac{a \ \mathrm{I}_3 \cos \gamma}{g \ \mathrm{A}} \ldots\ldots\ldots\ldots\ldots (3.)$$

Now, as already shown in Article 586, the quantity whose square root is extracted in equation 2 is the reciprocal of the product of the squares of the semidiameter and normal of the ellipsoid of inertia; and by inspecting the equations of Article 586, it is evident, that the square root itself, in equation 2 of this Article, is the *resultant* of the moment of inertia and moment of deviation proper to the axis of rotation; so that equation 2 may be expressed in the following form :—

$$\mathrm{A} = \frac{a}{g \, n \, s} = \frac{a}{g} \sqrt{(\mathrm{I}^2 + \mathrm{K}^2)}; \ldots\ldots\ldots\ldots\ldots\ldots (4.)$$

n being, as before, the normal, and s the semidiameter of the ellipsoid of inertia at the point cut by the axis of rotation; for which the moments of inertia and of deviation are I and K.

Further, the direction cosines of the axis of angular momentum, in the formula 3, which may otherwise be expressed as follows :—

$$\frac{I_1 \cos \alpha}{\sqrt{(I^2+K^2)}}; \ \frac{I_2 \cos \beta}{\sqrt{(I^2+K^2)}}; \ \frac{I_3 \cos \gamma}{\sqrt{(I^2+K^2)}}; \ldots\ldots (5.)$$

are the direction cosines of the normal of the ellipsoid of inertia. Hence the *axis of angular momentum at any instant is in the direction of the normal let fall from the centre of the ellipsoid of inertia upon a plane touching that ellipsoid at the end of that diameter which is the axis of rotation; and the angular momentum itself is directly as the angular velocity of rotation, and inversely as the product of the normal and semidiameter.*

The angle between the axes of rotation and of angular momentum is the angle already denoted by θ in Article 586, whose value is given by the equation

$$\cos \theta = \frac{n}{s} = \frac{I}{\sqrt{I^2+K^2}} \ldots (6.)$$

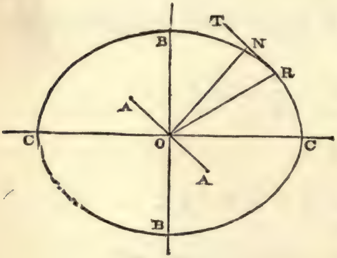

Fig. 241.

By the following geometrical construction, the preceding principles are represented to the eye:—

In fig. 241, let O be the point about which the body rotates, and A B C A B C its ellipsoid of inertia, whose semi-axes have the proportions

$$\overline{OA} : \overline{OB} : \overline{OC} :: \frac{1}{\sqrt{I_1}} : \frac{1}{\sqrt{I_2}} : \frac{1}{\sqrt{I_3}} \ldots\ldots\ldots\ldots (7.)$$

Let O R be the axis of rotation, whether permanent or instantaneous, $\overline{OR}$ being the semidiameter of the ellipsoid of inertia. Let R T be part of a plane touching the ellipsoid at R, and $\overline{ON}$ a normal upon that plane from O. Then the moment of inertia, the moment of deviation, and their resultant, the *total moment*, have the following proportions:—

$$\left. \begin{array}{l} I : K : \sqrt{I^2+K^2} \\ :: \dfrac{1}{\overline{OR}^2} : \dfrac{\overline{RN}}{\overline{OR}^2 \cdot \overline{ON}} : \dfrac{1}{\overline{OR} \cdot \overline{ON}}; \end{array} \right\} \ldots\ldots\ldots (8.)$$

the direction of the axis of angular momentum is O N; and its amount is proportional to $\dfrac{a}{g \cdot \overline{OR} \cdot \overline{ON}}$.

589. **The Actual Energy of Rotation** of a body rotating about its

centre of gravity, being the sum of the masses of its particles, each multiplied into one-half of the square of its velocity, is found as follows:—a being the angular velocity of rotation, the linear velocity of any particle whose distance from the axis of rotation is r, is

$$v = a\,r;$$

and the actual energy of that particle, its weight being W, is

$$\frac{\mathrm{W}\,v^2}{2\,g} = \frac{\mathrm{W}\,a^2 r^2}{2\,g};\ldots\ldots\ldots\ldots(1.)$$

being the *moment of inertia* of the particle multiplied by $\frac{a^2}{2\,g}$. Hence for the whole body the actual energy of rotation is

$$\mathrm{E} = \frac{a^2\,\mathrm{I}}{2\,g};\ldots\ldots\ldots\ldots(2.)$$

that is to say, *actual energy bears the same relation to angular velocity and moment of inertia that it does to linear velocity and weight.*

Referring again to fig. 241, it appears that the actual energy of rotation is proportional to

$$\frac{a^2}{2\,g\cdot\overline{\mathrm{O\,R}}^2}\ldots\ldots\ldots\ldots(3.)$$

Conceive, as in the last Article, the angular velocity a to be resolved into three components about the three axes of inertia respectively, viz.:—

$$a\cos\alpha,\ a\cos\beta,\ a\cos\gamma;$$

then the quantities of actual energy due to those three component rotations are

$$\frac{a^2\,\mathrm{I}_1\cos^2\alpha}{2\,g};\ \frac{a^2\,\mathrm{I}_2\cos^2\beta}{2\,g};\ \frac{a^2\,\mathrm{I}_3\cos^2\gamma}{2\,g};\ldots\ldots\ldots\ldots(4.)$$

which being added together, reproduce the amount of actual energy given in formula 2; showing that *the actual energy of rotation about a given axis is the sum of the actual energies due to the components of that rotation about the three axes of inertia.*

590. **Free Rotation** is that of a body turning about its centre of gravity under no force. The principles of the conservation of angular momentum (Article 563), and of the conservation of internal energy (Article 565), being applied to free rotation, show that it is governed by the following laws:—

I. *The direction of the axis of angular momentum is fixed.*
II. *The angular momentum is constant.*
III. *The actual energy is constant.*

The first law shows, that the direction of the normal $\overline{\mathrm{ON}}$, fig. 241, is fixed; and consequently, that unless that normal coincides with the axis of rotation $\overline{\mathrm{OR}}$, which takes place for axes of inertia only, the axis of rotation is not a fixed direction, and is therefore an *instantaneous axis* only (Articles 385 to 393). Hence the axes of inertia are sometimes called "*permanent axes of rotation.*"

The second and third laws are expressed by the following equations:—

$$\left.\begin{aligned} \mathrm{A} &= \frac{a}{g}\sqrt{(\mathrm{I}^2+\mathrm{K}^2)} = \text{constant}; \\ \mathrm{E} &= \frac{a^2\,\mathrm{I}}{2\,g} = \text{constant}. \end{aligned}\right\} \ldots\ldots\ldots\ldots(1.)$$

To find how these laws regulate the *changes of direction* of the instantaneous axis, eliminate the angular velocity as follows:—

$$\frac{g\,\mathrm{A}^2}{2\,\mathrm{E}} = \frac{\mathrm{I}^2+\mathrm{K}^2}{\mathrm{I}} = \frac{\mathrm{I}_1^2\cos^2\alpha + \mathrm{I}_2^2\cos^2\beta + \mathrm{I}_3^2\cos^2\gamma}{\mathrm{I}_1\cos^2\alpha + \mathrm{I}_2\cos^2\beta + \mathrm{I}_3\cos^2\gamma}$$

$$= \text{constant}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.)$$

Now, referring to fig. 241, and to equation 8 of Article 588, it appears that $\mathrm{I}^2 + \mathrm{K}^2 \propto 1 \div \overline{\mathrm{OR}}^2 \cdot \overline{\mathrm{ON}}^2$, and that $\mathrm{I} \propto 1 \div \overline{\mathrm{OR}}^2$; whence

$$\frac{\mathrm{I}^2+\mathrm{K}^2}{\mathrm{I}} \propto \frac{1}{\overline{\mathrm{ON}}^2} \propto \text{constant};\ldots\ldots\ldots\ldots(3.)$$

That is to say, *the normal* $\overline{\mathrm{ON}}$ *is constant in length as well as fixed in direction;* and therefore *a body rotating freely moves in such a manner, that its ellipsoid of inertia always touches a fixed plane* (viz., the plane T N R), *the instantaneous axis traversing the point of contact.*

The second of the equations (1.) further shows, that the angular velocity, being given by the equation

$$a = \sqrt{\frac{2\,g\,\mathrm{E}}{\mathrm{I}}},\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4.)$$

is at each instant proportional to the semidiameter $\overline{\mathrm{OR}}$.

If the instantaneous axis O R and the position of the body are known at any instant of the rotation, the *invariable plane* T N R, and the length and direction of the fixed normal $\overline{ON}$, are also known.

Conceive a curve to be drawn on the ellipsoid of inertia through all the points whose tangent planes are at the same perpendicular distance $\overline{ON}$ from the centre: then the instantaneous axis $\overline{OR}$ will always traverse that curve, and will always be found in the surface of a cone of the second order *fixed relatively to the axes of inertia*, whose equation is

$$\left(I_1^2 - \frac{I_1}{\overline{ON}^2}\right)\cos^2\alpha + \left(I_2^2 - \frac{I_2}{\overline{ON}^2}\right)\cos^2\beta + \left(I_3^2 - \frac{I_3}{\overline{ON}^2}\right)\cos^2\gamma = 0 \ldots (5.)$$

Let this be called *the rolling cone.* Then the motion of the body will be such as would be produced by the rolling of the rolling cone upon a fixed cone generated by the motion of $\overline{OR}$ relatively to $\overline{ON}$.

As free rotation is of unusual occurrence in practical mechanics, I shall refrain from applying its principles to special examples here, and shall refer the reader to the work of M. Poinsot on Rotation, and to a paper by Professor Clerk Maxwell in *The Transactions of the Royal Society of Edinburgh*, vol. xxi.

591. **Uniform Rotation about a Fixed Axis.**—When a body rotates about a fixed axis traversing its centre of gravity, with an uniform angular velocity, its actual energy is still represented, as in the case of free rotation, by

$$E = \frac{a^2 I}{2g} = \text{constant}; \ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

and its angular momentum by

$$A = \frac{a}{g}\sqrt{(I^2 + K^2)} = \text{constant}; \ldots\ldots\ldots\ldots (2.)$$

but *unless the axis of rotation is an axis of inertia*, the axis of angular momentum O N is no longer fixed, but revolves about the fixed axis of rotation O R with the angular velocity *a*. In order to produce that continual change in the direction of the axis of angular momentum, a continual angular impulse, or continuously acting couple, must be applied to the body; and unless that couple be applied, the axis of rotation will not remain fixed.

592. The **Deviating Couple,** as the couple required for the above purpose is called, must have its axis always perpendicular to the axis of angular momentum, otherwise it would alter the amount of the angular momentum, contrary to the condition of uniform rotation. The axis of the deviating couple must also be always per-

pendicular to the axis of rotation, because, in order that it may not alter the actual energy of the body (contrary to the condition of uniform rotation), the pair of equal and opposite forces composing it must act through points having no motion; that is, through points in the axis of rotation. (In machines, the forces constituting the deviating couples are supplied by the pressures of the bearings against the axles.) It appears, therefore, that the axis of the deviating couple must always be perpendicular to the plane O R N, which contains the axes of rotation and of angular momentum; and that the pair of forces constituting it must always act in that plane, changing their direction as the body rotates, with an angular velocity equal to that of the body. The direction of the deviating couple must be such as would of itself tend to turn O N *towards* O R.

To determine the amount of the deviating couple, let θ, as before, denote the angle R O N. Then in the indefinitely short interval of time dt, the direction of the axis of angular momentum is shifted through the indefinitely small angle

$$a\,dt \cdot \sin\theta,$$

and the result differs to an indefinitely small extent from that which would be produced by combining with the actual angular momentum A, an angular momentum about the axis of the deviating couple represented by

$$\mathrm{A}\,a \cdot \sin\theta \cdot dt = \frac{a^2}{g}\sqrt{\mathrm{I}^2+\mathrm{K}^2} \cdot \sin\theta \cdot dt;$$

and this is the angular impulse to be supplied in the interval dt by the deviating couple; therefore the deviating couple is

$$\mathrm{M} = \mathrm{A}\,a \cdot \sin\theta = \frac{a^2}{g}\sqrt{\mathrm{I}^2+\mathrm{K}^2} \cdot \sin\theta;$$

but $\sin\theta = \frac{\mathrm{K}}{\sqrt{\mathrm{I}^2+\mathrm{K}^2}}$; therefore

$$\mathrm{M} = \frac{a^2\,\mathrm{K}}{g}; \qquad \text{(1)}$$

and if Q be the magnitude of each of the forces constituting this couple, and l the length of the arm on which they act (being the distance between their points of application to the axis), so that $\mathrm{M} = \mathrm{Q}\,l$, then

$$\mathrm{Q} = \frac{\mathrm{M}}{l} = \frac{a^2\,\mathrm{K}}{g\,l}; \qquad \text{(2.)}$$

which being compared with the expression for *deviating force* in Article 537, shows that the force of a deviating couple bears the same relation to the angular velocity a, the *moment of deviation* K, and the arm l, which a simple deviating force bears to the linear velocity v, the weight W, and the radius vector r.

To represent these principles graphically, it is to be observed that in fig. 241, the ratio of moment of deviation to the moment of inertia is

$$\mathrm{K} : \mathrm{I} :: \overline{\mathrm{R\,N}} : \overline{\mathrm{O\,N}}\,;\ldots\ldots\ldots\ldots\ldots\ldots(3.)$$

and that this also expresses the ratio of the *deviating couple* to *double the actual energy*, viz. :—

$$\frac{\mathrm{M}}{2\,\mathrm{E}} = \frac{\mathrm{K}}{\mathrm{I}} = \tan\theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4.)$$

The reaction of the axis of the rotating body on its bearings, equal and opposite to the deviating couple,—that is, tending to turn the axis of those bearings towards the axis of angular momentum O N, —is called the CENTRIFUGAL COUPLE. It is balanced, in machines, by the strength and rigidity of the framework.

The amount and direction of the deviating couple might have been determined by finding the resultant couple of the deviating forces required to make each particle of the body revolve in a circle about O R with the common angular velocity; and the result would have been exactly the same.

593. **Energy and Work of Couples.**—The *energy exerted* by a couple is the product of the common magnitude of its pair of forces into the sum of the distances through which their points of application move in the interval of time under consideration; and as that sum is the product of the length of the arm of the couple into the angle through which it rotates about its axis in that time, the energy exerted may be expressed by

$$\mathrm{F}\,l\,d\,i = \mathrm{M}\,d\,i = \mathrm{M}\,a\,d\,t, \ldots\ldots\ldots\ldots\ldots\ldots(1.)$$

$d\,i$ being the angle of rotation about the axis of the couple in the interval $d\,t$, with the angular velocity a. When the couple acts *against* the direction of rotation, the above expression becomes negative, and represents *work performed.*

If a couple be applied to a rotating body whose axis of rotation makes an angle ϕ with the axis of the couple, then the energy exerted may be found either by resolving the couple into two components, one about the axis of rotation, which is either an accelerating or a resisting couple, gives rise to energy exerted or work performed, as the case may be, and may be called the *direct* couple, and the other about an axis perpendicular to the axis of rotation, which may be called the *lateral* couple,—or by resolving the rotation into components about the axis of the couple and about an axis perpendicular to it, and multiplying the former component by the couple.

The result obtained by either method is expressed by

$$\mathrm{M}\, a \cos \phi \cdot d\, t, \qquad \text{(2.)}$$

which represents energy exerted or work performed, according as the couple acts with or against the rotation.

When the direct couples applied to a rotating body are balanced, the actual energy of the body remains constant, the potential energy exerted in any interval of time is equal to the work performed; that is

$$\Sigma \cdot \mathrm{M} \cos \phi = 0; \qquad \text{(3.)}$$

and the same law holds for the energy exerted and work performed during each *period* in the motions of a body or system, whose motions vary periodically; but it is unnecessary to enter in detail into the consequences of these propositions, which are only a particular form of expressing a part of the general principles already explained in Articles 518, 519, 520, and 553, further than to state that the principle of *virtual velocities* (Article 520), when applied to a system of bodies in equilibrio, capable of rotating with angular velocities bearing given ratios to each other, takes the form,

$$\Sigma \cdot \mathrm{M}\, a \cos \phi = 0, \qquad \text{(4.)}$$

where a is either the uniform angular velocity of which the body acted on by the couple M is capable about an axis making the angle ϕ with the axis of M, or any number proportional to that angular velocity.

SECTION 3.—*On Varied Rotation.*

594. The **Law of Varied Rotation** is the Theorem already stated in Article 562, of the equality of each variation of angular momentum to the angular impulse producing it; a principle which has already been applied to the finding of the deviating force required to produce uniform rotation about a fixed axis.

To express this mathematically, let x, y, z, denote three fixed rectangular axes, with which the axis of angular momentum makes the angles λ, μ, ν; and let the angular momentum be resolved into three components about those three axes,

$$\mathrm{A}_x = \mathrm{A} \cos \lambda; \ \mathrm{A}_y = \mathrm{A} \cos \mu; \ \mathrm{A}_z = \mathrm{A} \cos \nu;$$

also, let the unbalanced couple which acts on the body be resolved into three rectangular components denoted by

$$\mathrm{M}_x;\ \mathrm{M}_y;\ \mathrm{M}_z;$$

then

$$\frac{d\,\mathrm{A}_x}{d\,t} = \mathrm{M}_x;\ \frac{d\,\mathrm{A}_y}{d\,t} = \mathrm{M}_y;\ \frac{d\,\mathrm{A}_z}{d\,t} = \mathrm{M}_z \ldots\ldots\ldots\ldots (1.)$$

Those three equations express the relations between the unbalanced couple and the rate of change of the angular momentum. Those relations may otherwise be expressed as follows:—let ψ be the angle made by the axis of the unbalanced couple with the axis of angular momentum; then the couple may be resolved into two components,

$$\mathrm{M} \cos \psi \text{ and } \mathrm{M} \sin \psi,$$

of which the former produces *variation in the amount* of angular momentum, and the latter, *deviation of the axis* of angular momentum, according to the following laws:—

$$\frac{d\,\mathrm{A}}{d\,t} = \mathrm{M} \cos \psi;\ \mathrm{A}\,\frac{d\,\chi}{d\,t} = \mathrm{M} \sin \psi; \ldots\ldots\ldots\ldots (2.)$$

in the latter of which equations, $d\,\chi$ denotes the angle through which the axis of angular momentum deviates in the indefinitely small interval $d\,t$, in the plane which contains that axis and the axis of the couple M, and in a direction towards the latter axis. This equation of deviation of angular momentum has in fact been already employed in Article 592, to find the deviating couple required in order to fix the axis of rotation, when that differs from the axis of angular momentum.

The equations 1, or their equivalents 2, are not of themselves sufficient to determine the variations of motion of a body rotating without a fixed axis; for in such a body, the angular momentum may change by a change of the *direction of its axis relatively to the body*, as well as by a variation of amount, or a deviation of its axis in absolute direction. This is expressed by putting for the angular momentum its value in terms of the moments of inertia and deviation relatively to the instantaneous axis, viz., $\mathrm{A} = \frac{a\sqrt{\mathrm{I}^2 + \mathrm{K}^2}}{g}$; when the equations 1 take the following form:—

$$g\,\mathrm{M}_x = \frac{d}{d\,t}\left\{ a \cos \lambda \cdot \sqrt{\mathrm{I}^2 + \mathrm{K}^2} \right\};\ \text{and analogous equations for}$$

$$g\,\mathrm{M}_y \text{ and } g\,\mathrm{M}_z; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.)$$

while the equations 2 become

$$\left.\begin{aligned} g\,\mathrm{M}\cos\psi &= \frac{d}{dt}\left(a\sqrt{\mathrm{I}^2+\mathrm{K}^2}\right); \\ g\,\mathrm{M}\sin\psi &= a\sqrt{\mathrm{I}^2+\mathrm{K}^2}\cdot\frac{d\chi}{dt}. \end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots(4.)$$

It is therefore necessary to have an additional equation to complete the data for the solution of the problem; and this is afforded by the law of the *conservation of energy*, in virtue of which the actual energy stored or restored by the rotating body is equal to the energy exerted or consumed by the unbalanced couple, according as it acts with or against the rotation, as the following equation expresses, where ϕ is the angle between the axis of the unbalanced couple and the instantaneous axis of rotation.

$$\mathrm{M}a\cos\phi = \frac{1}{2g}\cdot\frac{d\cdot a^2\mathrm{I}}{dt}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5.)$$

The equations 3 or 4, together with 5, and with the relations between the positions of the axes of rotation and of angular momentum demonstrated in the two preceding sections, serve to solve the problem of varied rotation in its utmost generality, and give rise to some exceeding complex mathematical investigations. In the present treatise, however, it will be sufficient to show the solution of some of the more simple cases.

595. **Varied Rotation about a Fixed Axis.**—When a body rotates about a fixed axis traversing its centre of gravity, and is acted upon by a couple M, whose axis makes an angle ϕ with the axis of rotation, that couple is to be resolved into a *direct couple*, M cos ϕ, about the axis of rotation, which will be an accelerating or retarding couple according as it acts with or against the motion, and a *lateral couple*, M sin ϕ, which tends to deviate the axis of rotation, but is balanced by the resistance of the bearings. The entire amount of the couple to be resisted by the bearings at any instant is the resultant of this lateral couple and of the centrifugal couple (Article 592), due to the deviation (if any) of the axis of angular momentum.

The effect of the direct couple in varying the angular velocity is found by means of the law of the conservation of energy, observing that I in this case is constant; that is to say,

$$\mathrm{M}a\cos\phi = \frac{a\,\mathrm{I}\,da}{g\,dt};\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.)$$

and by dividing this equation by a, and observing that $a\,dt = di$, where di is any indefinitely small angle of rotation, it is made to assume the following forms:—

$$\mathrm{M} \cdot \cos \phi = \frac{\mathrm{I}\, d\, a}{g\, d\, t} = \frac{\mathrm{I}}{g} \cdot \frac{d^2 i}{d\, t^2} = \frac{\mathrm{I}\, a\, d\, a}{g\, d\, i}; \ldots\ldots\ldots\ldots\ldots (2.)$$

showing that the direct couple is equal at once to the *variation of angular momentum about the fixed axis divided by the time*, and to the *variation of actual energy divided by the angular motion.*

596. **Analogy of Varied Rotation and Varied Translation.**—When the equation of Article 554 is compared with equation 2 of Article 595, it appears that those equations are exactly analogous to each other, and that the former is transformed into the latter, when for

$$\mathrm{P}, \quad \mathrm{W}, \quad s, \quad v,$$

there are respectively substituted

$$\mathrm{M} \cos \phi, \quad \mathrm{I}, \quad i, \quad a;$$

that is to say, a direct couple for a direct force, moment of inertia for weight, angular motion for linear motion, and angular velocity for linear velocity.

Consequently, by making those substitutions, any equation relating to the varied translation produced by a direct force, may be transformed into a corresponding equation respecting the varied rotation of a body about a fixed axis traversing its centre of gravity produced by a direct couple. Examples of this principle are given in the two following Articles.

597. **Uniform Variation** of angular velocity is produced by a constant couple, and is analogous to the vertical motion of a heavy body, as given in Article 533. In that Article, g is the proportion of the moving force to the mass of the body. Let M be the couple, and let $\phi = 0$; that is, let the couple be altogether about the axis of rotation. Then for g is to be substituted

$$\frac{g\,\mathrm{M}}{\mathrm{I}};$$

which is to be considered positive when in the direction of the initial angular velocity a_0; and for h is to be substituted i. Then equations 1 and 3 of Article 533, being transformed, give for the angular velocity and total angular motion at the end of a given time t, the expressions

$$\left.\begin{aligned} a &= a_0 + \frac{\mathrm{M}\, g\, t}{\mathrm{I}}; \\ i &= a_0 t + \frac{\mathrm{M}\, g t^2}{2\,\mathrm{I}}. \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

Equation 4 gives

$$\mathrm{M}\, i = \frac{(a^2 - a_0^2)\,\mathrm{I}}{2\, g}; \ldots\ldots\ldots\ldots\ldots\ldots (2.)$$

which is also the result of applying to the present case the law of the conservation of energy; the right hand side of the equation being the potential energy exerted, and the left hand side the actual energy stored.

To find through what angle a body will turn before stopping against a constant resistance, its initial angular velocity being a_0, it is to be considered that if R is the resistance, and l its perpendicular distance from the fixed axis, the resisting couple is

$$\mathrm{M} = -\mathrm{R}\,l\,;$$

and that a is to be made $=0$; whence equation 2 gives

$$\mathrm{R}\,l\,i = \frac{a_0^2\,\mathrm{I}}{2\,g} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.)$$

598. **Gyration** about a fixed axis, or **Angular Oscillation,** is alternate rotation to one side and to the other of a middle position. Let a straight line be conceived to be drawn perpendicular to the axis of the gyrating body, to serve as an index; let its middle position be denoted by 0, and its angular displacement from that position by i, positive or negative according as it is to one side or to the other; and let i_1 be the *semi-amplitude* of gyration, or extreme displacement. To produce gyration, the body must be acted upon by a couple directed towards the middle position; that is, contrary to the displacement i. In most cases which occur, the couple is either exactly or nearly proportional to the displacement. Supposing it to be exactly proportional, let M_1 be its extreme *magnitude irrespective of sign;* then

$$\mathrm{M} = -\frac{\mathrm{M}_1\,i}{i_1}\,;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

the negative sign showing that the couple is contrary to the displacement, tending to restore the body to its middle position.

It is obvious from this equation, that gyration is analogous to *straight oscillation,* explained in Article 542; and that the equations of that Article are to be transformed by substituting respectively for

$$r,\quad x,\quad \mathrm{Q},\quad \frac{\mathrm{W}\,a^2}{g},\quad \mathrm{Q}_x,\quad \frac{d\,x}{d\,t},\quad a^2,$$

$$i_1,\quad i,\quad \mathrm{M}_1,\quad \frac{\mathrm{M}_1}{i_1},\quad \mathrm{M},\quad a,\quad \frac{g\,\mathrm{M}_1}{i_1\,\mathrm{I}}.$$

For brevity's sake, let the substitute for a^2 be thus expressed :—

$$\frac{g\,\mathrm{M}_1}{i_1\,\mathrm{I}} = k^2\,;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.)$$

then by transforming equation 4 of Article 542, it appears that the *number of double gyrations per second* is

$$n = \frac{k}{2\pi}; \text{..............................}(3.)$$

which is independent of the semi-amplitude i_1 so long as M_1 is proportional to i_1, and I is constant. This constitutes *isochronism,* and is the property aimed at in the balance wheels of watches, where I is the moment of inertia of the wheel, and the couple is derived from the elasticity of the balance spring.

The equations 2 and 3 being transformed, give for the angle and angular velocity of displacement at any instant,

$$\left.\begin{aligned} i &= i_1 \cos k t; \\ a &= \frac{d i}{d t} = -k i_1 \sin k t; \end{aligned}\right\} \text{...............} (4.)$$

and the maximum couple M_1, in terms of the number of double oscillations per second n, is given by the equation

$$M_1 = \frac{k^2 i_1 I}{g} = \frac{4\pi^2 n^2 i_1 I}{g} \text{........................}(5.)$$

599. A **Single Force** applied to a body with a fixed axis causes the bearings of the axis to exert a pressure equal, opposite, and parallel; so that if the line of action of the force traverses the fixed axis, it is balanced; and if not, a couple is formed whose moment is the product of the force into its perpendicular distance from the axis, and whose effects are such as have been already described.

SECTION 4.—*Varied Rotation and Translation Combined.*

600. **General Principles.**—All rotation of a body about an axis, fixed or instantaneous, which does not traverse the centre of gravity of the body, is to be considered as compounded of rotation about a parallel axis traversing the centre of gravity, and translation of the centre of gravity with a velocity equal to the product of the angular velocity into the distance of the centre of gravity from the actual axis of rotation.

Consequently, every variation of the motion of a body, which consists in a variation of the angular velocity about an axis, fixed or instantaneous, and not traversing the centre of gravity, is to be considered as producing a change of the *momentum,* which is the product of the mass of the entire body into the velocity of its centre of gravity, and a simultaneous change of the *angular momentum* due to the rotation of the body with the given angular velocity about an axis traversing its centre of gravity parallel to the actual axis of rotation; and the force required to produce the given variation of motion will be the resultant of the force required to produce the change of momentum, applied at the centre of gravity, and the couple required to produce the change of angular momentum.

601. **Properties of the Centre of Percussion.**—In fig. 239, Article 581, page 520, let G be the centre of gravity of a rigid body whose weight is W, X X the axis about which, in the interval dt, a change of angular velocity denoted by $d\,a$ takes place, and $\overline{\mathrm{G\,C}} = r_0$, the perpendicular distance of the centre of gravity from that axis. Then the force, in a direction perpendicular to the plane of X X and G C, required at G to produce the change of momentum, is

$$\mathrm{F} = \frac{\mathrm{W}\, r_0\, d\,a}{g\, d\,t};\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.)$$

and the couple required to produce the change of angular momentum due to the change of angular velocity $d\,a$ about the axis G D || X X is

$$\mathrm{M} = \frac{\mathrm{I}_0\, d\,a}{g\, d\,t};\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.)$$

and the resultant of that force and couple (according to Article 41) is a force acting in the same plane with them, parallel and equal to F, and in the same direction, but acting through a point whose distance from G, in a direction opposite to G C, is

$$\frac{\mathrm{M}}{\mathrm{F}} = \frac{\mathrm{I}_0}{\mathrm{W}\, r_0} = \frac{\varrho_0^2}{r_0} = \overline{\mathrm{G\,B}};\ldots\ldots\ldots\ldots\ldots\ldots(3.)$$

that is, the resultant of the force and couple is a *single force* F *acting through the centre of percussion* B *corresponding to the given axis.* (See Article 581, equation 4.)

Now suppose, as in Article 581, that the weight of the body is distributed in two rigidly connected masses, one concentrated at C and the other at B, and having their common centre of gravity still at G. Then in producing the same change of angular velocity $d\,a$ about the axis X C X, the momentum of C is unchanged, while that of B undergoes the change

$$\mathrm{B} \cdot \overline{\mathrm{B\,C}} \cdot \frac{d\,a}{g} = \mathrm{W}\, r_0 \cdot \frac{d\,a}{g},$$

being the exact change of momentum already given in equation 1; a consequence, indeed, of the fact, that the centre of gravity is not changed by the concentration of the masses at B and C; and to produce this change of momentum in the interval $d t$, there is required the same force F applied at B, which has already been found; which proves the following

THEOREM I. *If the mass of a body be conceived to be concentrated at two rigidly connected points, one at a given axis, and the other at the corresponding centre of percussion, so as not to alter the position of the centre of gravity of the body, the force required to produce a given change of angular velocity in the body about the given axis is the same, in magnitude, direction, and line of action, with that required to produce the corresponding change of motion in that part of the mass which is conceived to be concentrated at the centre of percussion.*

This proposition might also have been arrived at by considering

THEOREM II. *If a body rotates about a given axis not traversing its centre of gravity, and the mass of that body be conceived to be concentrated at the axis of rotation and centre of percussion so as not to alter the centre of gravity, the momentum, the angular momentum, and the actual energy of the body are not changed by that concentration of mass.*

For the centre of gravity being unchanged, the momentum is unchanged; and because (by the definition of the centre of percussion) the moment of inertia about the axis of rotation is unchanged, the angular momentum and actual energy are unchanged.—Q. E. D.

COROLLARY. From Theorem I., and from equation 5 of Article 581, it follows, that the action of an impulse upon a free body at either of the points B or C, produces a rotation about an axis traversing the other point.

602. **Fixed Axis.**—When the axis of rotation X X is fixed, an impulse applied to the centre of percussion B, in a direction perpendicular to the plane B X X, simply alters the angular velocity according to the principles explained in the last Article, without causing any additional pressure between the axis and its bearings. But should the force giving the impulse not traverse the centre of percussion, or traverse it in a different direction, it is to be resolved by the principles of statics into two components, one traversing the centre of percussion in the required direction, and the other traversing the axis of rotation; when the former will produce change of motion, and the latter will be balanced by the resistance of the bearings of the axis.

603. **The Deviating Force** of a body rotating about a fixed axis not traversing its centre of gravity is the resultant of the deviating force due to the revolution of the whole mass conceived as concentrated at its centre of gravity, found as in Article 540, combined with the deviating couple due to the rotation of the body with the same angular velocity about a parallel axis traversing the centre of gravity, found as in Article 592. This resultant deviating force is

supplied by the resistance of the bearings of the axis, and an equal and opposite CENTRIFUGAL FORCE is exerted by the axis against the bearings.

604. A **Compound Oscillating Pendulum** is a body supported by a horizontal fixed axis, about which it is free to swing under the action of its own weight, its centre of gravity not being in the axis.

Now, by Article 601, Theorem II., the momentum and angular momentum of the body are at every instant the same as if its mass were concentrated at the axis and at the centre of oscillation in the proportions given by Article 581, equations 1 and 6; and by the definition of the centre of oscillation, the statical moment of the weight of the body with respect to the axis, being the couple which causes the motion, is in every position the same as if the mass were concentrated in these proportions; therefore, the motion of the body is exactly the same as if it were so concentrated; that is to say, it oscillates in the same time and according to the same laws, with a simple oscillating pendulum as defined in Article 544, whose length is the distance from the axis X C X to the centre of oscillation B, as given by equation 3 of Article 581, viz.:—

$$\overline{\mathrm{B\,C}} = \frac{\varrho_0^2}{r_0} + r_0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

Such a simple pendulum is called the *equivalent simple pendulum.*

It is obvious that, for a given body swinging about all possible axes parallel to a given direction in the body, the shortest equivalent simple pendulum is that whose length is the minimum value of $\overline{\mathrm{B\,C}}$ as given by the above equation. That minimum length corresponds to the condition,

$$\left.\begin{array}{ll} & \varrho_0 = r_0; \\ \text{whence,} & \text{min. } \overline{\mathrm{B\,C}} = 2\,\varrho_0; \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.)$$

that is to say, the least period of oscillation of a pendulous body takes place when the distance of its centre of gravity from its axis is equal to the radius of gyration about a parallel axis traversing the centre of gravity; and the length of the equivalent simple pendulum is double of that radius of gyration.

If for a given direction of axis, a pair of points be so related that each is the centre of percussion for an axis in the given direction traversing the other (as shown by Article 581, equation 5), then the period of oscillation about either axis is the same.

From the properties of the centre of percussion explained in this Article, it is sometimes called the CENTRE OF OSCILLATION.

605. **Compound Revolving Pendulum.**—To avoid unnecessary complexity in the theory of a compound revolving pendulum, let the body of which it consists be of such a figure and so suspended, that the straight line C G B (fig. 239), traversing the point of suspension C and the centre of gravity G, shall be one of the axes of inertia, and that the moments of inertia about the other two axes shall be equal. Then for every axis traversing the centre of gravity at right angles to C G B, the radius of gyration is the same; and consequently, for every axis traversing the point of suspension C at right angles to C G B, the centre of percussion B is the same; and the body moves exactly like a simple revolving pendulum of the length $\overline{\mathrm{C\,B}}$, and height $\overline{\mathrm{C\,B}} \cdot \cos \theta$, if θ is the angle which it makes with the vertical.

It is to be borne in mind, that in order that a pendulum may revolve according to the above law, it must have *no rotation* about its longitudinal axis B G C, but must swing as if hung by a double universal joint at C (Article 492).

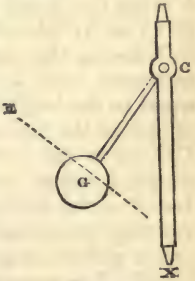

Fig. 242.

606. A **Rotating Pendulum** (fig. 242) is a body C G suspended by a point C not in the centre of gravity G, and rotating about a vertical axis C X traversing the point of suspension. To avoid needless complexity, as before, let C G, and E G perpendicular to it in the vertical plane of C G and C X, be two of the axes of inertia of the pendulum. Let I_1 be its moment of inertia about G E, and I_2 its moment of inertia about G C, and ϱ_1, ϱ_2, the corresponding radii of gyration. Let the angle X C G $= \alpha$; let C G $= r_0$; and let the weight of the pendulum be W. Then, a being the angular velocity of rotation about the vertical axis, it appears from Articles 592 and 586 that the deviating couple due to rotation about a vertical axis traversing G is

$$\frac{a^2}{g}(\mathrm{I}_1 - \mathrm{I}_2) \cos \alpha \sin \alpha = \frac{\mathrm{W}\, a^2}{g}(\varrho_1^2 - \varrho_2^2) \cos \alpha \sin \alpha;$$

to which has to be added, the couple due to the deviating force of W revolving along with the centre of gravity G, and to the leverage $r_0 \cos \alpha$, being the height of C above G; that is to say,

$$\frac{\mathrm{W}\, a^2}{g} \cdot r_0^2 \cos \alpha \sin \alpha;$$

making for the entire deviating couple

$$\frac{\mathrm{W}\, a^2}{g}(\varrho_1^2 - \varrho_2^2 + r_0^2) \cos \alpha \sin \alpha;$$

and this couple has to be supplied by means of the weight of the pendulum acting with the leverage $r_0 \sin \alpha$; that is, it must be equal to

$$\mathrm{W}\, r_0 \sin \alpha.$$

Dividing by this quantity, we find

$$\frac{a^2}{g}\left(\frac{\varrho_1^2 - \varrho_2^2}{r_0} + r_0\right) \cos \alpha = 1;\text{..................}(1.)$$

and putting for a^2 its value, $4\,\pi^2\,\mathrm{T}^2$, where T is the number of turns per second, this leads to the equation

$$\left(\frac{\varrho_1^2 - \varrho_2^2}{r_0} + r_0\right) \cos \alpha = \frac{g}{4\,\pi^2\,\mathrm{T}^2} = h;\text{..............}(2.)$$

h being the *height of the equivalent simple* revolving pendulum, as given in Article 539, equation 2.

When ϱ_2, the radius of gyration about C G, is insensibly small compared with ϱ_1, the radius of gyration about G E, h becomes equal to the height of the simple pendulum equivalent to the pendulum in the figure, when made to revolve without rotation about C G, as in the last Article. When $\varrho_2 = \varrho_1$, the height becomes simply $r_0 \cos \alpha$, being the same as if the whole mass were concentrated at the centre of gravity. This is very nearly the case in the rotating pendulums used as GOVERNORS for prime movers, which are in general large heavy spheres hung by slender rods.

607. The **Ballistic Pendulum** is used to measure the momentum of projectiles, and the impulse of the explosion of gunpowder. To measure the momentum of a projectile, such as a rifle ball, the pendulum must consist of a mass of material in which the ball can lodge, such as a block of wood, or a box full of moist clay, hung by rods from a horizontal axis. Suppose the ball to be of the weight b, and to move with the velocity v in a line of flight whose perpendicular distance from the axis of suspension is r'. Then the angular momentum of the ball relatively to the axis of suspension is

$$\frac{b\,v\,r'}{g};\text{.............................}(1.)$$

and because the ball lodges in the pendulum, this angular momentum is wholly communicated to the joint mass consisting of the ball and the pendulum, which swings forward, carrying with it an index that remains, and points out on a scale the extreme angular displacement. Let this be denoted by i. Let l denote the length of the simple pendulum equivalent to that mass, which can be found by means of Article 544, equation 1, from the number of oscillations in a given time; let W be the joint weight of the pendulum and ball, and r_0 the distance of their common centre of gravity from the axis; then

$$B = \frac{W r_0}{l}, \text{..........................} (2.)$$

is the portion of the joint weight to be treated as if concentrated at the centre of oscillation.

Let V be the velocity of the centre of oscillation at the lowest point of its arc of motion; this is the velocity due to the height, $l \cdot \text{versin}\, i$; that is to say,

$$V = \sqrt{(2 g l \cdot \text{versin}\, i)} = 2 \sin \frac{i}{2} \cdot \sqrt{g l}; \text{.........} (3.)$$

and the corresponding angular momentum of the combined mass is $\frac{B V l}{g}$; which, being equated to the angular momentum of the ball before the collision (1), gives the equation

$$b v r' = B V l; \text{......................} (4.)$$

giving for the velocity, momentum, and actual energy of the ball, respectively,

$$\left. \begin{array}{c} v = \dfrac{B V l}{b r'}; \\ \dfrac{b v}{g} = \dfrac{B V l}{g r'}; \dfrac{b v^2}{2 g} = \dfrac{B^2 V^2 l^2}{2 g b r'^2}. \end{array} \right\} \text{..............} (5.)$$

The energy of the combined mass after the collision being $\frac{B V^2}{2 g}$, and less than that of the ball before the collision in the proportion of $b r'^2 : B l^2$, shows, that an amount of energy denoted by

$$\frac{b v^2}{2 g} \left(1 - \frac{b r'^2}{B l^2}\right), \text{......................} (6.)$$

disappears in producing heat and molecular changes in the ball and in the soft mass in which it is lodged.

To measure the impulse produced by the explosion of gunpowder, the gun to be experimented on is to be fixed to and form part of the pendulum, and a ball is to be fired from it. The gas produced by the explosion exerts equal pressures during the same time,—that is, equal impulses,—forwards against the ball, and backwards against the gun, and the pendulum swings back through a certain angle, which is registered by an index as before, and from which the maximum velocity of the centre of percussion of the pendulum can be calculated as before by equation 3. Let r' now denote the distance from the axis of suspension to the axis of the gun, and P the pressure exerted by the explosive gas at any instant; the total impulse exerted by the gas is $\int \mathrm{P}\, d\, t$; and the angular impulse $r' \cdot \int \mathrm{P}\, d\, t$; which being equated to the angular momentum produced in the pendulum, gives

$$r' \int \mathrm{P}\, d\, t = \frac{\mathrm{B\,V}\, l}{g}; \quad\text{.......................}(7.)$$

in which it is to be observed, that B does not now include the weight of the ball. The impulse exerted by the powder is therefore

$$\int \mathrm{P}\, d\, t = \frac{\mathrm{B\,V}\, l}{g\, r'}; \quad\text{.......................}(8.)$$

and the velocity of the ball b on leaving the gun is consequently

$$v = \frac{g \int \mathrm{P}\, d\, t}{b} = \frac{\mathrm{B\,V}\, l}{b\, r'} \quad\text{.......................}(9.)$$

The *energy* exerted by the exploding powder is

$$\int \mathrm{P}\, d\, s = \frac{b\, v^2}{2\, g} + \frac{\mathrm{B\,V}^2}{2\, g}; \quad\text{.................}(10.)$$

of which the portions communicated to the ball and to the pendulum are indicated by the two terms, being in the ratio

$$b\, v^2 : \mathrm{B\,V}^2 :: \mathrm{B}\, l^2 : b\, r'^2 \quad\text{...................}(11.)$$

In the preceding calculations, the momentum and energy produced in the explosive gases themselves are not considered; but it is very doubtful whether any attempt to take them into account, hypothetical as it must be, adds to the practical correctness of the result. As a probable approximation, the following may be employed:—Let w be the weight of powder used. Divide this into two parts proportional to b and B, viz.:—

$$\frac{b\, w}{b + \mathrm{B}} \text{ and } \frac{\mathrm{B}\, w}{b + \mathrm{B}};$$

consider the smaller part to move with half the velocity of B, and the larger with half the velocity of b; that is to say, in equations 7, 8, and 9, put,

$$\left.\begin{array}{ll}\text{instead of B,} & \mathrm{B}+\dfrac{b\,w}{2\,(b+\mathrm{B})};\\ \text{and instead of } b, & b+\dfrac{\mathrm{B}\,w}{2\,(b+\mathrm{B})}.\end{array}\right\}\quad\text{(12.)}$$

The equation 10, in its original form, will still show the actual energies of the pendulum and of the ball, and their sum; but that sum will be *exclusive* of the energy exerted in giving motion to the explosive gases themselves.

The ballistic pendulum was invented by Robins, celebrated for his investigations on gunnery.

CHAPTER IV.

MOTIONS OF PLIABLE BODIES.

608. **Nature of the Subject; Vibration.**—The motion of each particle of a pliable body may always be resolved into three components: that which it has in common with the centre of gravity of the body, being the motion due to translation of the whole body; that which it has about the centre of gravity of the body, being the motion due to rotation of the whole body; and a third component, being the motion due to alterations of the volume and figure of the body and of its parts. This third component is alone to be considered in the present chapter.

The *cinematical* branch of the present subject,—that is to say, the branch which comprehends the relations amongst the displacements of the particles in a strained solid from their free positions, and the strains or disfigurements of its parts accompanying such displacements,—has already been treated of generally in Articles 248, 249, 250, 260, and 261; with reference to bending, in part of 293, part of 300, 301, part of 303, part of 304, part of 307, part of 309, part of 312, and part of 319; with reference to twisting, in part of 321 and part of 322;—and again with reference to bending, in part of Article 340.

The *dynamical* branch of the subject has been, to a certain extent anticipated in Article 244, where *resilience* is defined; in Article 252, where *potential energy of elasticity* is defined;* in Articles 266 and 269, which relate to the *resilience of a stretched bar* and the effect of a *sudden pull;* in Article 305, which relates to the *resilience of a beam;* in Article 306, which relates to the effect of a *suddenly applied transverse load;* and in Article 323, which relates to the *resilience of an axle.*

The motions due to strains amongst the particles of pliable bodies being all of limited extent, and consisting in changes of the displacement of each particle from the position which it would occupy in a state of equilibrium, which displacement is limited and generally small, are of the kind called VIBRATIONS, and are more or less analogous to the oscillations already treated of in Articles 542 and 543.

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* In Article 252, the first employment of this function is correctly ascribed to Mr. Green; but it is right also to mention, that its use was independently discovered by M. Clapeyron.

The complete theory of vibration embraces all the phenomena of the production and transmission of sound, and all those of the propagation of light, as well as those of the visible and tangible vibrations of bodies. Many of its branches are foreign to the objects of this treatise; and therefore in the present chapter there will be given only an outline of the general principles of the theory of vibration, and an explanation of such of its applications as are of importance in practical mechanics.

609. **Isochronous Vibrations** of an elastic body are those in which each particle of the body performs a complete oscillation in the same period of time, so that all the particles return to the same relative situations at the end of each equal period of time, and that whether the oscillations are of greater or of less amplitude. Isochronous vibrations being communicated to the ear produce the sensation of a sound of uniform pitch, or musical tone. In order that oscillations of different amplitudes may be performed by equal masses in the same time, it is evidently necessary that the forces under which they are performed should be *proportional, and directly opposed, to the displacements at each instant.* This is the CONDITION OF ISOCHRONISM, and has already been illustrated in Articles 542, 543, 544, 545, and 557, Example III., for the case of a single particle acted on by a single force, and in Article 598 for the analogous case of a gyrating rigid body, where angular is substituted for linear displacement, and a couple for a force. To express that condition by an equation suited to the present class of questions, let $W \div g$ be the mass of a particle, δ its displacement from its position of equilibrium at any given instant, F an unbalanced force by which it is urged directly towards that position, and a^2 a numerical constant, expressed as a square for reasons which will presently appear; then the condition of isochronism is expressed as follows:—

$$F = -\frac{W a^2 \delta}{g}; \quad \ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

an equation identical with equation 1 of Article 542; while from equation 4 of the same Article it appears that the number of double oscillations per second is expressed by

$$\left.\begin{array}{l} n = \dfrac{a}{2\pi}, \\ \text{and the period of a double oscillation by} \\ \dfrac{1}{n} = \dfrac{2\pi}{a}. \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots (2.)$$

All the equations of Article 542 and Article 557, Example III., are made applicable to the present case, by substituting respectively for

$$\begin{array}{cccc} Q \text{ or } Q_1, & Q_x, & r \text{ or } x_1, & x, \\ F_1, & F, & \delta_1, & \delta, \text{ respectively,} \end{array}$$

where F_1 represents the maximum force, corresponding to δ_1, the maximum displacement, or semi-amplitude; consequently, if in order to make the formulæ more general we represent by t_0 any instant of time at which the particle reaches the extremity of an oscillation, we have

$$\left. \begin{array}{r} \delta = \delta_1 \cos a\,(t-t_0); \\ \dfrac{d\,\delta}{d\,t} = -\,a\,\delta_1 \sin a\,(t-t_0). \end{array} \right\} \ldots\ldots\ldots\ldots\ldots\ldots (3.)$$

When the restoring force corresponding to a given displacement is known, the constant a^2 is computed by the formula

$$a^2 = \frac{-g\,F}{W\,\delta}\,; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4.)$$

in which the negative sign denotes, that although F being contrary to δ in direction, their quotient is implicitly negative, it is to have that negativity reversed and to be treated as positive.

The equations 2 and 4 show, that *the square of the number of oscillations made by a particle in a second, is inversely as the mass of the particle, and directly as the ratio of the restoring force to the displacement.*

610. **Vibrations of a Mass held by a Light Spring.**—The deflection of a straight spring or elastic beam under any load is given by the equations of Article 303 for those cases in which it is sensibly proportional to the load.

The position of equilibrium of the spring, if not affected by a lateral transverse load (for example, if it is placed vertically), may be straight; or if there be a permanent transverse load, that position may be more or less deflected. In either case, the production of an independent deflection, δ, of the point for which deflections are computed by the formulæ, to one side or to the other of the position of equilibrium, provided the limits of perfect elasticity are not exceeded, causes the spring to exert a restoring force F, whose value is found by applying to this case equation 4 of Article 303; that is to say,

$$\left. \begin{array}{l} \delta = -\dfrac{n'''\,F\,c^3}{n'\,E\,b\,h^3} \;\therefore\; F = -\dfrac{n'\,E\,b\,h^3}{n'''\,c^3}\cdot\delta\,; \\ \quad = -f\delta \text{ for brevity's sake}; \end{array} \right\} \ldots\ldots\ldots (1.)$$

in which *f* may be called the *stiffness* of the spring.

Now suppose that there is attached to the point of the spring for which δ is calculated, a mass $W \div g$, in comparison with which the mass of the spring is inappreciably small. Then if that mass be drawn to one side or to the other of the position of equilibrium, and left free to vibrate, the spring will make it vibrate according to the law already explained in Article 609, putting for the constant a the value

$$a = \sqrt{\frac{fg}{W}} \dots\dots\dots\dots\dots\dots\dots\dots (2.)$$

If the mass gyrates about a fixed axis traversing its centre of gravity, let l denote the distance from that axis to the point upon which the spring acts; then in the equations of motion, substitutions are to be made according to the principles of Article 598, when the above equation becomes

$$k = \sqrt{\frac{flg}{I}} \dots\dots\dots\dots\dots\dots\dots\dots (3.)$$

If the mass oscillates about a fixed axis not traversing its centre of gravity, the above equation is still applicable, when the proper value is put for the moment of inertia I.

The *restoring couple* F l for a gyrating body may be supplied by the resistance of a rod or wire to torsion; in which case fl is to be taken to represent the ratio of the moment of torsion to the angle of torsion, which, for a cylindrical rod or wire, is given in Article 322, case 2, equation 4, viz.:—

$$fl = \frac{M}{i} = \frac{\pi C h^4}{32 x}; \dots\dots\dots\dots\dots\dots\dots (4.)$$

x being the length, and h the diameter of the rod or wire, and C the co-efficient of transverse elasticity of the material.

By the aid of the principles here explained, experiments on the numbers of vibrations per second made by springs and wires loaded with masses great in proportion to the masses of the springs and wires, may be used to determine the co-efficients of elasticity E and C.

611. **Superposition of Small Motions.**—If the restoring force of a particle for vibrations in a given direction be opposite and proportional to the displacement, and if the same be the case for one or more other directions of vibration, then for a displacement which is the resultant of two or more displacements in the given directions, the force acting on the particle will evidently be the resultant of the separate forces corresponding to the component displacements, and the velocity the resultant of the component velocities.

This is called the principle of the SUPERPOSITION OF SMALL MOTIONS.

If the co-efficient a of Article 609 is the same for the different directions of the component displacements, the component vibrations will not only be isochronous in themselves, but isochronous with each other, or *simultaneous*, and so also will be the resultant vibration. This has already been sufficiently illustrated in Articles 542 and 543, where circular and elliptic oscillations are treated as compounded of a pair of straight oscillations in directions perpendicular to each other. Such, for example, is the oscillation of a mass placed at the end of a spring whose stiffness is the same for all directions of deflection.

If the co-efficient a has different values for the different directions of the component vibrations, they will no longer be isochronous with each other; the resultant restoring force will not at every instant act directly towards the position of equilibrium, and the resultant vibration will take place in a complex curve which may have a great variety of figures. For example, let a mass $W \div g$ be fixed at the end of a spring whose cross section is a rectangle of unequal dimensions, so that its stiffness is different for displacements in the directions of two rectangular axes, denoted by x and y. Let f_x, f_y, be the two values of the stiffness of the spring for those two directions of displacement; and let ξ and η denote component displacements in those two directions respectively, and ξ_1 and η_1 their maximum values or semi-amplitudes. Then the equations of motion of the mass are the following:—

$$\left.\begin{aligned}\xi &= \xi_1 \cos a_x\,(t - t_{0,x});\\ \eta &= \eta_1 \cos a_y\,(t - t_{0,y});\end{aligned}\right\} \text{.....................(1.)}$$

where

$$a_x = \sqrt{\frac{f_x g}{W}};\ a_y = \sqrt{\frac{f_y g}{W}}; \text{...............(2.)}$$

and $t_{0,x}$ and $t_{0,y}$ are two arbitrary constants. Thus the numbers in a second of the two series of component vibrations, viz.,

$$n_x = \frac{a_x}{2\pi},\ n_y = \frac{a_y}{2\pi}, \text{......................(3.)}$$

are proportional to the square roots of the stiffnesses of the spring in the directions of the two rectangular axes; that is, they are proportional to its thicknesses in these two directions respectively.

If n_x and n_y are commensurable, the path of the vibrating mass is a closed curve; for example, to take the simplest case, if $n_x = 2\,n_y$, that path is such a curve as is represented in fig. 243. If n_x

and n_y are incommensurable, the path is of indefinite length; but in every case it is wholly inscribed within the rectangle whose sides are the amplitudes $2\,\xi_1$, $2\,\eta_1$, of the component vibrations.

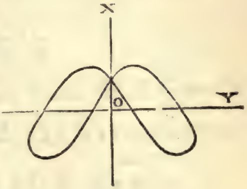

Fig. 243.

612. **Vibrations not Isochronous** can only be expressed mathematically by conceiving them to be compounded of a number of superposed vibrations, each isochronous in itself, but not isochronous with each other, as in the last example of the preceding Article; and the forces under which such vibrations take place are in like manner to be conceived to be resolved into component forces, each proportional to a parallel component of the displacement. The art of resolving displacements of any kind whatsoever into components, each of which separately satisfies the conditions of isochronism, is a mathematical process which it will not be necessary to exemplify in this treatise.

613. **Vibrations of an Elastic Body in General.**—The general equations of the vibration of an elastic body are found by the aid of D'Alembert's principle (Article 568), by conceiving the body to be divided into indefinitely small rectangular or other regularly shaped molecules, and equating the components of the rate of variation of momentum of each molecule to the corresponding components of the restoring force caused by the internal stresses, *which restoring force, for each molecule, is at each instant equal and opposite to the share belonging to that molecule, of a distributed external load that would, in a state of equilibrium, produce the actual state of disfigurement of the body at the instant.* The condition of isochronism is expressed by making each restoring force proportional and opposite to the displacement of the molecule to which it is applied; and the displacements, velocities, and forces for vibrations not isochronous are expressed by sums of series of corresponding quantities for isochronous vibrations.

By the application of D'Alembert's principle as stated above, every equation concerning the equilibrium of an elastic body under external forces distributed amongst its molecules can be converted into a corresponding equation concerning its vibration.

Example I. *General Differential Equations.*—In Article 116, illustrated by fig. 58, are given the equations of internal equilibrium (2.) of an elastic solid for a rectangular molecule $d\,x\,d\,y\,d\,z$, expressing the three components of the external force *per unit of volume* of that molecule, in terms of the equal and opposite components of the internal forces arising from the variations of the six elementary stresses, pulls being considered as positive, and thrusts as negative. Those equations are converted into general equations of vibration of the same molecule by substituting, at the right-hand sides of the three equations respectively,

$$\text{for}\quad \left.\begin{array}{ccc} 0, & 0, & w, \\ \frac{w}{g}\cdot\frac{d^2\xi}{dt^2}, & \frac{w}{g}\cdot\frac{d^2\eta}{dt^2}, & \frac{w}{g}\cdot\frac{d^2\zeta}{dt^2}; \end{array}\right\}\text{..............(1.)}$$

where $\frac{w}{g}$ is the *mass per unit of volume,* and ξ, η, ζ, are the three components of the displacement of the molecule *from its position of equilibrium.*

To make use of the three equations thus obtained, each of the six elementary stresses is to be expressed in terms of the six elementary strains multiplied by the proper co-efficients of elasticity of the substance (Article 253); then each of the six elementary strains is to be expressed as in Article 250, by means of the differential co-efficients of the three component displacements ξ, η, ζ, and thus the three original equations are converted into *three linear differential equations of the second order* in ξ, η, and ζ, by the integration of which, with due regard to the circumstances of each particular problem, all questions respecting vibration are solved. It is unnecessary here to enter into details respecting those integrations. The most complete compendium of the processes which they involve and the results to which they lead, is contained in M. Lamé's *Leçons sur l'Elasticité des Corps solides.*

Example II. *Case of an Axis of Vibration.*—In figs. 244 and 245,

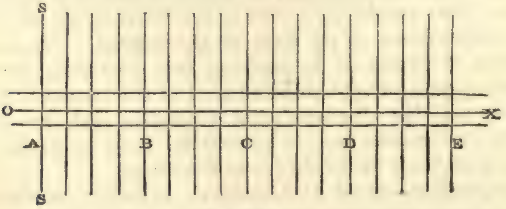

Fig. 244.

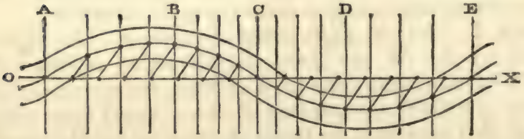

Fig. 245.

let S S and the lines parallel to it represent a series of planes parallel to each other, and let the mode of vibration of the particles of the body be such, that all the particles in any one of those planes have equal displacements in parallel directions at the same instant. A straight line O X, perpendicular to all those surfaces, may be called an *axis of vibration.* Let the displacement of each particle, denoted at any instant by δ, take place in a direction making an angle θ with O X, in the plane of $x\,y$; so that its component displacements are

$$\left.\begin{aligned}\xi &= \delta \cos \theta,\\ \eta &= \delta \sin \theta.\end{aligned}\right\} \dots\dots\dots\dots\dots\dots (2.)$$

In the condition of equilibrium, conceive a square prism to extend along the axis O X, as in fig. 244, and to be divided into cubical molecules, each of the volume $d\,x\,d\,y\,d\,z$, and mass $\frac{w}{g}\,d\,x\,d\,y\,d\,z$. At a given instant in the state of vibration, let those molecules be displaced in the manner shown in fig. 245, the displacement of each point in each molecule depending, according to some law yet to be determined, upon the lapse of time and upon the distance, when in a state of equilibrium, of the plane of equal displacement containing it from O, which distance is denoted by x; that is, let

$$\delta = \text{function of } (t, x) \dots\dots\dots\dots\dots\dots (3.)$$

Then it is evident, that each molecule, originally cubical, becomes directly strained and distorted; the direct strain along x (an elongation if positive) being represented at any instant by

$$\alpha = \frac{d\,\xi}{d\,x} = \frac{d\,\delta}{d\,x} \cos \theta; \dots\dots\dots\dots\dots\dots (4.)$$

and the distortion, in the plane of $x\,y$, by

$$\nu = \frac{d\,\eta}{d\,x} = \frac{d\,\delta}{d\,x} \sin \theta \dots\dots\dots\dots\dots\dots (5.)$$

The vibrating substance will be supposed to be *isotropic* as to elasticity, according to the definition given in Article 256, A being its direct and C its transverse elasticity. Then at a given plane of equal displacement, and at a given instant, there is a direct stress (tension being positive) of the intensity

$$p_{xx} = \mathrm{A}\,\alpha = \mathrm{A}\,\frac{d\,\xi}{d\,x} = \mathrm{A}\,\frac{d\,\delta}{d\,x} \cos \theta; \dots\dots\dots\dots (6.)$$

and a tangential stress of the intensity

$$p_{xy} = \mathrm{C}\,\nu = \mathrm{C}\,\frac{d\,\eta}{d\,x} = \mathrm{C}\,\frac{d\,\delta}{d\,x} \sin \theta; \dots\dots\dots\dots (7.)$$

and applying to these data the reasoning of the preceding example, we find that the components of the moving force, per unit of volume, acting on a given molecule, at a given instant, are as follows :—

$$\left.\begin{array}{ll}\text{Longitudinal,} & Q_x = A\dfrac{d^2\xi}{dx^2} = A\dfrac{d^2\delta}{dx^2}\cos\theta; \\ \text{transverse,} & Q_y = C\dfrac{d^2\eta}{dx^2} = C\dfrac{d^2\delta}{dx^2}\sin\theta;\end{array}\right\}\ldots\ldots\ldots\ldots(8.)$$

so that if we make

$$\frac{gA}{w} = a^2;\ \frac{gC}{w} = c^2;\ldots\ldots\ldots\ldots\ldots\ldots(9.)$$

we find for the equations of vibration,

$$\text{longitudinal,}\qquad \frac{d^2\xi}{dt^2} = a^2\frac{d^2\xi}{dx^2};\ldots\ldots\ldots\ldots\ldots\ldots(10.)$$

$$\text{transverse,}\qquad \frac{d^2\eta}{dt^2} = c^2\frac{d^2\eta}{dx^2}\ldots\ldots\ldots\ldots\ldots\ldots(11.)$$

The general integral of those two equations is given by the pair of equations,

$$\left.\begin{array}{l}\xi = \phi(at+x) + \psi(at-x); \\ \eta = \chi(ct+x) + \omega(ct-x);\end{array}\right\}\ldots\ldots\ldots\ldots(12.)$$

where ϕ, ψ, χ, ω, represent *any functions whatsoever.* But to obtain definite results, which can be used in calculation, the conditions of isochronism are to be applied; and they lead to the following consequences :—

First, in order that vibrations may be isochronous, the restoring force must act along the direction of vibration; that is, we must have

$$Q_x : Q_y :: \cos\theta : \sin\theta;\ldots\ldots\ldots\ldots\ldots\ldots(13.)$$

and because for every known substance, A and C are unequal, this condition can only be fulfilled when either $\cos\theta$ or $\sin\theta$ is nothing; that is to say, *in an isotropic substance, isochronous vibrations are either wholly longitudinal, or wholly transverse.*

Secondly, the moving force acting on a particle must be proportional and opposite to its displacement; a condition expressed for longitudinal and transverse vibrations respectively, by

$$\frac{d^2\xi}{dt^2} = a^2\frac{d^2\xi}{dx^2} = -b^2\xi;\ldots\ldots\ldots\ldots\ldots\ldots(14.)$$

$$\frac{d^2\eta}{dt^2} = c^2\frac{d^2\eta}{dx^2} = -b'^2\eta\ldots\ldots\ldots\ldots\ldots\ldots(15.)$$

where b^2 and b'^2 are two arbitrary positive constants. The most convenient way of expressing those constants, for reasons which will afterwards appear, is the following :—

$$b = \frac{2\pi}{\lambda}\,;\; b' = \frac{2\pi}{\lambda'}\,;$$

λ and λ' being *arbitrary lengths.* Then it is easily seen, that to satisfy the equations 14 and 15, the displacements must be expressed as follows :—

$$\xi = \xi_1 \cdot \cos \frac{2\pi}{\lambda}(x - x_0) \cdot \cos \frac{2\pi a}{\lambda}(t - t_0);\ldots\ldots(16.)$$

$$\eta = \eta_1 \cdot \cos \frac{2\pi}{\lambda'}(x - x'_0) \cdot \cos \frac{2\pi c}{\lambda'}(t - t'_0)\ldots\ldots\ldots(17.)$$

ξ_1, η_1, λ, λ', x_0, x'_0, t_0, and t'_0 being arbitrary constants, having values depending on the circumstances of each particular problem. These constants have the following meanings :—

ξ_1 and η_1 are the *maximum semi-amplitudes* of vibration.

$\frac{\lambda}{2\pi a}$ and $\frac{\lambda'}{2\pi c}$, are the *periodic times* of a complete oscillation.

λ and λ' are the distances (for the longitudinal and transverse vibrations respectively) between a pair of planes in which the particles are in the same *phase* of vibration at the same instant; such as the planes A and E in figs. 244, 245.

Nodal planes are planes in which the particles have no displacement, $x - x_0$, or $x - x'_0$, being an odd multiple of $\frac{\lambda}{4}$ or $\frac{\lambda'}{4}$. Their distance apart is $\frac{\lambda}{2}$ or $\frac{\lambda'}{2}$ (A, C, and E, in the figures).

Ventral planes are those of maximum displacement, $x - x_0$, or $x - x'_0$, being a multiple of $\frac{\lambda}{2}$ or $\frac{\lambda'}{2}$ (B and D in the figures). They lie midway between the nodal planes.

The following quantities for isochronous vibrations are deduced from equations 16 and 17 :—For longitudinal vibrations,

$$\left.\begin{aligned} &\left.\begin{matrix}\text{velocity of}\\ \text{a particle,}\end{matrix}\right\}\ \frac{d\xi}{dt} = -\frac{2\pi a}{\lambda}\,\xi_1 \cdot \cos\frac{2\pi}{\lambda}(x - x_0)\cdot \sin\frac{2\pi a}{\lambda}(t - t_0);\\ &\text{direct strain, } \frac{d\xi}{dx} = -\frac{2\pi}{\lambda}\cdot \xi_1 \sin\frac{2\pi}{\lambda}(x - x_0)\cdot \cos\frac{2\pi a}{\lambda}(t - t_0). \end{aligned}\right\}\ (18.)$$

For transverse vibrations,

$$\left.\begin{array}{l} \left.\begin{array}{l}\text{velocity of}\\ \text{a particle,}\end{array}\right\} \dfrac{d\eta}{dt} = -\dfrac{2\pi c}{\lambda'}\eta_1 \cdot \cos\dfrac{2\pi}{\lambda}(x-x'_0)\cdot\sin\dfrac{2\pi c}{\lambda'}(t-t'_0); \\ \text{Distortion, } \dfrac{d\eta}{dx} = -\dfrac{2\pi}{\lambda'}\eta_1\cdot\sin\dfrac{2\pi}{\lambda}(x-x'_0)\cdot\cos\dfrac{2\pi c}{\lambda'}(t-t'_0). \end{array}\right\} (19.)$$

Vibrations may exist in which the displacements, strains, velocities, and forces, are the resultants of combinations of isochronous vibrations, having any number of different sets of arbitrary constants, and having only in common the co-efficients a and c.

The results of the preceding investigation, so far as they relate to longitudinal vibrations, are applicable to fluids as well as to solids. Transverse vibrations are impossible in fluids, because in them there is no transverse elasticity.

614. **Waves of Vibration** consist in the transmission of a vibratory state from particle to particle through a body. Let OX denote the direction in which the vibratory state is transmitted, being, as in the last Article and its figures, an *axis of vibration*, or line perpendicular to a series of surfaces of simultaneous and equal displacement, which surfaces do not now remain stationary, but advance from particle to particle with a velocity called the *velocity of transmission* or of *propagation*. With respect to wave motion in general, it has already been explained in Article 416, that the condition of motion of any particle, whose distance from the origin is x, is expressed by a function of $a\,t-x$, where t is the time elapsed from a given instant, and a the *velocity of transmission*. Applying this to the displacements in longitudinal and transverse vibrations respectively, we find the equations

$$\left.\begin{array}{l}\xi = \varphi\,(a\,t-x);\\ \eta = \psi\,(c\,t-x);\end{array}\right\}\dots\dots\dots\dots\dots\dots(1.)$$

where a and c are the velocities of transmission of longitudinal and transverse vibrations respectively. Now the equations 1 have already been shown in Article 613 to be forms of the integrals of the general equations of vibratory motion, a and c having the values there given, viz. :—

$$a = \sqrt{\frac{g\,\mathrm{A}}{w}};\ \ c = \sqrt{\frac{g\,\mathrm{C}}{w}}\dots\dots\dots\dots\dots(2.)$$

which accordingly are the respective velocities of transmission of waves of longitudinal and transverse vibration in a medium whose weight per unit of volume is w, and its direct and transverse elasticities A and C. In a fluid, for which $\mathrm{C} = 0$, the transmission of waves of transverse vibration is impossible.

It may here be observed, that it is essential to the exactness of the values given above for the velocities of the transmission of waves, that the *surfaces of simultaneous displacement* (called sometimes *wave-surfaces*) should also be surfaces of *equal amplitude of vibration.* If the amplitude varies at different points of the same wave-surface, the velocity of transmission becomes less than that given by the equations 2, according to a law which it is unnecessary here to explain in detail.

615. **Velocity of Sound.**—Longitudinal vibrations, being those which can be transmitted through all substances, solid and fluid, are the ordinary means of transmitting sound; so that the velocity of sound in a given medium is the co-efficient a in the equations 2 of Article 614; being the velocity which a body would acquire in falling from the height $\mathrm{A} \div 2\,w$; that is, a height equal to half the length of a prism of the substance of the base unity, whose weight is equal to the co-efficient of longitudinal elasticity.

The velocity of sound, as determined by experiment, is,

In water, at 61° Fahr..... 4,708 feet per second;
In dry air, at 32° Fahr.... 1,092 ... ...

In air and other gases, the velocity of sound depends on the pressure, density, and temperature in the following manner:—When a nearly perfect gas has its density changed, and is kept at a constant temperature, the pressure varies nearly in proportion to the density simply. But with every change of density which takes place under circumstances such that the gas cannot gain or lose heat by conduction, a variation of temperature occurs depending on the change of density in such a manner, that the pressure, instead of varying simply as the density, varies as a power of the density higher than the first. Let γ denote the index of that power, p the pressure, and w the density of the gas; then

$$p \propto w^{\gamma}; \dots\dots\dots\dots (1.)$$

so that the co-efficient of elasticity A has the following value:—

$$\mathrm{A} = \frac{dp}{dw} = \frac{\gamma p}{w} \dots\dots\dots\dots (2.)$$

The value of the index γ for air is

$$\gamma = 1{\cdot}408; \dots\dots\dots\dots (3.)$$

it is nearly the same for oxygen, hydrogen, carbonic oxide, and other nearly perfect gases; but has smaller values for carbonic acid, sulphurous acid, and other gases which deviate considerably from the perfectly gaseous condition.

Now, if p be taken in pounds on the square foot, and w in pounds per cubic foot, and if T be the temperature of the air in degrees of Fahrenheit (see Article 122),

$$\frac{p}{w} = 26214 \cdot \frac{\mathrm{T} + 461^\circ{\cdot}2}{493^\circ{\cdot}2}\ ;\ldots\ldots\ldots\ldots\ldots(4.)$$

and for gases nearly perfect in general, if p_0 represent one atmosphere—that is, 2116·4 lbs. per square foot,—and w_0 the weight of a cubic foot of the gas at 32° Fahrenheit, and under that pressure,

$$\frac{p}{w} = \frac{p_0}{w_0} \cdot \frac{\mathrm{T} + 461^\circ{\cdot}2}{493^\circ{\cdot}2}, \text{ nearly};\ldots\ldots\ldots\ldots(5.)$$

whence the velocity of sound in a nearly perfect gas is

$$a = \sqrt{\frac{g\,\gamma\,p}{w}} = \sqrt{\left\{\frac{g\,\gamma\,p_0\,(\mathrm{T} + 461^\circ{\cdot}2)}{w_0 \cdot 493^\circ{\cdot}2}\right\}}\ ;\ldots(6.)$$

and in air

$$a = 1092\sqrt{\left(\frac{\mathrm{T} + 461^\circ{\cdot}2}{493^\circ{\cdot}2}\right)}\ldots\ldots\ldots\ldots(7.)$$

616. **Impact and Pressure; Pile Driving.**—The impact or blow of a body which has acquired momentum by the action of a certain force during a greater time, is used to overcome a greater force during a less time; as when the ram of a pile engine, having acquired momentum by the action of its weight during a short but sensible interval of time, overcomes the resistance of a pile to being driven, many times greater than the weight of the ram, and during an interval too short to be measured.

If the ratio of the times could be ascertained, the ratio of the forces could be inferred from it; but as one of the times is always insensibly short, the ratio of the forces has to be computed from the *spaces* through which they act, by considering how the *energy* of the blow is distributed.

Let W be the weight of the ram; h, the height from which it falls. Then

$$\mathrm{W}\,h$$

is the energy of the blow.

That energy is employed—

1. In compressing the ram;
2. In compressing the pile;
3. In giving actual energy of motion to the ram and pile;
4. In driving the pile against the resistance of the ground.

The compression of the ram is inappreciable in practice; and so also are the velocities of the ram and pile after the collision. The second and fourth ways of expending the energy have therefore alone to be considered.

Let R be the *effective* resistance of the ground; that is, its total resistance less the weights of the pile and ram. Let S be the area of the head of the pile, and P the pressure exerted at any instant between it and the ram. At first, P is nothing; it increases as the pile becomes compressed, until at length it becomes equal to R; then the compression of the pile ceases; it begins to penetrate into the ground, and continues to do so until the energy of the blow is all expended. The mean value of P is $\frac{R}{2}$. The distance through which it is overcome in compressing the pile is the compression due to its maximum value, viz., $\frac{RL}{ES}$, where E is the modulus of elasticity of the pile, and L the length of a post, which, if uniformly compressed throughout its length, would be as much shortened as the pile. Considering that the pile is held in a great measure by friction against its sides, L may be made equal to *half* its length.

Then the work performed in compressing the pile is $\frac{R^2 L}{2ES}$; and the work performed in driving it deeper is $R x$, where x is the depth through which it is driven by a blow; and equating these to the energy of the blow, we find

$$W h = \frac{R^2 L}{2ES} + R x \text{.....................} (1.)$$

When x has been ascertained by observation, R is found by solving a quadratic equation, viz.,

$$R = \sqrt{\left\{ \frac{2ESWh}{L} + \frac{E^2 S^2 x^2}{L^2} \right\}} - \frac{ESx}{L} \text{......} (2.)$$

Piles are in general driven till R amounts to between 2,000 and 3,000 lbs. per square inch of the area of head S, and are loaded with from 200 to 1,000 lbs. per square inch; so that the factor of safety is from 10 to 3.

The overcoming of any resistance by blows is analogous to the example here given, which is extracted, and somewhat modified, from a section by Mr. Airy in Dr. Whewell's treatise on Mechanics.

CHAPTER V.

MOTIONS OF FLUIDS.

617. **Division of the Subject.**—The principles of dynamics, as applied to fluids, so far as small and rapid changes of density are concerned, have already been discussed under the head of vibratory motions. Now the only changes of density which occur during the motions of *liquids* are small and rapid; so that in the present chapter those motions of liquids are alone to be considered in which the density is constant, and whose cinematical principles have been treated of in Part III., Chapter III., Section 2. In the motions of *gases*, great and continuous changes of density occur, such as those whose cinematical principles have been treated of in section 3 of the chapter already referred to; and the dynamical laws of motions affected by such changes have still to be considered. One mode of division, therefore, of hydrodynamics, is founded on the distinction between the motions of liquids, regarded as of constant density, and those of gases.

Another mode of division is founded on the distinction between motions not sensibly affected by friction, and those which are so affected. The motions of fluids not sensibly affected by friction, and therefore governed by pressure and weight only, take place according to laws which are exactly known; so that any difficulty which exists in tracing their consequences, in particular cases, arises from mathematical intricacy alone. The laws of the friction of fluids, on the other hand, are only known approximately and empirically; and the mode of operation of that force amongst the particles of a fluid is not yet thoroughly understood; so that the solution of a particular problem has often to be deduced, not from first principles representing the condensed results of all experience, but from experiments of a special class, suited to the problem under consideration.

The laws of the mutual impulses exerted between masses of fluid and solid surfaces require to be considered separately.

The following is the division of the subject of this chapter :—

I. Motions of Liquids under Gravity and Pressure alone.
II. Motions of Gases under Gravity and Pressure alone.
III. Motions of Liquids affected by Friction.
IV. Motions of Gases affected by Friction.
V. Mutual Impulses of Fluid Masses and Solid Surfaces.

SECTION 1.—*Motions of Liquids without Friction.*

618. **General Equations.**—In Articles 414 and 415 have been given the three general equations, by which the rates of variation of the components of the velocity of an individual particle of liquid are expressed in terms of those of the velocity at a point given in position; and in Article 412 has been given the *equation of continuity* which connects the components of the latter velocity with each other. To obtain the general dynamical equations of the motion of a liquid, the first three equations are to be converted into expressions for the rates of variation of the components of the *momentum* of a particle, and the results equated to the unbalanced forces which act upon it.

Let $d\,x\,d\,y\,d\,z$ denote the volume of a rectangular molecule, and p the intensity of the pressure of the liquid at a point whose coordinates are x, y, z. Let z be vertical, and positive downwards. w being used to denote one of the components of the velocity at a point, the symbol ϱ will now be employed to denote the *weight of an unit of volume.* Then the forces by which the molecule is acted upon are

$$\left.\begin{array}{c}\text{along } x, -\dfrac{d\,p}{d\,x}\cdot d\,x\,d\,y\,d\,z;\ \text{along } y, -\dfrac{d\,p}{d\,y}\cdot d\,x\,d\,y\,d\,z;\\ \text{along } z, \left(\varrho-\dfrac{d\,p}{d\,z}\right) d\,x\,d\,y\,d\,z.\end{array}\right\}\ (1.)$$

Let the rates of variation of the components of the momentum of the molecule be found by multiplying the three rates of variation of the components of the velocity in Article 415, equation 2, each by $\frac{\varrho\,d\,x\,d\,y\,d\,z}{g}$; then equating these respectively to the three forces in equation 1 above, dividing by $d\,x\,d\,y\,d\,z$, so as to reduce the equation to the unit of volume, and then by ϱ, so as to reduce them to the unit of weight, the following results are obtained :—

$$\left.\begin{array}{l}-\dfrac{d\,p}{\varrho\,d\,x}=\dfrac{1}{g}\cdot\dfrac{d^2\,\xi}{d\,t^2}=\dfrac{1}{g}\left\{\dfrac{d\,u}{d\,t}+u\dfrac{d\,u}{d\,x}+v\dfrac{d\,u}{d\,y}+w\dfrac{d\,u}{d\,z}\right\};\\ -\dfrac{d\,p}{\varrho\,d\,y}=\dfrac{1}{g}\cdot\dfrac{d^2\,\eta}{d\,t^2}=\dfrac{1}{g}\left\{\dfrac{d\,v}{d\,t}+u\dfrac{d\,v}{d\,x}+v\dfrac{d\,v}{d\,y}+w\dfrac{d\,v}{d\,z}\right\};\\ 1-\dfrac{d\,p}{\varrho\,d\,z}=\dfrac{1}{g}\cdot\dfrac{d^2\,\zeta}{d\,t^2}=\dfrac{1}{g}\left\{\dfrac{d\,w}{d\,t}+u\dfrac{d\,w}{d\,x}+v\dfrac{d\,w}{d\,y}+w\dfrac{d\,w}{d\,z}\right\}.\end{array}\right\}\ (2.)$$

Combining with those three equations of motion the equation of continuity, viz. :—

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0; \dots\dots\dots\dots\dots\dots (3.)$$

we have the data for solving all dynamical questions as to liquids without friction. These equations are adapted to the case of *steady motion* by making

$$\frac{du}{dt} = \frac{dv}{dt} = \frac{dw}{dt} = 0; \dots\dots\dots\dots\dots\dots (4.)$$

as in Article 413.

619. **Dynamic Head.**—The quotient $\frac{p}{\varrho}$ is what is called the *height, or head, due to the pressure;* that is, the height of a column of the liquid, of the uniform specific gravity ϱ, whose weight per unit of base would be equal to the pressure p. Now as the vertical ordinate z is measured *positively downwards* from a datum horizontal plane, ϱz is the weight of a column of liquid per unit of base extending down from that plane to a particle under consideration; $p - \varrho z$ is the difference between the intensity of the actual pressure at that particle and the pressure due to its depth below the datum horizontal plane; and

$$\frac{p}{\varrho} - z = h \dots\dots\dots\dots\dots\dots\dots\dots (1.)$$

is the *height* or *head due to that difference of intensity*, being what will be termed the *dynamic head.* When z is measured *positively upwards* from a datum horizontal plane, its sign is to be changed; so that the expression for the dynamic head in that case becomes

$$\frac{p}{\varrho} + z = h \dots\dots\dots\dots\dots\dots\dots\dots (2.)$$

620. **General Dynamic Equations in Terms of Dynamic Head.**—If instead of the rates of variation of the pressure in the equations 2 of Article 618, there are substituted their values in terms of the dynamic head, those equations take the following forms:—

$$\left.\begin{aligned} -\frac{dh}{dx} &= \frac{1}{g} \cdot \frac{d^2 \xi}{dt^2} = \frac{1}{g}\left\{\frac{du}{dt} + u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz}\right\}; \\ -\frac{dh}{dy} &= \frac{1}{g} \cdot \frac{d^2 \eta}{dt^2} = \frac{1}{g}\left\{\frac{dv}{dt} + u\frac{dv}{dx} + v\frac{dv}{dy} + w\frac{dv}{dz}\right\}; \\ -\frac{dh}{dz} &= \frac{1}{g} \cdot \frac{d^2 \zeta}{dt^2} = \frac{1}{g}\left\{\frac{dw}{dt} + u\frac{dw}{dx} + v\frac{dw}{dy} + w\frac{dw}{dz}\right\}. \end{aligned}\right\} (1.)$$

621. **Law of Dynamic Head for Steady Motion.**—From these equations is deduced the following consequence, in the case of

steady motion, in which there is no variation of the dynamic head at a particle, except that arising from the change of position of the particle.

Let V be the velocity of a given particle. Its value, in terms of its rectangular components, is given by the equation

$$\mathrm{V}^2 = \frac{d\,\xi^2}{d\,t^2} + \frac{d\,\eta^2}{d\,t^2} + \frac{d\,\zeta^2}{d\,t^2}; \ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

which, being divided by $2\,g$, gives the height due to the velocity; so that the variation of that height, in a given indefinitely short interval of time, is

$$\left.\begin{aligned} d \cdot \frac{\mathrm{V}^2}{2\,g} &= \frac{1}{g}\left(\frac{d\,\xi}{d\,t} \cdot \frac{d^2\,\xi}{d\,t^2} + \frac{d\,\eta}{d\,t} \cdot \frac{d^2\,\eta}{d\,t^2} + \frac{d\,\zeta}{d\,t} \cdot \frac{d^2\,\zeta}{d\,t^2}\right) d\,t \\ &= -\left(\frac{d\,h}{d\,x} \cdot \frac{d\,\xi}{d\,t} + \frac{d\,h}{d\,y} \cdot \frac{d\,\eta}{d\,t} + \frac{d\,h}{d\,z} \cdot \frac{d\,\zeta}{d\,t}\right) d\,t = -\,d\,h. \end{aligned}\right\} (2.)$$

This principle might otherwise be stated thus :—*In steady motion, the sum of the height due to the velocity of a particle and of its dynamic head is constant*, or symbolically

$$\frac{\mathrm{V}^2}{2\,g} + h = \text{constant} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.)$$

This equation applies to the particles which successively occupy the same fixed point, as well as to each individual particle.

622. The **Total Energy** of a particle of a moving liquid without friction is expressed by multiplying the expression in equation 3 of the last Article by the weight of the particle W, thus :—

$$\frac{\mathrm{W\,V}^2}{2\,g} + \mathrm{W}\,h; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

in which $\frac{\mathrm{W\,V}^2}{2\,g}$ is the *actual energy* of the particle, and W h is its *potential energy;* because, from the last Article it appears, that by the diminution of W h, $\frac{\mathrm{W\,V}^2}{2\,g}$ may be increased by an equal amount, and *vice versa;* so that *the dynamic head of a particle is its potential energy per unit of weight.* In the case of *steady motion*, the total energy of each particle is constant; and the total energy of each of the equal particles which successively occupy the same position is the same.

In the case of unsteady motion of a liquid mass, the total internal energy of the entire mass is constant; that is, if the centre of gravity of the mass, or a point either fixed or moving uniformly, with respect to that centre of gravity, is taken as the fixed point to which the motions of all the particles are referred, the following equation is fulfilled:—

$$\Sigma \cdot \mathrm{W} \left(\frac{\mathrm{V}^2}{2g}+h\right) \text{ or } \int\int\int\left(\frac{\mathrm{V}}{2g}+h\right)\rho \cdot dxdydz = \text{constant}\ldots(2.)$$

623. The **Free Surface** of a moving liquid mass, being that which is in contact with the air only, is characterized by the pressure being uniform all over it, and equal to that of the atmosphere. Let p_1 be the atmospheric pressure, z_1 the vertical ordinate, measured *positively upwards* from a given horizontal plane, of any point in the free surface of the liquid, and h_1 the dynamic head at the same point; then it appears from Article 619, equation 2, that for that surface,

$$h_1 - z_1 = \frac{p_1}{\rho} = \text{constant}\ldots\ldots\ldots\ldots\ldots\ldots(1.)$$

624. A **Surface of Equal Pressure** is characterized by an analogous equation,

$$h - z = \frac{p}{\rho} = \text{constant};\ldots\ldots\ldots\ldots\ldots\ldots(1.)$$

and all surfaces of equal pressure fulfil the differential equation,

$$dh = dz;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.)$$

which, for *steady motion*, becomes

$$dz = dh = -d \cdot \frac{\mathrm{V}^2}{2g};\ldots\ldots\ldots\ldots\ldots\ldots(3.)$$

expressing that the variations of actual energy are those due to the variations of level simply.

625. **Motion in Plane Layers** is a state which is either exactly or approximately realized in many ordinary cases of liquid motion;

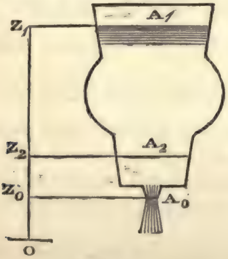

Fig. 246.

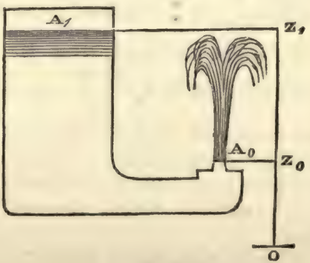

Fig. 247.

and the assumption of which is often used as a first approximation to the solution of various questions in hydraulics. It consists in the motions of all the particles in one plane being parallel to each other, perpendicular to the plane, and equal in velocity. It is illustrated by the three figures 246, 247, and 248, each of which represents a reservoir containing liquid up to the elevation $\overline{\mathrm{OZ}_1} = z_1$ above a given datum, and discharging the liquid from an orifice A_0 at the smaller elevation $\overline{\mathrm{OZ}_0} = z_0$. The liquid moves exactly or nearly in plane layers at the upper surface A_1 and at the orifice A_0. Let these symbols denote the areas of the upper surface and of the issuing stream respectively.

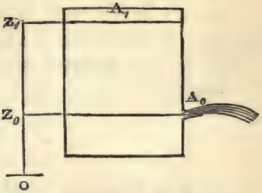

Fig. 248.

Let Q denote the rate of flow per second, v_1 the velocity of descent of the liquid at the upper surface, v_0 its velocity of outflow from the orifice; then, according to Article 405, the equation of continuity is

$$\left.\begin{array}{r} v_1 \mathrm{A}_1 = v_0 \mathrm{A}_0 = \mathrm{Q}\,; \\ \text{or } v_1 = \dfrac{\mathrm{Q}}{\mathrm{A}_1}\,;\ v_0 = \dfrac{\mathrm{Q}}{\mathrm{A}_0}. \end{array}\right\} \dots\dots\dots\dots (1.)$$

The pressures at the upper surface and at the orifice respectively are each equal to the atmospheric pressure; hence the difference of dynamic head is simply the difference of elevation; that is to say,

$$h_1 - h_0 = z_1 - z_0\,;$$

therefore, according to Article 621, equations 2 and 3,

$$\frac{v_0^2 - v_1^2}{2\,g} = \frac{v_0^2}{2\,g}\left(1 - \frac{\mathrm{A}_0^2}{\mathrm{A}_1^2}\right) = z_1 - z_0 \dots\dots\dots\dots (2.)$$

This gives for the velocity of outflow,

$$v_0 = \sqrt{\left\{\frac{2\,g\,(z_1 - z_0)}{1 - \frac{\mathrm{A}_0^2}{\mathrm{A}_1^2}}\right\}}\,; \dots\dots\dots\dots (3.)$$

from which can be computed the rate of flow or discharge by means of equation 1.

The general equation of motion, for every part of the vessel or channel at which the motion takes place in plane layers, is, according to Article 621, equation 3,

$$\frac{v^2}{2\,g} + h = \text{constant} = \frac{v_0^2}{2\,g} + h_0 = \frac{v_0^2}{2\,g} + z_0 + \frac{p_0}{\varrho} \dots\dots (4.)$$

The motion may be considered to take place in plane layers at any part of the channel whose sides are nearly straight and parallel, such as A_2 in fig. 246, whose elevation above the datum is z_2. To find the dynamic head, and thence the pressure, at this intermediate section of the channel, the velocity through it is to be computed by the formula

$$v_2 = \frac{Q}{A_2} = \frac{v_0 A_0}{A_2}; \qquad (5.)$$

whence the dynamic head relatively to the datum O is obtained by the equation

$$h_2 = h_0 + \frac{v_0^2 - v_2^2}{2g}; \qquad (6.)$$

and thence the pressure by the formula

$$p_2 = \varrho\,(h_2 - z_2) \qquad (7.)$$

When a large vessel discharges liquid through a small orifice, the ratio $\frac{A_0^2}{A_1^2}$ is often so small a fraction, that it may be neglected in equations 2 and 3.

626. The **Contracted Vein** is the name given to a portion of a jet of fluid at a short distance from an orifice in a plate, which is smaller in diameter and in area than the orifice, owing to a spontaneous contraction which the jet undergoes after leaving the orifice.

The area of the narrowed part of the contracted vein is in every case to be considered as the *virtual* or *effective outlet*, and used for A_0 in the equations of the last Article.

The ratio of the area of the contracted vein, or effective orifice, to that of the actual orifice, is called the *co-efficient of contraction.* For sharp edged orifices in thin plates, it has different values for different figures and proportions of the orifice, ranging from about 0·58 to 0·7, and being on an average about $\frac{5}{8}$. It diminishes somewhat for great pressures, and for dynamic heads of six feet and upwards may be taken at about 0·6. The most elaborate table of those co-efficients is that of Poncelet and Lesbros.

For orifices with edges that are not sharp and thin, the discharge is modified sensibly by friction.

627. **Vertical Orifices** of discharge, whose vertical dimensions are not small in comparison with their depths below the upper surface of the reservoir, are treated as having a mean velocity of discharge through their contracted veins due to the *mean value of the square root of the dynamic head* for the several parts of the orifice. For example, let y be the horizontal breadth of an orifice at any given

elevation z above the datum, z' the elevation of the lower, and z'' that of the upper edge of the orifice, so that

$$A_0 = c \int_{z'}^{z''} y\,d\,z \text{ .... } (1.)$$

is its effective area, c being the co-efficient of contraction. Then that orifice is to be treated as if its depth below the upper surface A_1 were

$$z_1 - z_0 = \left\{ \frac{\int_{z'}^{z''} y\,\sqrt{z_1 - z}\cdot d\,z}{\int_{z'}^{z''} y\,d\,z} \right\}^2 ; \text{ .... } (2.)$$

and the formulæ of Article 625 applied accordingly. For a rectangular orifice for which y is constant, this gives

$$\sqrt{z_1 - z_0} = \frac{2}{3}\cdot\frac{(z_1 - z')^{\frac{3}{2}} - (z_1 - z'')^{\frac{3}{2}}}{z'' - z'} ; \text{ .... } (3.)$$

and if it is a *notch*, or a rectangular orifice extending to the upper surface, so that $z'' = z_1$,

$$\sqrt{z_1 - z_0} = \frac{2}{3}\cdot\sqrt{z_1 - z'} \text{ .... } (4.)$$

628. **Surfaces of Equal Head,** which for steady motion are also SURFACES OF EQUAL VELOCITY, are ideal surfaces traversing a fluid mass, at each of which the dynamic head is uniform. Their positions are related to the direction, velocity, curvature, and variation of velocity of the fluid motion in the following manner :—

In fig. 249, let $H_1 H_1$, $H_2 H_2$, represent a pair of such surfaces, very near each other; their normal distance apart being $d\,n$, measured forwards from H_1 towards H_2, and the difference of dynamic head at them being $d\,h$. Let A B be part of the moving fluid, forming an elementary stream whose velocity is V, its radius of curvature r, its thickness $d\,r$, and the variation of its velocity $d\,V$; velocities from A towards B being positive, and curvature concave towards H_2 being positive. Then the equations 2 and 3 of Article 621 give, as before,

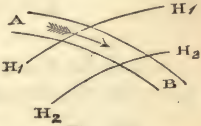

Fig. 249.

$$\frac{V\,d\,V}{g} = -\,d\,h ; \text{ or } \frac{V^2}{2\,g} + h = \text{constant} ; \text{........} (1.)$$

and in order that the variation of head may supply the deviating force necessary to produce the curvature of the stream A B, the radius of curvature must be in a plane perpendicular to the surfaces of equal head, and the following equation must be fulfilled :—

$$\left.\begin{aligned} \frac{V^2 dr}{gr} &= -\frac{dh}{dn} \cdot dr \cos \widehat{nr}; \\ \text{or} \quad \frac{V^2}{gr} &= -\frac{dh}{dn} \cdot \cos \widehat{nr}. \end{aligned}\right\} \text{..............(2.)}$$

629. In a **Radiating Current,** flowing towards or from an axis, as described in Article 407, the surfaces of equal dynamic head and equal velocity are cylinders described about the axis. The equation of continuity, 1 of Article 407, putting b instead of h to denote the depth, parallel to the axis, of the cylindrical space in which the current flows, gives for the velocity the formula

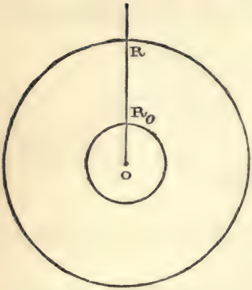

Fig. 250.

$$v = \frac{Q}{2\pi b r} = \frac{v_0 r_0}{r}; \text{......(1)}$$

where r_0 is the radius of the cylindrical surface R_0, fig. 250, at which the radiating part of the current begins or ends, according as it flows outwards or inwards. The radiating current may extend indefinitely in all directions beyond this surface, the velocity being at any point inversely as the distance from the axis O. Let h_0 be the dynamic head at R_0; then at any other cylindrical surface of the radius $\overline{OR} = r$, we have the dynamic head,

$$h = h_0 + \frac{v_0^2}{2g} - \frac{v^2}{2g} = h_0 + \frac{v_0^2}{2g}\left(1 - \frac{r_0^2}{r^2}\right) \text{..... ....(2.)}$$

Let h_1 be the limit towards which the dynamic head approximates as the distance from the axis is indefinitely increased; then

$$\left.\begin{aligned} h_1 &= h_0 + \frac{v_0^2}{2g} = h + \frac{v^2}{2g}; \\ h &= h_1 - \frac{v^2}{2g} = h_1 - \frac{v_0^2 r_0^2}{2gr^2}. \end{aligned}\right\} \text{..............(3.)}$$

630. **Free Circular Vortex.**—In the cylindrical space of fig. 250, lying outside of the surface R_0, let the particles of the fluid revolve in a circular current round the axis O; and let the velocity of each circular current be such, that if, owing to a slow radial movement, particles should find their way from one circular current to another, they would assume freely the velocities proper to the several cur-

rents entered by them, without the action of any force but weight and fluid pressure. This last condition is what constitutes a *free* vortex, and is a condition towards which every vortex not acted on by external forces tends, because of the tendency to the intermixture of the particles of adjoining circular currents. It is expressed mathematically by

$$h + \frac{v^2}{2g} = h_1 = \text{constant} \ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

h_1 will be called the *maximum head.*

Conceive a portion of a thin circular current of the mean radius r, contained between two cylindrical surfaces at the indefinitely small distance apart $d\,r$, and of the area *unity*, the current having the velocity v. Then the centrifugal force of that portion of the current is

$$\frac{v^2 \varrho\, d\,r}{g\,r},$$

which is equal and opposite to the deviating force

$$\varrho\, d\,h;$$

that is to say,

$$\frac{d\,h}{d\,r} = \frac{v^2}{g\,r} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.)$$

But by the condition of freedom in equation 1, we have $\frac{v^2}{g} = 2\,(h_1 - h)$, which being substituted in equation 2, gives

$$\frac{d\,h}{d\,r} = \frac{2\,(h_1 - h)}{r},$$

whence

$$h_1 - h = \frac{v^2}{2g} \propto \frac{1}{r^2}; \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.)$$

or, *the velocity is inversely as the distance from the axis,* exactly as in a radiating current. Then let v_0 be the velocity of revolution, and h_0 the dynamic head, at the inner boundary R_0 of the vortex; we have for the general equations amongst the dynamic heads and velocities at all points,

$$\left.\begin{aligned} h_1 &= h_0 + \frac{v_0^2}{2g} = h + \frac{v^2}{2g} = h + \frac{v_0^2}{2g} \cdot \frac{r_0^2}{r^2}; \\ h &= h_1 - \frac{v^2}{2g} = h_1 - \frac{v_0^2}{2g} \cdot \frac{r_0^2}{r^2}. \end{aligned}\right\} \ldots\ldots (4.)$$

631. **Free Spiral Vortex.**—As the equations of the motion of a free circular vortex are exactly the same with those of a radiating current, it follows that they also apply to a vortex in which the motion is compounded of those two motions in any proportions, *so long as the velocity is inversely as the distance from the axis.* To fulfil this condition, the currents of liquid must have a form that is at every point equally inclined to the radius drawn from the axis; a property of the logarithmic spiral. Let v be the velocity of the current in a free spiral vortex at any point, and θ the constant inclination of the current to the radius vector; then the component of the motion whose velocity is $v \cos \theta$, is analogous to the motion of a radiating current, and that whose velocity is $v \sin \theta$ is analogous to the motion of a free circular vortex.

632. A **Forced Vortex** is one in which the velocity of revolution of the particles follows any law different from that of a free vortex; but the kind of forced vortex which it is most useful to consider, is one in which the particles revolve with equal angular velocities of revolution, as if they belonged to a rotating solid body; so that if r_0 be the radius of the *outer* boundary of the vortex, where the velocity is v_0,

$$v = \frac{v_0 r}{r_0} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

The equation of deviating force, 2 of Article 630, is applicable to all vortices, forced as well as free. Introducing into it the value of v from equation 1, above, we find,

$$\frac{dh}{dr} = \frac{v_0^2 r}{g r_0^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.)$$

which being integrated, with the understanding that the dynamic head is to be reckoned relatively to the axis of the vortex, gives

$$h = \frac{v_0^2 r^2}{2 g r_0^2} = \frac{v^2}{2g}; \ h_0 = \frac{v_0^2}{2g}; \ldots\ldots\ldots\ldots (3.)$$

from which it appears, that in a rotating vortex, *the dynamic head at any point is the height due to the velocity, and the energy of any particle is half actual and half potential.*

633. A **Combined Vortex** consists of a free vortex without and a forced vortex within a given cylindrical surface, such as R_0 in fig. 250. In order that such a combined vortex may exist, the velocity v_0 and the dynamic head h_0 at the surface of junction must be the same for the two vortices; consequently, as the dynamic head of the forced vortex is equal to the height due to its velocity, and

the sum of those heights for the surface of junction is equal to the *maximum head* h_1 of the free vortex, we have this principle:—*In a combined vortex, the maximum dynamic head is double of the dynamic head at the surface of junction, each being measured relatively to the axis of the vortex;* or symbolically,

$$h_1 = 2\,h_0 = \frac{v_0^2}{g} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

To illustrate this geometrically, let a combined vortex revolve about a vertical axis, O Z_0 Z_1, fig. 251, the upper surface of the liquid being free, and represented in section by D B O B D. Let A B, A B, be the cylindrical surface of junction between the free and the forced vortices. Let A O A be a horizontal plane, touching the upper surface at its lowest point, which is at the axis, and let vertical ordinates be measured from this plane. The pressure of the atmosphere being equal at all points, may be left out of consideration; so that if z be the height of any point in the surface of the vortex above A O A, we shall have simply

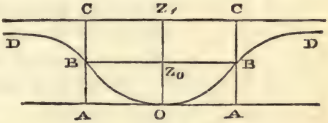

Fig. 251.

$$z = h \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.)$$

Then for the forced vortex,

$$z = \frac{v_0^2\,r^2}{2\,g\,r_0^2}; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.)$$

so that B O B is a paraboloid of revolution with its vertex at O.

Make $\overline{A\,C} = 2\,\overline{A\,B} = 2\,z_0$; this will represent h_1, the maximum dynamic head; and for the free vortex,

$$z = h - \frac{v_0^2\,r_0^2}{2\,g\,r^2} = h_1 - \frac{z_0\,r_0^2}{r^2}; \ldots\ldots\ldots\ldots (4.)$$

and D B, D B, is a hyperboloid of the second order, described by the rotation round the vertical axis of a hyperbola of the second order, whose ordinate $h_1 - z$, measured *downwards* from C Z_1 C, is inversely as the square of the distance from the axis. The two surfaces have a common tangent at B B, where they join.

The velocity of any particle in the free vortex is that due to its depth below C C; that of any particle in the forced vortex is that due to its height above A A; and B, where those velocities are equal, is midway between C C and A A.

The theory of the combined vortex was made, by Professor James Thomson of Belfast, the principle of the action of his turbine or vortex water-wheel.

634. **Vertical Revolution.**—When a mass of liquid revolves in a vertical plane about a horizontal axis (like the water in a bucket of an overshot wheel), its upper surface is not horizontal, but assumes a figure depending on the deviating force required by its revolution.

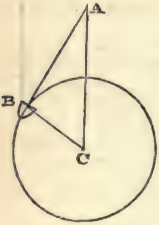

Fig. 252.

In fig. 252, let C represent a horizontal axis, and B a bucket of liquid revolving round it in a vertical circle of the radius $\overline{BC}$, with the angular velocity of revolution a. Let W be the weight of liquid in the bucket.

Then the deviating force required is given by the formula

$$\frac{W a^2}{g} \cdot \overline{BC}.$$

Take the radius $\overline{BC}$ itself to represent the deviating force, and $\overline{CA}$ vertically upwards from the axis to represent the weight; the height $\overline{CA}$ is given by the proportion

$$\overline{CA} : \overline{BC} :: W : \frac{W a^2}{g} \cdot \overline{BC},$$

that is,

$$\overline{CA} = \frac{g}{a^2} = \frac{g}{4 \pi^2 n^2}, \dots\dots\dots\dots (1.)$$

where n is the number of revolutions per second.

Now $\overline{AC}$ representing the weight, and $\overline{CB}$ the centrifugal force, *equal and opposite to the deviating force*, the *internal condition* of the liquid in the bucket, according to D'Alembert's principle, is the same as if it were under a force represented by $\overline{AB}$, the resultant of these two forces; therefore the surface of the liquid is perpendicular to $\overline{AB}$.

Now it appears from equation 1, that the height of A above C is independent of the radius of the wheel, and of every circumstance except the time of revolution; being, in fact, the height of a revolving pendulum which revolves in the same time with the wheel. (See Article 539.) Therefore the point A is the same for all buckets carried by the same wheel with the same angular velocity, and for all points in the surface of the liquid in the same bucket, whether nearer to or farther from the axis C; therefore the upper surface of the liquid in each bucket is part of a cylinder described about a horizontal axis passing through A and parallel to C.

The theory of rolling waves may be deduced from the above proposition. For a brief sketch of that theory, see Addendum,

SECTION 2.—*Motions of Gases without Friction.*

635. **Dynamic Head in Gases.**—The dynamical equations of motion of a gas are the same with those already given in Article 618, equation 2; and in their integration, it has to be observed that ρ, the density, is no longer constant, but depends on the pressure. The equations of continuity have been given in Articles 419 to 423.

In finding the DYNAMIC HEAD for a particle of a gas, instead of $\frac{p}{\rho}$ there is to be taken $\int_0^p \frac{dp}{\rho}$, as is evident from the general equations of fluid motion already referred to. Consequently, the dynamic head for a gaseous particle, at a given elevation z above a fixed horizontal plane, is, relatively to that plane,

$$h = \int_0^p \frac{dp}{\rho} + z; \text{..........................} (1.)$$

and the putting of this value for h in all the *dynamical equations* relating to liquids, transforms them into the corresponding equations for gases.

In most practical problems respecting the flow of gases, the differences of level of different points of the gaseous mass have little or no sensible effect on the motion; so that z may often be omitted from the preceding formula.

In determining the value of the integral in that formula, it is to be observed that almost all changes of velocity of gases take place so rapidly, that the particles have no time to receive or to emit heat to any sensible amount; and therefore the pressure and density of each particle are related to each other according to the law already explained in treating of the velocity of sound; that is to say,

$$p \propto \rho^{\gamma}; \text{..............................} (2.)$$

the exponent γ having the values therein stated, of which the most important is 1·408 for air. This gives for the value of the integral in equation 1,

$$h - z = \int_0^p \frac{dp}{\rho} = \frac{\gamma}{\gamma - 1} \cdot \frac{p}{\rho}; \text{..................} (3.)$$

in which, for air,

$$\frac{\gamma}{\gamma-1} \quad \frac{1\cdot408}{\cdot408} = 3\cdot451 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4.)$$

Let

$$\tau = \mathrm{T} + 461^\circ\cdot2 \text{ Fahr.} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(5.)$$

denote the *absolute temperature* of the gas, T being its temperature on the ordinary Fahrenheit's scale; and let

$$\tau_0 = 493^\circ\cdot2 \text{ Fahr.} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(6.)$$

be the absolute temperature of melting ice. Then for gases sensibly perfect,

$$\frac{p}{\varrho} = \frac{p_0\,\tau}{\varrho_0\,\tau_0};\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7.)$$

from which we have the following value of the integral in terms of the temperature:—

$$h - z = \int_0^p \frac{dp}{\rho} = \frac{\gamma}{\gamma-1} \cdot \frac{p_0}{\rho_0} \cdot \frac{\tau}{\tau_0};\ldots\ldots\ldots\ldots\ldots(8.)$$

so that it is *simply proportional to the absolute temperature.*

It is known by the science of thermodynamics, that the above expression is equivalent to

$$\mathrm{J}\,c'\,\tau;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9.)$$

where c' is the *specific heat of the gas at constant pressure,* and J is "*Joule's equivalent,*" or the height from which a given weight must fall, in order to produce by friction as much heat as will raise the temperature of an equal weight of water by one degree. For Fahrenheit's scale,

$$\mathrm{J} = 772 \text{ feet}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10.)$$

The following are the values of $\frac{p_0}{\rho_0}$ and c' for certain gases and vapours:—

	$\frac{p_0}{\rho_0}$ feet.	c'.
Air,	26,214	0·238
Oxygen,	23,710	
Hydrogen,	378,819	
Steam,	42,141*	0·480
Æther vapour,	10,110	0·481
Bisulphuret of carbon vapour,	9,902	0·1575
Carbonic acid, if a perfect gas,	17,264	
Do., actually,	17,145	0·217

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* This is an ideal result, arrived at not by direct experiment, but by calculation from the chemical composition of steam.

The variations of pressure, volume, and absolute temperature of a gas during rapid changes of motion, are connected by the proportional equation

$$\tau \propto \rho^{\gamma-1} \propto p^{\frac{\gamma-1}{\gamma}} \ldots\ldots\ldots\ldots(11.)$$

The equations in this Article are all adapted to perfect gases. Actual gases deviate from the perfectly gaseous condition more or less; but in most practical questions of hydrodynamics the equations for perfect gases may be applied to them without material error.

636. **The Equation of Continuity for a Steady Stream of Gas** takes the following form, when the laws stated in the last Article are taken into account. The original equation, as given in Article 421, being equivalent to

$$\mathrm{Q}\,\rho = \mathrm{A}\,v\,\rho = \text{constant}, \ldots\ldots\ldots\ldots(1.)$$

we have to consider that, by the equations of the last Article, we have

$$\rho \propto p^{\frac{1}{\gamma}} \propto \tau^{\frac{1}{\gamma-1}} \propto (h-z)^{\frac{1}{\gamma-1}} \ldots\ldots\ldots\ldots(2.)$$

the exponents having, for air, the values

$$\frac{1}{\gamma} = 0{\cdot}71\,;\ \frac{1}{\gamma-1} = 2{\cdot}451 \ldots\ldots\ldots\ldots(3.)$$

Hence the equation of continuity, in terms of the pressure, of the absolute temperature, and of the dynamic head respectively, takes the following forms:—

$$\mathrm{Q}\,p^{\frac{1}{\gamma}} = \mathrm{A}\,v\,p^{\frac{1}{\gamma}} = \text{constant}\,; \ldots\ldots\ldots\ldots(4.)$$

$$\mathrm{Q}\,\tau^{\frac{1}{\gamma-1}} = \mathrm{A}\,v\,\tau^{\frac{1}{\gamma-1}} = \text{constant}\,; \ldots\ldots\ldots\ldots(5.)$$

$$\mathrm{Q}\,(h-z)^{\frac{1}{\gamma-1}} = \mathrm{A}\,v\,(h-z)^{\frac{1}{\gamma-1}} = \text{constant}\,; \ldots\ldots(6.)$$

637. **Flow of Gas from an Orifice.**—Let the pressure of a gas within a receiver be p_1, and without, p_2; let A be the *effective* area of an orifice with thin edges; that is, the product of the actual area by a *co-efficient of contraction,* whose value is

0·6, nearly.

Let the receiver be so large that the velocity within it is insensible. Let the absolute temperature and density of the gas within the receiver be τ_1, ρ_1, and those of the issuing jet τ_2, ρ_2. The latter are not the same with those of the *still* gas outside, for reasons to be stated afterwards. Then

$$\frac{\tau_2}{\tau_1} = \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}};\ \frac{\varrho_2}{\varrho_1} = \left(\frac{p_2}{p_1}\right)^{\frac{1}{\gamma}};\ldots\ldots\ldots\ldots\ldots(1.)$$

and by equation 8 of Article 635, and equation 3 of Article 621, we have for the height due to the velocity of outflow,

$$\left.\begin{aligned}\frac{v^2}{2g} &= h_1 - h_2 = \frac{\gamma}{\gamma-1}\cdot\frac{p_0}{\varrho_0}\cdot\frac{\tau_1-\tau_2}{\tau_0}\\ &= \frac{\gamma}{\gamma-1}\cdot\frac{p_0}{\varrho_0}\cdot\frac{\tau_1}{\tau_0}\left\{1-\left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}};\right\}\end{aligned}\right\}\ldots\ldots.(2.)$$

from which the velocity itself, and the *flow of volume* $Q = v\,A$ at the contracted vein, are easily computed. To find the *flow of weight*, the last quantity is to be multiplied by

$$\varrho_2 = \varrho_1\cdot\left(\frac{p_2}{p_1}\right)^{\frac{1}{\gamma}} = \frac{\varrho_0\,\tau_0\,p_1}{p_0\,\tau_1}\cdot\left(\frac{p_2}{p_1}\right)^{\frac{1}{\gamma}};\ldots\ldots\ldots\ldots(3.)$$

giving the following results :—

$$\left.\begin{aligned}\varrho_2\,Q &= v\,A\,\varrho_2\\ = A\,p_1\cdot\sqrt{\frac{\gamma\,\varrho_0\,\tau_0}{(\gamma-1)\,p_0\,\tau_1}}&\cdot\sqrt{\left\{1-\left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}}\right\}\cdot\left(\frac{p_2}{p_1}\right)^{\frac{1}{\gamma}}.}\end{aligned}\right\}(4.)$$

For small differences of pressure, such that $\frac{p_2}{p_1}$ is nearly $= 1$, the following approximate formula may be used where great accuracy is not required :—

$$\frac{v^2}{2g} = \frac{p_0}{\varrho_0}\cdot\frac{\tau_1}{\tau_0}\cdot\frac{p_1-p_2}{p_1}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5.)$$

When the motion of the jet is finally extinguished by friction, heat is reproduced sufficient to raise the absolute temperature nearly to its original value, τ_1.

637 A. **Maximum Flow of Gas.**—When $\frac{p_2}{p_1}$ is indefinitely diminished, the velocity of outflow given by equation 2 of Article 637 increases towards the limit

$$\sqrt{\left\{\frac{2\,\gamma\,g\,p_0\,\tau_1}{(\gamma-1)\,\rho_0\,\tau_0}\right\}},\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.)$$

being greater than the velocity of sound in the ratio $\sqrt{\frac{2}{\gamma-1}} : 1$, whose value for air is 2·21, giving for the limiting velocity of flow of air

$$2{,}413 \text{ feet per second } \times \sqrt{\frac{\tau_1}{\tau_0}} \ldots\ldots\ldots\ldots\ldots (2.)$$

The *flow of weight*, however, as given by equation 4 of Article 637, does not continuously increase as $\frac{p_2}{p_1}$ is indefinitely diminished, but reaches a maximum for the value

$$\left.\begin{array}{l} \dfrac{p_2}{p_1} = \left(\dfrac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}} \\ \text{corresponding to} \\ \dfrac{\rho_2}{\rho_1} = \left(\dfrac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}}; \ \dfrac{\tau_2}{\tau_1} = \dfrac{2}{\gamma+1}. \end{array}\right\} \ldots\ldots\ldots\ldots (3.)$$

The values of these ratios for air are

$$\frac{p_2}{p_1} = 0{\cdot}527\,; \ \frac{\varrho_2}{\varrho_1} = 0{\cdot}6345\,; \ \frac{\tau_2}{\tau_1} = 0{\cdot}8306 \ldots\ldots (4.)$$

and the corresponding velocity of flow is

$$v = \sqrt{\left\{\frac{2\,\gamma\,g\,p_0\,\tau_1}{(\gamma+1)\,\rho_0\,\tau_0}\right\}}, \ldots\ldots\ldots\ldots\ldots\ldots (5.)$$

being less than the velocity of sound in the ratio $\sqrt{\frac{2}{\gamma+1}} : 1$, whose value for air is 0·912; giving for the velocity of flow of air corresponding to the greatest flow of weight through a given orifice from a receiver where the pressure and temperature are given,

$$v = 997 \text{ feet per second } \times \sqrt{\frac{\tau_1}{\tau_0}} \ldots\ldots\ldots\ldots (6.)$$

It is often convenient to express the flow of weight in the following manner:—

$$\rho_2 \mathrm{Q} = \frac{v\,\rho_2}{\rho_1} \cdot \mathrm{A}\,\rho_1\,; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7.)$$

in which $\frac{v\,\rho_2}{\rho_1}$ is what is called the *reduced velocity*, being the velocity of a current of a density equal to that of the gas in the receiver, whose flow of weight would be equal to that of the actual current.

The maximum reduced velocity corresponds to the maximum flow; and its value is

$$\frac{v\,\rho_2}{\rho_1} = \text{velocity of sound} \times \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \text{..........(8.)}$$

whose value for air is

$$\text{velocity of sound} \times 0{\cdot}579 = 632 \text{ feet per sec.} \times \sqrt{\frac{\tau_1}{\tau_0}} \text{...(9.}$$

The investigations in this and the preceding Article are substantially the same with those originally communicated to the Royal Society in May, 1856, by Dr. Joule and Dr. Thomson; and the results differ by small quantities arising mainly from those gentlemen having taken $\gamma = 1{\cdot}41$, instead of $1{\cdot}408$.

Messrs. Joule and Thomson tested the theoretical result as to the *maximum reduced velocity* given in equation 9, by experiments on the flow of air through orifices in plates of copper of 0·029, 0·053, and 0·084 of an inch in diameter, at the temperature of 57° Fahrenheit, for which $\frac{\tau_1}{\tau_0} = \frac{518{\cdot}2}{493{\cdot}2}$, and the calculated maximum reduced velocity is 647 feet per second.

The maximum reduced velocity found by experiment was 550 feet per second, or 0·84 of that found by theory; but in calculating the velocity from the experiments, the actual area of the orifice was employed; so that the difference probably arises from contraction. The corresponding value of the ratio $p_2 : p_1$, as found by experiment, was 0·375 instead of 0·527; a difference produced by friction.

SECTION 3.—*Motions of Liquids with Friction.*

638. **General Laws of Fluid Friction.**—It is known by experiment, that between a fluid, and a solid surface over which it glides, there is exerted a resistance to their relative motion which is proportional to their surface of contact, and to the density of the fluid, and is approximately proportional to the square of the velocity of the relative motion; that is, the resistance is approximately proportional *to the weight of a prism of the fluid, whose base is the surface of contact, and its height the height due to the relative velocity.*

Let S be the surface of contact, v the velocity, ϱ the weight of an unit of volume of the fluid, and f a factor called the *co-efficient of friction;* then

$$\mathrm{R} = f\,\varrho\,\mathrm{S}\,\frac{v^2}{2\,g}, \text{........................(1.)}$$

is the amount of the friction at the surface S.

The co-efficient f is not absolutely constant at different velocities. The mode of calculation employed in practice, where the velocity is one of the unknown quantities to be determined, is to find an approximate value of the velocity from the *mean* value of f; then to compute the value of f corresponding to that approximate velocity, and use it to compute the velocity more exactly.

The following are some of the values of the co-efficients of friction, according to different authorities, for streams of WATER, gliding over various surfaces; v being the *mean velocity of the stream*, in feet per second:—

Iron pipes (Darcy). Let d = diameter of pipe in feet; then,

$$f = 0{\cdot}0043 \left(1 + \frac{1}{9\,d}\right) + \frac{0{\cdot}001}{v} \left(1 + \frac{1}{18\,d}\right);$$

or for velocities that are not very small,

$$f = 0{\cdot}005 \left(1 + \frac{1}{12\,d}\right).$$

Iron pipes, value of f for first approximation, 0·0064

Beds of rivers (Weisbach),... $f = a + \frac{b}{v}$; a = 0·0074.

b = 0·00023 foot.

Beds of rivers, value of f for first approximation,........................ 0·0076.

A collection of numerous formulæ for fluid friction, proposed by different authors, together with tables of the results of the best formulæ, is contained in Mr. Neville's work on hydraulics. The formulæ of many authors, though differing in appearance, are founded on the same, or nearly the same, experimental data, being chiefly those of Du Buat, with additions by subsequent inquirers; and their practical results do not materially differ. The two formulæ given above, on the authority of Darcy, for iron pipes, are based on his experiments as recorded in his treatise *du Mouvement de l'Eau dans les Tuyaux.*

639. **Internal Fluid Friction.**—Although the particles of fluids have no transverse elasticity—that is, no tendency to recover a certain figure after having been distorted—it is certain that they resist being made to slide over each other, and that there is a lateral communication of motion amongst them; that is, that there is a tendency of particles which move side by side in parallel lines to

assume the same velocity. The laws of this lateral communication of motion, or internal friction, of fluids, are not known exactly; but its effects are known thus far :—that the energy due to differences of velocity, which it causes to disappear, is replaced by heat in the proportion of one thermal unit of Fahrenheit's scale for 772 foot pounds of energy, and that it causes the friction of a stream against its channel to take effect, not merely in retarding the film of fluid which is immediately in contact with the sides of the channel, but in retarding the whole stream, so as to reduce its motion to one approximating to a motion in plane layers perpendicular to the axis of the channel (Article 625).

640. **Friction in an Uniform Stream.**—It is this last fact which renders possible the existence of an open stream of uniform section, velocity, and declivity. In hydraulic calculations respecting the resistance of this, or any other stream, the value given to the velocity is its mean value throughout a given cross-section of the stream A,

$$v = \frac{Q}{A} \dots\dots\dots\dots\dots\dots\dots\dots (1.)$$

The greatest velocity in each cross-section of a stream takes place at the point most distant from the rubbing surface of the channel. Its ratio to the mean velocity is given by the following empirical formula of Prony, where V is the greatest velocity in feet per second :—

$$\frac{v}{V} = \frac{7 \cdot 71 + V}{10 \cdot 25 + V} \dots\dots\dots\dots\dots\dots (2.)$$

In an uniform stream, the dynamic head which would otherwise have been expended in producing increase of actual energy, is wholly expended in overcoming friction. Consider a portion of the stream whose length is l, and fall z. The loss of head is equal to the fall of the surface of the stream, according to Article 623; and the expenditure of potential energy in a second is accordingly

$$z \varrho Q = z \varrho v A.$$

Equating this to the work performed in a second in overcoming friction, viz., v R, we find

$$z \varrho v A = f \varrho S \frac{v^3}{2g};$$

or dividing by common factors, and by the area of section A, we find for the value of the fall in terms of the velocity

$$z = f \cdot \frac{S}{A} \cdot \frac{v^2}{2g} \dots\dots\dots\dots\dots\dots (3.)$$

Let s be what is called the *wetted perimeter* of the cross-section of the stream; that is, the cross-section of the rubbing surface of the stream and channel; then

$$S = l\,s;$$

and dividing both sides of equation 3 by l, we find for the relation between the rate of declivity and the velocity,

$$\sin i = \frac{z}{l} = f\,\frac{s}{A}\cdot\frac{v^2}{2g} \quad \text{(4.)}$$

$\frac{A}{s}$ is what is called the "HYDRAULIC MEAN DEPTH" of the stream; and as the friction is inversely proportional to it, it is evident that the figure of cross-section of channel which gives the least friction is that whose hydraulic mean depth is greatest, viz., a semicircle. When the stability of the material limits the side-slope of the channel to a certain angle, Mr. Neville has shown that the figure of least friction consists of a pair of straight side-slopes of the given inclination connected at the bottom by an arc of a circle whose radius is the depth of liquid in the middle of the channel; or, if a flat bottom be necessary, by a horizontal line touching that arc. For such a channel, the hydraulic mean depth is half of the depth of liquid in the middle of the channel.

641. **Varying Stream.**—In a stream whose area of cross-section varies, and in which, consequently, the mean velocity varies at different cross-sections, the loss of dynamic head is the sum of that expended in overcoming friction, and of that expended in producing increased velocity, when the velocity increases, or the difference of those two quantities when the velocity diminishes, which difference may be positive or negative, and may represent either a loss or a gain of head. The following method of representing this principle symbolically is the most convenient for practical purposes. In fig. 253, let the origin of co-ordinates be taken at a point O *completely below* the part of the stream to be considered; let horizontal abscissæ x be measured *against* the direction of flow, and vertical ordinates to the surface of the stream, z, upwards. Consider any indefinitely short portion of the stream whose horizontal length is $d\,x$; in practice this may almost always be considered as equal to the actual length. The fall in that portion of

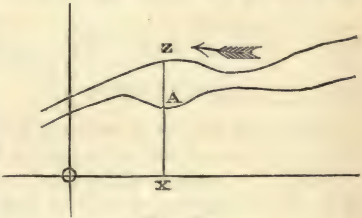

Fig. 253.

the stream is dz, and the acceleration $-dv$, because of v being opposite to x. Then modifying the expression for the loss of head due to friction in equation 3 of Article 640 to meet the present case, and adding the loss of head due to acceleration, we find

$$dz = -\frac{v\,dv}{g} + f \cdot \frac{s\,dx}{\mathrm{A}} \cdot \frac{v^2}{2g} \dots\dots\dots\dots\dots (1.)$$

In applying this differential equation to the solution of any particular problem, for v is to be put $\mathrm{Q} \div \mathrm{A}$, and for A and s are to be put their values in terms of x and z. Thus is obtained a differential equation between x and z, and the constant quantity Q, the flow per second. If Q is known, then it is sufficient to know the value of z for one particular value of x, in order to be able to determine the integral equation between z and x. If Q is unknown, the values of z for two particular values of x, or of z and $\frac{dz}{dx}$ (the declivity), for one particular value of x, are required for the solution, which comprehends the determination of the value of Q.

642. **The Friction in a Pipe Running Full** produces loss of dynamic head according to the same law with the friction in a channel, except that the dynamic head is now the sum of the elevation of the pipe above a given level, and of the height due to the pressure within it. The differential equation which expresses this is as follows:—Let dl be the length of an indefinitely short portion of a pipe measured *in* the direction of flow, s its internal circumference, A its area of section, z its elevation above a given level, p the pressure within it, h the dynamic head. Then the loss of head is

$$-dh = -dz - \frac{dp}{\rho} = \frac{v\,dv}{g} + f \cdot \frac{s\,dl}{\mathrm{A}} \cdot \frac{v^2}{2g} \dots\dots\dots (1.)$$

The ratio $\frac{dh}{dl}$ is called the *virtual* or *hydraulic declivity*, being the rate of declivity of an open channel of the same flow, area, and hydraulic mean depth. This may differ to any extent from the *actual declivity* of the pipe, $\frac{dz}{dl}$.

When the pipe is of uniform section, $dv = 0$, and the first term of the right-hand side of equation 1 vanishes.

When the section of the pipe varies, s and A are given functions of l. If Q is given, $v = \mathrm{Q} \div \mathrm{A}$ is also a given function of l; and to solve the equation completely, there is only required in addition the value of h for one particular value of l. If Q is unknown, the values of h for two particular values of l, or of h and $\frac{dh}{dl}$ for one particular value of l, are required for the solution, which comprehends the determination of Q.

643. **Resistance of Mouthpieces.**—A mouthpiece is the part of a channel or pipe immediately adjoining a reservoir. The internal friction of the fluid on entering a mouthpiece causes a loss of head equal to the height due to the velocity multiplied by a constant depending on the figure of the mouthpiece, whose values for certain figures have been found empirically; that is to say, let $-\Delta h$ be the loss of head; then

$$-\Delta h = \frac{f' v^2}{2g}, \qquad (1.)$$

f' being a constant.

For the mouthpiece of a cylindrical pipe, issuing from the flat side of a reservoir, and making the angle i with a normal to the side of the reservoir, according to Weisbach,

$$f' = 0{\cdot}505 + 0{\cdot}303 \sin i + 0{\cdot}226 \sin^2 i. \qquad (2.)$$

644. **The Resistance of Curves and Knees** in pipes causes a loss of head equal to the height due to the velocity multiplied by a coefficient, whose values, according to Weisbach, are given by the following formulæ:—For *curves*, let i be the arc to radius unity, r the radius of curvature of the centre line of the pipe, and d its diameter.

Then for a circular pipe,

$$f'' = \frac{i}{\pi}\left\{0{\cdot}131 + 1{\cdot}847\left(\frac{d}{2r}\right)^{\frac{7}{2}}\right\};$$

and for a rectangular pipe,

$$f'' = \frac{i}{\pi}\left\{0{\cdot}124 + 3{\cdot}104\left(\frac{d}{2r}\right)^{\frac{7}{2}}\right\}. \qquad (1.)$$

For *knees*, or sudden bends, let i be the angle made by the two portions of the pipe at either side of the knee with each other; then

$$f'' = 0{\cdot}9457 \sin^2 \frac{i}{2} + 2{\cdot}047 \sin^4 \frac{i}{2}. \qquad (2.)$$

645. A **Sudden Enlargement** of the channel in which a stream of liquid flows, causes a sudden diminution of the mean velocity in the same proportion as that in which the area of section is increased. Thus, let v_1 be the velocity in the narrower portion of the channel, and let m be the number expressing the ratio in which the channel is suddenly enlarged: the velocity in the enlarged part

is $\frac{v_1}{m}$. Now it appears from experiment, that the actual energy due to the velocity of the narrow stream *relatively* to the wide stream, that is, to the difference $v_1\left(1-\frac{1}{m}\right)$, is expended in overcoming the internal fluid friction of eddies, and so producing heat; so that there is a *loss of total head,* represented by

$$\frac{v_1^2}{2g}\left(1-\frac{1}{m}\right)^2 \text{..........................} (1.)$$

646. **The General Problem** of the flow of a stream with friction is thus expressed:—Let $h_1 + \frac{v_1^2}{2g}$, and $h_2 + \frac{v_2^2}{2g}$, be the total heads at the beginning and end of the stream respectively; then the loss of total head is represented by

$$h_1 - h_2 + \frac{v_1^2 - v_2^2}{2g} = \Sigma \cdot \mathrm{F}\frac{v^2}{2g} \text{..................} (1.)$$

where the right-hand side of the equation represents the sum of all the losses of head due to the friction in various parts of the channel.

SECTION 4.—*Flow of Gases with Friction.*

647. **The General Law** of the friction of gases is the same with that of the friction of liquids as expressed by equation 1, Article 638, the value of the co-efficient f being

0·006, nearly,

for friction against the sides of the pipe or channel. For a cylindrical mouthpiece, the co-efficient of resistance is 0·83; for a conical mouthpiece diminishing from the reservoir, 0·38.

When the pressures at the beginning and end of a stream of gas do not differ by more than $\frac{1}{10}$ of their mean amount, problems respecting its flow may be solved approximately by means of the above data, treating it as if it were a liquid of the density due to the lesser pressure, as in the approximate equation of Article 637.

In seeking the exact solution of the flow of a gas with friction, it is necessary to take into account the effect of the friction in producing heat, and so raising the temperature of the gas above what it would be if there were no friction, as supposed in Section 2. In the flow of a perfect gas with friction, if the heat produced by the friction is not lost by conduction, the friction causes no loss of total head; so that if at the beginning and end of a stream the velocities of a perfect gas are the same, its temperatures must also be the same. In an imperfect gas, there is a small depression of temperature, which has been employed by Dr. Joule and Dr. Thomson as a means of determining or verifying the laws of the deviation of different gases from the condition of perfect gas.

SECTION 5.—*Mutual Impulse of Fluids and Solids.*

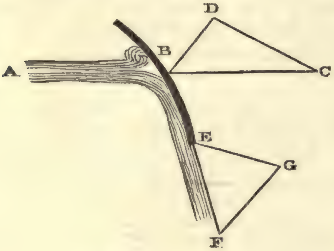

Fig. 254.

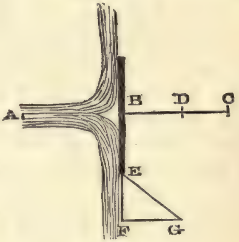

Fig. 255.

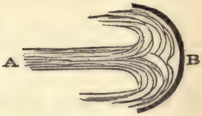

Fig. 256.

648. **Pressure of a Jet against a Fixed Surface.**—A jet of fluid A, fig. 254, striking a smooth surface, is deflected so as to glide along the surface in that path B E which makes the smallest angle with its original direction of motion A B, and at length glances off at the edge E in a direction tangent to the surface. To simplify the question, the surface is supposed to be curved in such a manner as to guide the jet to glance off it in one definite direction. The friction between the jet and the surface is supposed insensible. This being the case, as the particles of fluid in contact with the surface move along it, and the only sensible force exerted between them and the surface is perpendicular to their direction of motion, that force cannot accelerate or retard the motion of the particles, but can only deviate it. Let v, then, be the velocity of the particles of fluid, Q the volume discharged per second, ρ the density, and β the angle by which the direction of motion is deflected; then

$$\frac{\rho Q v}{g}$$

is the momentum of the quantity of fluid whose motion is deflected per second. Also conceive an isosceles triangle whose legs are each equal to the velocity v, and make with each other the angle β; then the base of that triangle, whose value is

$$2\,v \sin \frac{\beta}{2},$$

represents the change of velocity undergone by each particle of fluid; so that the change of momentum per second is

$$\mathrm{F} = \frac{2\,\varrho\,\mathrm{Q}\,v}{g} \cdot \sin \frac{\beta}{2};\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.)$$

and this also is the amount of the total pressure acting between the fluid and the surface, in the direction of a line which is parallel to the base of the isosceles triangle before mentioned; that is, which makes equal angles in opposite directions with the original and new directions of motion of the jet.

The force represented by F may be resolved into two components, F_x and F_y, respectively parallel and perpendicular to the original direction of the jet. The values of the resultant and its two components evidently bear to each other the proportions,

$$\mathrm{F} : \mathrm{F}_x : \mathrm{F}_y :: 2 \sin \frac{\beta}{2} : 1 - \cos\beta : \sin\beta \ldots\ldots\ldots\ldots(2.)$$

whence the components have the values,

$$\mathrm{F}_x = \frac{\varrho\,\mathrm{Q}\,v}{g}(1 - \cos\beta);\ \mathrm{F}_y = \frac{\varrho\,\mathrm{Q}\,v}{g}\sin\beta \ldots\ldots\ldots(3.)$$

If the surface struck by the jet is of a symmetrical figure about the original direction of the jet as an axis, the quantity of fluid Q which strikes the surface in each second spreads and glides off in various directions distributed symmetrically round the axis, and making equal angles β with it; so that the forces exerted perpendicular to the axis by the different parts of the spread and diverted jet balance each other, and nothing remains but the sum of the components parallel to the axis, whose value is F_x, as given in the first of the equations 3.

By substituting A v for Q, the forces may be expressed in terms of the sectional area of the jet.

As a particular case, let the surface be a plane, as in fig. 255. The jet, on striking the surface, spreads and glances off in all directions at right angles to its original direction, so that $\beta = 90°$, $\cos\beta = 0$, and

$$\mathrm{F}_x = \frac{\varrho\,\mathrm{Q}\,v}{g} = \frac{\varrho\,\mathrm{A}\,v^2}{g};\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4.)$$

being equal to the weight of a column of fluid whose base is the sectional area of the jet, and its height *double* of the height due to the velocity. This result is confirmed by experiment.

As another case, let the surface be a hollow hemisphere (fig. 256), so that the jet in spreading is turned directly backwards. Then $\beta = 180°$, $-\cos\beta = +1$, and

$$\mathrm{F}_x = \frac{2\,\rho\,\mathrm{Q}\,v}{g} = \frac{2\,\rho\,\mathrm{A}\,v^2}{g}\,; \ldots\ldots\ldots\ldots\ldots\ldots (5.)$$

being equal to the weight of a column of fluid whose base is the sectional area of the jet, and its height *four times* the height due to the velocity.

649. **The Pressure of a Jet against a Moving Surface** is found by substituting in the equations of the preceding Article, the motion of the jet *relatively to the surface* for its motion relatively to the earth. In this case there is energy transmitted from the jet to the solid surface or from the solid surface to the jet; and the determination of the amount of energy so transmitted per second forms an important part of the problem.

CASE 1. *When the surface has a motion of translation parallel to the original direction of the jet*, let u be the velocity of that motion, positive if it is along with the motion of the jet, and negative if against it; let v_1 be the original velocity of the jet; then $v_1 - u$ is the velocity of the jet relatively to the surface. Consequently, the component force acting between the fluid and the solid surface, in the direction of motion of the latter, is

$$\mathrm{F}_x = \frac{\rho\,\mathrm{Q}\,(v_1 - u)}{g}\,(1 - \cos\beta)\,; \ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

representing also the equal and opposite force which must be applied to the solid to make its motion uniform; and the energy transmitted per second is

Q reaching [illegible]

$$\mathrm{F}_x\,u = \frac{\rho\,\mathrm{Q}\,u\,(v_1 - u)}{g}\,(1 - \cos\beta)\,; \ldots\ldots\ldots\ldots\ldots (2.)$$

which, if u is positive, is transmitted from the fluid to the solid, and if u is negative, from the solid to the fluid.

The energy thus transmitted per second is equal to the difference of the actual energies of the volume Q of fluid before and after acting on the solid. Let v_2 be the velocity of the fluid after the collision; this being the resultant of u, and of $v_1 - u$ in the deviated direction, its square is given by the equation

$$\begin{aligned} v_2^2 &= u^2 + (v_1 - u)^2 + 2\,u\,(v_1 - u)\cdot\cos\beta \\ &= v_1^2 - 2\,u\,(v_1 - u)\,(1 - \cos\beta)\,; \ldots\ldots\ldots\ldots\ldots (3.) \end{aligned}$$

2 Q

by comparing which with equation 2 it is evident that

$$F_x u = \frac{\varrho\, Q\, (v_1^2 - v_2^2)}{2\,g}, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4.)$$

as has been stated.

The maximum transmission of energy from the fluid to the solid, for a given velocity of jet, is obviously given by the velocity,

$$\left.\begin{array}{l} u = \dfrac{v_1}{2}; \\ \text{which gives} \\ F_x = \dfrac{\varrho\, Q\, v_1}{2\,g}\,(1 - \cos\beta);\ F_x u = \dfrac{\varrho\, Q\, v_1^2}{4\,g}\,(1 - \cos\beta). \end{array}\right\} \ldots (5.)$$

If $\beta = 90°$, as in fig. 255, the maximum energy transmitted is $\varrho\, Q\, v_1^2 \div 4\,g$, or *half* of the original actual energy of the fluid. If $\beta = 180°$, as in fig. 256, the maximum energy transmitted is $\varrho\, Q\, v_1^2 \div 2\,g$, or *the whole* of the original actual energy of the fluid, which, after the collision, is left at rest.

CASE 2. *When the surface has a motion of translation in any direction*, with the velocity u, let B D, fig. 254, represent that direction and velocity, and B C the direction and original velocity v_1 of the jet. Then D C represents the direction and velocity of the original motion of the jet relatively to the surface. Draw E F = D C tangent to the surface at E, where the jet glances off; this represents the relative velocity and direction with which the jet leaves the surface. Draw F G ∥ and = B D, and join E G; this last line represents the direction and velocity relatively to the earth, with which the jet leaves the surface, being the resultant of E F and F G.

The total force exerted between the fluid and the surface might be determined by finding the change of the momentum of the volume of fluid Q, due either to the change of direction and velocity relatively to the earth, viz., from B C to E G; or to that relatively to the surface, viz., from D C to E F. But the force which it is most important to determine is that to which the transmission of energy is due, viz., the force parallel to B D, which will be denoted by F_x. This force is equal to the change in one second of the component momentum of the fluid in the direction B D. Let $\alpha = \angle$ D B C, denote the angle between the direction of the jet and that of the body's translation; then the component, in the direction B D, of the original velocity of the jet is

$$v_1 \cos \cdot \alpha.$$

Let $w = \overline{DC}$ be the velocity of the jet relatively to the surface; then

$$w^2 = u^2 + v_1^2 - 2\,u\,v_1 \cdot \cos\alpha \ldots\ldots\ldots\ldots\ldots\ldots (6.)$$

Let γ = supplement of $\angle$ E F G, denote the angle which a tangent to the surface at the edge where the fluid leaves it makes with the direction of translation. Then the component, in the direction B D, of the new velocity of the jet is

$$u + w\cos\gamma\,;$$

and the change of momentum in that direction in one second is

$$F_x = \frac{\varrho\,Q}{g}\,(v_1\cos\alpha - u - w\cdot\cos\gamma) \ldots\ldots\ldots\ldots\ldots (7.)$$

which gives for the energy transferred per second,

$$F_x\,u = \frac{\varrho\,Q}{g}\,u\,(v_1\cos\alpha - u - w\cdot\cos\gamma) \ldots\ldots\ldots\ldots (8.)$$

Let v_2 be the resultant velocity of the fluid after the collision; then

$$v_2^2 = u^2 + w^2 + 2\,u\,w\cdot\cos\gamma \ldots\ldots\ldots\ldots\ldots\ldots (9.)$$

and it is easily verified that

$$F_x\,u = \frac{\varrho\,Q\,(v_1^2 - v_2^2)}{2\,g} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (10.)$$

650. **Pressure of a Forced Vortex Against a Wheel.**—In a free vortex (Article 630, 631), because the velocity of each particle is inversely as its distance from the axis, the *angular momentum* of every particle of equal weight is the same; and a particle in moving nearer to or farther from the axis of the vortex, preserving its angular momentum, requires no external force to be applied to it in order to make it assume the motion proper to each part of the vortex at which it arrives.

If, in a forced vortex, there is at the same time a radiating current by which the fluid moves towards or from the axis, then by means of solid surfaces, such as those of the vanes of a wheel, there must be applied to the fluid in the vortex a couple sufficient in each second to produce the requisite change of angular momentum in the quantity of fluid which flows *radially* through the vortex in a second, and the fluid will react upon the wheel with an equal and opposite couple.

Symbolically, let r_0, r_1, be the radii of the cylindrical surfaces at which a forced vortex begins and ends; v_0, v_1, the velocities of the revolving motion at these two surfaces; Q, the flow of the radial current; then the moment of the couple exerted between the vortex and the wheel is

$$M = \frac{\varrho Q}{g} (v_0 r_0 - v_1 r_1) \dots\dots\dots\dots\dots\dots (1.)$$

A vortex-wheel, or turbine, when working in the most favourable manner, receives the fluid at ends of its vanes which have a velocity of revolution equal to that of the particles of fluid in contact with them; so that *relatively to the wheel*, the motion of the fluid is at first radial. The fluid glances off from the vanes at their other ends, which are of such a figure and position that they leave the fluid behind them with only a radial motion relatively to the earth; so that the whole of the energy due to the *revolution* of the fluid is transmitted to the wheel. That is to say, let a be the angular velocity of the wheel; then we must have

$$\left. \begin{array}{c} v_0 = a\, r_0;\ v_1 = 0; \\ M = \dfrac{\varrho Q a r_0^2}{g};\ M a = \dfrac{\varrho Q a^2 r_0^2}{g} = \dfrac{\varrho Q v_0^2}{g} \end{array} \right\} \dots\dots\dots (2.)$$

The last quantity, M a, is the energy transmitted in each second from the fluid to the wheel, which, in the case supposed, is the whole energy due to the motion of revolution and centrifugal pressure of the weight ϱ Q of fluid in a rotating forced vortex, as already shown in Article 632.

The ends of the vanes which receive the fluid should be radial, because the motion of the fluid relatively to them is radial. The ends of the vanes where the fluid glances off should be inclined backwards so as to make with the radii intersecting them, an angle θ given by the following equation :—Let $u = \frac{Q}{2 \pi r_1 b}$ be the velocity of the radial current at the ends of the vanes now in question; then

$$\tan \theta = \frac{a r_1}{u} = \frac{2 \pi a r_1^2 b}{Q}; \dots\dots\dots\dots\dots\dots (3.)$$

b being the depth of the wheel in a direction parallel to the axis.

Fig. 257 represents part of Thomson's vortex water-wheel, designed on these principles. The water is supplied to the wheel from a large external casing, in which it forms a free spiral vortex; it is directed by guide blades, C, against the outer circumference of the wheel, where the vanes are radial, and is discharged at the central orifice of the wheel, the inner ends of the vanes being directed backwards at the angle θ above described. The guide blades are moveable about pivots at A, in order to adjust the angle of obliquity of the external free spiral vortex at pleasure, and so to adapt the flow Q of the radial current to the work to be performed

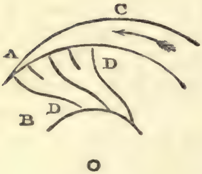

Fig. 257.

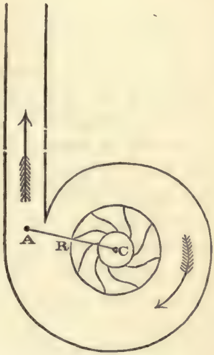

Fig. 258.

A vortex-wheel has been applied to steam by Mr. William Gorman of Glasgow.

651. A **Centrifugal Pump** consists mainly of a vortex-wheel which communicates motion to the water, so as to make it form a forced vortex of the radius $\overline{\mathrm{C\,R}} = r_0$, fig. 258. The water is supplied by a radiating current proceeding *outwards* from the central orifice towards the circumference. The inner ends of the vanes should make with the radii traversing them the angle already denoted by θ, Article 650, equation 3, that they may *cleave* the fluid as it moves radially outwards, without *striking* it, which would cause agitation, and waste of energy in friction. The outer ends of the vanes should be radial. Beyond the wheel, the water forms a free spiral vortex in a casing, from which it is discharged at A through a pipe. The surface velocity $a\, r_0 = v_0$ of the wheel is regulated by the total head required, consisting of the elevation at which the water is to be delivered, the height due to its velocity of delivery, and the head lost in overcoming friction; that is to say, according to the principles of Article 630 to 633,

$$\frac{v_0^2}{g} = h_1 = z + \frac{\mathrm{V}^2}{2\,g}\,(1 + \Sigma \cdot f) \dots\dots\dots\dots\dots\dots (1.)$$

where z is the elevation of the point of delivery, V the velocity in the discharge pipe, and $\Sigma \cdot f$ the sum of the various quantities by which the height due to that velocity is to be multiplied to find the loss of head from various causes of friction. The ratio of $\overline{\mathrm{C\,A}}$ to $\overline{\mathrm{C\,R}} = r_0$ is regulated by the law that in a free vortex the velocity is inversely as the radius; that is to say,

$$\overline{\mathrm{C\,A}} = \frac{r_0\,v_0}{\mathrm{V}} \dots\dots\dots\dots\dots\dots \dots\dots\dots (2.)$$

Guide blades in the free vortex are here unnecessary.

A blowing fan is a centrifugal pump applied to air.

652. The **Pressure of a Current** upon a solid body floating or immersed in it would be equal in opposite directions, and have nothing for its resultant, if fluids moved without friction. But because of the energy of the diverted streams which glance from the body being to a greater or less extent expended in fluid friction, the pressure on the back of the solid body becomes less intense than the pressure on the front; and to the resultant pressure in the direction of the current thus arising, has to be added the resultant of the direct friction of the fluid against the surface of the solid body.

Our knowledge of the laws of the force exerted by a current against a solid body is almost wholly empirical.

It is known that that force can be approximately represented by a formula of this kind :—

$$\mathrm{F} = k\,\varrho\,\mathrm{A} \cdot \frac{v^2}{2\,g}; \dots\dots\dots\dots\dots\dots\dots\dots\dots (1.)$$

being the product of the height due to the velocity of the current, the area A of the greatest cross-section of the solid body; the weight ϱ of an unit of volume of the fluid, and a co-efficient k depending on the figure of the body. The values of this co-efficient have been found experimentally for a few figures. The following, according to Duchemin, are some of its values for rectangular prisms and cylinders, placed with their axes along the current :—

Let L be the length of the prism or cylinder, A its transverse area, b and d its transverse dimensions, if a rectangular prism, or its axes, if a cylinder. Then for

$\mathrm{L} \div \sqrt{b\,d} =$	0,	1,	2,	3.
$k =$	1·864,	1·477,	1·347,	1·328.

The value headed 0 is applicable to very thin plates.

653. The **Resistance of Fluids** to the motion of bodies floating or immersed in them is subject to the same remarks which have been made respecting the pressure of currents against solid bodies. It is also capable in many cases of being approximately represented by the formula

$$R = k \varrho A \frac{v^2}{2g} \text{.........................} (1.)$$

The co-efficient k is less for a solid moving in a fluid, than for a fluid moving past the same solid. The following values are given chiefly on the authority of Duchemin. For prisms and cylinders, moving in the direction of their axes, the symbols having the same meaning as in the last Article:—

$L \div \sqrt{b\,d} =$	0,	1,	2,	3;	average above 3.
$k =$	1·254,	1·282,	1·306,	1·330;	1·4.

These results are also given by the empirical formula,

$$k = 1{\cdot}254 \left(1 + \frac{0{\cdot}227\, L}{9 \sqrt{b\,d} + L}\right) \text{..............} (2.)$$

k for a cylinder, moving sideways, about 0·77;

for a sphere, „ „ .... „ 0·51;

for a thin hollow hemisphere moving with the hollow foremost,............................about 2·0;

for a prism with wedge-formed ends = k for same prism with flat ends, $\times (1 - \cos \beta)$,

where $\beta = \frac{1}{2}$ angle of wedge (doubtful).

The following are results deduced from Mr. Bashforth's experiments on elongated projectiles at velocities of from 1,300 to 1,500 feet per second (see *Proceedings of the Royal Society*, Feb., 1868):

$$R = \frac{c\, A\, v^3}{g};$$

where A is in square feet, and v in feet per second; and c has the following values, according to the shape of the head of the projectile, —hemispherical, 0·0000245; oval and pointed, from 0·0000191 to 0·0000204.

From the results of observations of the engine power required to propel various steam vessels of different sizes and figures at different velocities, there is reason to think it probable, that when ships are built of such figures that the water glides round their surfaces without forming surge or large eddies, the principal part, if not the only appreciable part, of the resistance, is due to the direct friction between the water and the bottom of the ship. The opinion that the resistance to the motion of ships which are not very bluff consists almost wholly of friction, has been confirmed by subsequent experiments. The co-efficient of the friction between water and the bottom of an iron ship is nearly the same with that of water in iron pipes. The friction varies nearly as the square of the velocity of rubbing between the water and the ship's bottom. That velocity is different at different points of the ship's bottom, and bears to the speed of the ship a ratio at each point depending on the ship's figure and on the position of the point in question. The average velocity of rubbing exceeds the speed of the ship; and the excess is the greater the bluffer her shape. Thus, though a long and sharp vessel presents a greater rubbing surface than a short and bluff vessel of the same size, the average velocity of rubbing is less in the longer vessel at the same speed; so that there is a certain degree of sharpness which gives the least resistance for a given size and speed. What that degree of sharpness is cannot yet be fixed with any great precision; but in general it does not greatly differ from that which is given by making the sum of the lengths of the bow and stern equal to about seven times the greatest breadth.

The following formula has been found to agree well with experiments on the resistance of ships:—Let G be the mean immersed girth; L, the length on the water line; s^2, the mean of the squares of the sines of the angles of obliquity of the *stream lines*, or lines which the particles of water follow in gliding over the ship's bottom; let v be the velocity of the ship in feet per second, and f a co-efficient, whose value for a clean painted iron bottom is about 0·004; then the resistance is nearly

$$R = \frac{f v^2}{2g} \, L \, G \, (1 + 4s^2 + s^4) \ldots\ldots\ldots\ldots\ldots (3.)$$

The factor, $L\,G\,(1 + 4s^2 + s^4)$, is called the "augmented surface." See *Civil Engineer and Architect's Journal*, October, 1861; *Phil. Trans.*, 1862, 1863; *Trans. of the Institution of Naval Architects*, 1864; also *Shipbuilding, Theoretical and Practical*, by Watts, Rankine, Napier, and Barnes.

Mr. Scott Russell has proved that, when the length of a ship bears less than a certain proportion to that of the wave which naturally travels with the same speed, there is a rapidly increasing additional resistance. The least proper length in feet suitable for a given speed is about fifteen-sixteenths of the square of the speed in knots. (As to Waves, see page 631.)

654. **Stability of Floating Bodies.**—In Article 120 it has been shown, that in order that a body floating in a liquid may be in equilibrio, the weight of liquid displaced must be equal to the weight of the floating body, and the centre of buoyancy must be in the same vertical line with the centre of gravity of the floating body.

In order that the equilibrium of a floating body may be *stable*, every angular displacement of the body from the position of equilibrium must cause *a deviation of the centre of buoyancy, relatively to a* *vertical line traversing the centre of gravity, in the direction towards which the floating body heels;* so that the weight of the body acting through its centre of gravity, and the equal and opposite pressure of the liquid acting through the centre of buoyancy, may constitute a *restoring or righting couple*, tending to bring the body back to the position of equilibrium. Should the relative deviation of the centre of buoyancy take place in the opposite direction, a couple is produced tending to upset the body, which is accordingly unstable; should the centre of buoyancy continue to be in the same vertical line with the centre of gravity, the body continues to be in equilibrio in its new position, and its equilibrium is indifferent.

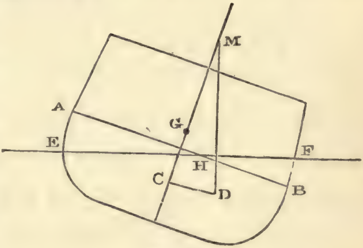

Fig. 259.

Let fig. 259 represent a cross-section of a ship, G her centre of gravity, A B the water line, and C the centre of buoyancy in the position of equilibrium. Let the ship heel through an angle θ, and let E F be the new water line, and D the new centre of buoyancy; and let the ship be kept in this position by a couple whose moment is known. Let W be the weight of the ship, and S the volume of water displaced by her, so that $W = \varrho S$ (ϱ being the weight of a cubic foot of water). Through D draw a vertical line D M, cutting the line C G, which was originally vertical, in M. The force of the righting couple is W, and its arm is the horizontal distance from G to the line D M; that is, $\overline{GM} \cdot \sin \theta$; consequently, the *moment of the righting couple*, equal and opposite to the moment of the heeling couple, is

$$W \cdot \overline{GM} \cdot \sin \theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

The *comparative stability* of a ship is proportional to the arm of the righting couple for the same angle of heel; and that arm is proportional to $\overline{GM}$, which length thus becomes a measure of the stability of the ship. The point M, when determined for an indefinitely small angle of heel, is called the METACENTRE; it may be the same, or it may be different for finite angles. When the position of M is variable, the angle of heel to be adopted in finding it should be the greatest which under ordinary circumstances is likely to occur; for different ships this varies from 6° to 20°.

If the metacentre is above the centre of gravity, the equilibrium is stable; if it coincides with the centre of gravity, the equilibrium is indifferent; if it is below the centre of gravity, the equilibrium is unstable.

Let H be the line of intersection of the planes of the two water lines A B, E F. The deviation C D of the centre of buoyancy is the same with the deviation of the centre of gravity of the mass of water displaced, which would arise from removing the wedge A H E into the position F H B. Let s be the volume of that wedge, ϱ its density, and let l denote the distance between the centres of gravity of its two positions, A H E and F H B. Draw C D parallel to the line joining those two centres of gravity; and, according to Article 77, make

$$\overline{\mathrm{C\,D}} = \frac{l \varrho s}{\mathrm{W}} = \frac{l s}{\mathrm{S}}; \text{......................} (2.)$$

then is D the new centre of buoyancy.

The angle which C D makes with the horizon is in general either exactly or very nearly $= \frac{\theta}{2}$; so that $\overline{\mathrm{C\,D}} = \overline{\mathrm{M\,C}} \cdot 2 \sin \frac{\theta}{2}$, approximately. Also, the volume s is in general either exactly or nearly proportional to $2 \sin \frac{\theta}{2}$; so that if c be a constant volume depending on the figure of the water line, $s = c \cdot 2 \sin \frac{\theta}{2}$, approximately. Consequently, to find the *height* $\overline{\mathrm{M\,C}}$ *of the point* M *above the centre of buoyancy, and its height* $\overline{\mathrm{M\,G}}$ *above the centre of gravity,* we have the approximate formulæ,—

$$\left.\begin{aligned} \overline{\mathrm{M\,C}} &= \overline{\mathrm{C\,D}} \div 2 \sin \frac{\theta}{2} = \frac{l c}{\mathrm{S}}; \\ \overline{\mathrm{M\,G}} &= \frac{l c}{\mathrm{S}} \mp \mathrm{G\,C}. \end{aligned}\right\} \text{............} (3.)$$

The sign $\mp$ denotes that $\overline{\mathrm{G\,C}}$ is to be subtracted or added according as G is above or below C. The product $l c$ is found approximately in the following manner, for those cases in which the water lines A B and E F are sensibly equal and similar figures, so that the line H, where their planes intersect, traverses the centre of gravity of each of those figures, and the wedges A H E, F H B, are similar as well as equal.

The product $l s = l c \cdot 2 \sin \frac{\theta}{2}$ is the double of the statical moment of one of the wedges relatively to the line H, supposing the density equal to unity. Let distances measured lengthways on the line H be denoted by x; let the perpendicular distance of any point in a water line plane bisecting the angle A H E from the line H be denoted by y, and let the thickness of the wedge at the point whose co-ordinates are x and y be $z = y \cdot 2 \sin \frac{\theta}{2}$. Then we have

$$\left.\begin{array}{l} s = 2 \sin \frac{\theta}{2} \cdot \int \int y \cdot d\, y\, d\, x;\ c = \int \int y \cdot d\, y\, d\, x; \\ \quad l\, s = 4 \sin \frac{\theta}{2} \cdot \int \int y^2 \cdot d\, y\, d\, x; \\ \text{and therefore} \\ \quad l\, c = 2 \int \int y^2 \cdot d\, y\, d\, x; \end{array}\right\} (4.)$$

being the *moment of inertia of the water line plane* about the axis H. To express this in a convenient form, let b be the breadth of the ship at the water line, at a given distance x, measured lengthways from an assumed origin. Then

$$2 \int y^2\, d\, y = \frac{b^3}{12}; \text{ and } l\, c = \frac{1}{12} \int b^3 \cdot d\, x \text{.........}(5.)$$

As to the moments of inertia of different plane figures, see Article 95. Thus, equation 3 becomes

$$\overline{\text{M G}} = \frac{\int b^3 \cdot d\, x}{12\ \text{S}} \mp \overline{\text{G C}} \text{.................}(6.)$$

The theory of the stability of ships was first investigated by Bossut, and was further developed by Atwood. The most important contributions to that theory, of later date, have been, the memoir of Dupin, *Sur la Stabilité des Corps Flottans*, a paper by Canon Moseley in the *Philosophical Transactions* for 1850, and various papers by Rawson, Froude, Merrifield, Barnes, and others.

655. **Oscillations of Floating Bodies.**—The theory of the oscillations of ships was investigated in an approximate manner by Bossut and other mathematicians, and was first brought into a complete state by Moseley, in the paper already referred to. Its details are of much complexity; and an outline of its leading principles, and of their results in the most simple cases, is all that needs be given in this treatise.

The oscillation of a ship may be resolved into rolling, or gyration about a longitudinal axis, pitching, or gyration about a transverse axis, and vertical oscillation, consisting in an alternate rising above and sinking below the position of equilibrium. The point of chief importance in practice is the *time occupied by a rolling oscillation.* If that time is too long, the ship is deficient in stability; if too short, her movements are abrupt, and tend to overstrain her.

If a ship is of such a figure that, when she rolls into a new position of equilibrium under the action of a couple, her centre of gravity does not alter its level, then her rolling gyrations are performed about a permanent longitudinal axis traversing her centre of gravity, and are not accompanied by vertical oscillations, and her moment of inertia is constant while she rolls. That condition is fulfilled if all the water line planes, such as A B and E F, are tangents to one sphere described about G. In what follows it will be supposed that this condition is fulfilled, and also that the position in the ship of the point M is sensibly constant.

According to Article 654, equation 1, the *righting couple* for a given angle of heel θ is

$$\mathrm{W} \cdot \overline{\mathrm{G\,M}} \cdot \sin \theta;$$

but in an approximate solution we may substitute θ for $\sin \theta$. Let I be the moment of inertia of the ship about her axis of rolling; then equations 2 and 3 of Article 598 give the following value for the time of a double gyration:—

$$\frac{2\,\pi}{k} = 2\,\pi \sqrt{\left(\frac{\mathrm{I}}{g\,\mathrm{W} \cdot \overline{\mathrm{G\,M}}}\right)} = \frac{2\,\pi\,\mathrm{R}}{\sqrt{g \cdot \overline{\mathrm{G\,M}}}};\text{........}(1.)$$

where R is the radius of gyration of the ship. This is the same with the time of a double oscillation of a simple pendulum whose length is $\mathrm{R}^2 \div \overline{\mathrm{G\,M}}$.

The researches of Mr. William Froude, first described to the British Association in July, 1860, and afterwards laid more fully before the Institute of Naval Architects, have shown, *first*, that the same forces which tend to keep a ship upright in still water tend to place her perpendicular to the surface of the water amongst waves, and thus to increase rolling; *secondly*, that the chief cause of excessive rolling is too near a coincidence between the periodic time of the vessel's rolling and that of her being acted upon by successive waves; and *thirdly*, that the most efficient method of preventing excessive rolling is to adjust the moment of inertia and the stability of a vessel, so that her periodic time of rolling shall be longer than the period of any waves she is likely to encounter, taking care at the same time to leave sufficient stability to prevent the risk of upsetting, or of heeling too far over with a side wind.

See *Trans. of the Institution of Naval Architects*, passim; also *Shipbuilding*, by Watts, Rankine, Napier, and Barnes. (As to Waves, see page 631.)

656. **The Action between a Fluid and a Piston**, consisting in the transmission of energy from the one to the other, has already been considered in a general way in Article 517. In the present Article it will be treated more in detail.

In figs. 260 and 261, let abscissæ measured parallel to the line O S represent the spaces successively occupied by a fluid in a cylinder provided with a piston, any such space being denoted by s; and let ordinates measured parallel to the line O P, perpendicular to O S, represent the intensities of the pressure exerted by the fluid against the piston, any such intensity being denoted by p.

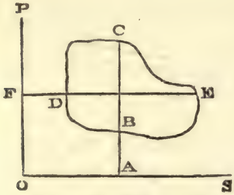

Fig. 260.

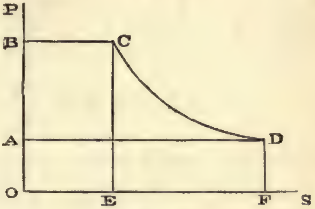

Fig. 261.

Let a given weight of a gaseous substance go through a succession of arbitrary changes of pressure and volume, so as to return in the end to the condition from which it set out. Such a succession of changes is called a *cycle* of changes; it is represented by a *closed curve*, such as D C E B in fig. 260, and the *area* of that curve represents the *energy transferred* during the cycle of changes. If the changes take place in the order D C E B, that is, if greater pressures are exerted during the expansion of the substance than during its compression, energy is transferred from the gas to the piston; if the changes take place in the order D B E C, that is, if greater pressures are exerted by the substance during its compression than during its expansion, energy is transferred from the piston to the gas.

The amount of energy transferred may be expressed in two ways. First, for any given volume $\overline{OA} = s$, let $\overline{AC} = p_1$ and $\overline{AB} = p_2$ be the greater and the less intensities of the pressure; then

$$\text{energy transferred} = \int (p_1 - p_2)\, d\, s \text{............} (1.)$$

Secondly, for any given pressure $\overline{OF} = p$, let $\overline{FE} = s_1$ and $\overline{FD} = s_2$ be the greater and the less of the spaces occupied; then

$$\text{energy transferred} = \int (s_1 - s_2)\, d\, p \text{............} (2.)$$

which is another expression for the same quantity.

Fig. 261 represents the case in which a given weight of an elastic substance occupying the space $\overline{OE} = s_1$ at the pressure $OB = p_1$, is introduced into a cylinder and made to drive a piston,—is then allowed to expand, its volume increasing to $\overline{\mathrm{OF}} = s_2$, and its pressure falling to $\overline{\mathrm{FD}} = p_2$, according to a law represented by the curve C D,—and is lastly expelled from the cylinder at the final pressure. In this case the energy transferred from the elastic substance to the piston is represented by

$$\text{area ABCD} = \int_{p_2}^{p_1} s\,d\,p = \mathrm{W}\int_{p_2}^{p_1}\frac{d\,p}{\varrho}; \ldots\ldots\ldots(3.)$$

being, in fact, as the last expression shows, equal to the weight of the elastic substance employed, W, multiplied by its loss of *dynamic head.*

The same equation gives the energy transferred from the piston to the elastic substance, when the latter is introduced into the cylinder at the lower pressure and expelled at the higher.

For a perfect gas (Article 635) this expression becomes

$$\int_{p_2}^{p_1} s\,d\,p = \frac{\gamma}{\gamma-1}\cdot s_1 p_1\left\{1-\left(\frac{s_1}{s_2}\right)^{\gamma-1}\right\}\ldots\ldots\ldots\ldots(4.)$$

If the fluid is discharged from the cylinder under a pressure p_3 less than that at which the expansion terminates, there is to be added to the preceding formula the term

$$s_2\,(p_2 - p_3)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5.)$$

If the fluid which acts on the piston is introduced in the state of saturated vapour, it is discharged as a mixture of saturated vapour at a lower pressure with more or less of liquid. In this case, the following equations belonging to the science of thermodynamics are to be used. Let p be the pressure of saturation of a vapour, and τ the corresponding boiling point of its liquid, in degrees reckoned from the *absolute zero*, 274° Centigrade or 493°·2 Fahrenheit below the melting point of ice. Then

$$\left.\begin{aligned} &\operatorname{Log} p = \mathrm{A} - \frac{\mathrm{B}}{\tau} - \frac{\mathrm{C}}{\tau^2};\\ &\frac{1}{\tau} = \sqrt{\left\{\frac{\mathrm{A}-\log p}{\mathrm{C}} + \frac{\mathrm{B}^2}{4\,\mathrm{C}^2}\right\}} - \frac{\mathrm{B}}{2\,\mathrm{C}}. \end{aligned}\right\}\ldots\ldots(6.)$$

(See *Edin. Philos. Jour.*, July, 1849; *Edin. Transac.*, xx; *Philos. Mag.*, Dec., 1854; *Nichol's Cyclopædia,* art. "Heat, Mechanical Action of.") The following are the values of some of the constants in the above formulæ, selected from a table in the *Philosophical Magazine* for Dec., 1854, p being in lbs. per *square foot,* and τ in degrees of Fahrenheit:—

	A	Log B	Log C	$\frac{B}{2C}$	$\frac{B^2}{4C^2}$
Water,...	8·2591	3·43642	5·59873	0·003441	0·00001184
Æther,...	7·5732	3·31492	5·21706	0·006264	0·00003924

Let L be the value, in foot pounds of energy, of the latent heat of evaporation, at the absolute temperature τ, of so much fluid as fills a cubic foot more in the state of vapour than it does in the state of liquid; D the weight of that fluid; H the value, in foot pounds of energy, of the latent heat of evaporation of one pound of the fluid at the absolute temperature τ; and J the equivalent in foot pounds of a British thermal unit, or 772; then

$$\left.\begin{aligned} &L = \tau\frac{dp}{d\tau} = p\left(\frac{B}{\tau} + \frac{2C}{\tau^2}\right)\cdot \text{hyp. log } 10 \\ &\qquad (\text{hyp. log. } 10 = 2{\cdot}3026); \\ &H = H_0 - J\,(c-b)\,(\tau-\tau_0) \\ &\qquad (\text{for water, } c - b = 0{\cdot}7); \\ &D = L \div H. \\ &\qquad (\text{for water at the temperature of} \\ &\qquad \text{melting ice, } H_0 = 842872.) \end{aligned}\right\}\cdots\cdots\cdots(7.)$$

J c denotes the value in foot pounds of the specific heat of the liquid, which for water is 772, and for æther, 399.

Let the suffixes 1, 2, and 3, denote the pressures and temperatures respectively, of the introduction of the vapour, the end of its expansion, and its final discharge, and quantities corresponding to them; s_1 and s_2 being, as before, the spaces filled by it at the beginning and end of its expansion. Then

$$\text{ratio of expansion, } \frac{s_2}{s_1} = \frac{\tau_2}{L_2}\left\{\frac{L_1}{\tau_1} + J\,c\,D_1\cdot\text{hyp log}\frac{\tau_1}{\tau_2}\right\};\ldots\ldots(8.)$$

$$\left.\begin{aligned} &\text{energy transferred, } U = \int_{p_3}^{p_1} s\,dp + s_2(p_2 - p_3) \\ &= s_2(p_2 - p_3) + s_1\left\{\frac{L\,(\tau_1 - \tau_2)}{\tau_1} + J\,c\,D_1\left(\tau_1 - \tau_2\left(1 + \text{hyp log}\frac{\tau_1}{\tau_2}\right)\right)\right\} \end{aligned}\right\}(9.)$$

$$\left.\begin{matrix}\text{heat expended} \\ \text{in foot pounds,}\end{matrix}\right\} H = s_1\left\{L_1 + J\,c\,D_1\,(\tau_1 - \tau_3)\right\}\ldots\ldots\ldots\ldots\ldots\ldots(10.)$$

These formulæ are demonstrated in a paper on Thermodynamics in the *Philosophical Transactions* for 1854.

The complexity of the preceding formulæ renders their use inconvenient, except with the aid of tables of the quantities p, L, and D, for different boiling points. In the absence of such tables, the following formulæ give approximate results for steam, where the pressure of its admission p_1 is from one to twelve atmospheres :—

$$\frac{s_2}{s_1} = \left(\frac{p_1}{p_2}\right)^{\frac{9}{10}}; \; \frac{p_1}{p_2} = \left(\frac{s_2}{s_1}\right)^{\frac{10}{9}}; \ldots\ldots\ldots\ldots\ldots (11.)$$

$$\left.\begin{aligned} \text{energy transferred, } \mathrm{U} &= \int_{p_2}^{p_1} s\,d\,p + s_2\,(p_2 - p_3) \\ &= p_1\,s_1 \cdot 10 \left\{ 1 - \left(\frac{s_1}{s_2}\right)^{\frac{1}{9}} \right\} + s_2\,(p_2 - p_3) \\ &= p_2\,s_2 \left\{ 10 \left(\frac{s_2}{s_1}\right)^{\frac{1}{9}} - 9 - \frac{p_3}{p_2} \right\}. \end{aligned}\right\} \ldots (12.)$$

The expenditure of heat in foot pounds may be computed roughly to about $\frac{1}{100}$, when the feed water is supplied to the boiler at about 100° Fahrenheit, by the formula

$$\mathrm{H} = \int_{p_2}^{p_1} s\,d\,p + n\,p_2\,s_2; \ldots\ldots\ldots\ldots\ldots\ldots (13.)$$

where n is a co-efficient whose value is, for condensing engines, 16; for non-condensing engines, 15.

Equations 11 and 12 are applicable to non-conducting cylinders without steam-jackets. For cylinders with steam-jackets, acting so as to keep the steam dry, it is more accurate to substitute 16 for 9, 17 for 10, and $\frac{16}{17}$, $\frac{17}{16}$, and $\frac{1}{16}$, respectively, for $\frac{9}{10}$, $\frac{10}{9}$, and $\frac{1}{9}$, throughout the equations 11 and 12.

For the exact theory of this case, see *A Manual of the Steam Engine and other Prime Movers;* also, *Philosophical Transactions*, 1859, Part I.

The following are the ordinary formulæ, which give a good approximation when the steam is slightly moist:—

$$\frac{s_2}{s_1} = \frac{p_1}{p_2}; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14.)$$

$$\mathrm{U} = p_1\,s_1 \text{ hyp. log. } \frac{s_2}{s_1} + s_2\,(p_2 - p_3) \ldots\ldots\ldots\ldots (15.)$$

The approximate formula (13) is applicable in all cases.

PART VI.

THEORY OF MACHINES.

657. **Nature and Division of the Subject.**—In the present Part of this work, machines are to be considered not merely as modifying motion, but also as modifying force, and transmitting energy from one body to another. The theory of machines consists chiefly in the application of the principles of dynamics to trains of mechanism; and therefore a large portion of the present part of this treatise will consist of references back to Part IV. and Part V.

There are two fundamentally different ways of considering a machine, each of which must be employed in succession, in order to obtain a complete knowledge of its working.

I. In the first place is considered the action of the machine during a certain period of time, with a view to the determination of its EFFICIENCY; that is, the ratio which the *useful* part of its work bears to the whole expenditure of energy. The motion of every ordinary machine is either uniform or periodical. Hence, as has been shown in Article 553, the principle of the equality of energy and work, as expressed in Article 518, is fulfilled either constantly or periodically at the end of each period or cycle of changes in the motion of the machine.

II. In the second place is to be considered the action of the machine during intervals of time less than its period or cycle, if its motion is periodic, in order to determine the law of the periodic changes in the motions of the pieces of which the machine consists, and of the periodic or reciprocating forces by which such changes are produced (Article 556).

The first chapter of the present Part relates to the work of machines moving uniformly or periodically, and the second chapter to variations of motion and force in machines. In a third chapter will be stated briefly the general principles of the action of the more important *prime movers*. With respect to those machines, it is impossible to enter fully into details within the limits of such a treatise as the present, especially as the most important of them all, the steam engine, depends on the laws of the phenomena of heat, which could not be completely explained except in a special treatise.

CHAPTER I.

WORK OF MACHINES WITH UNIFORM OR PERIODIC MOTION.

SECTION 1.—*General Principles.*

658. **Useful and Lost Work.**—The whole work performed by a machine is distinguished into *useful work*, being that performed in producing the effect for which the machine is designed, and *lost work*, being that performed in producing other effects.

659. **Useful and Prejudicial Resistance** are overcome in performing useful work and lost work respectively.

660. The **Efficiency** of a machine is a fraction expressing the ratio of the useful work to the whole work performed, which is equal to the energy expended. The limit to the efficiency of a machine is *unity*, denoting the efficiency of a perfect machine in which no work is lost. The object of improvements in machines is to bring their efficiency as near to unity as possible.

661. **Power and Effect; Horse Power.**—The *power* of a machine is the energy exerted, and the *effect*, the useful work performed, in some interval of time of definite length.

The unit of power called conventionally a *horse power*, is 550 foot pounds per second, or 33,000 foot pounds per minute, or 1,980,000 foot pounds per hour. The effect is equal to the power multiplied by the efficiency.

662. **Driving Point; Train; Working Point.**—The driving point is that through which the resultant effort of the prime mover acts. The train is the series of pieces which transmit motion and force from the driving point to the working point, through which acts the resultant of the resistance of the useful work.

663. **Points of Resistance** are points in the train of mechanism through which the resultants of prejudicial resistances act.

664. **Efficiencies of Pieces of a Train.**—The useful work of an intermediate piece in a train of mechanism consists in driving the piece which follows it, and is less than the energy exerted upon it by the amount of the work lost in overcoming its own friction. Hence the efficiency of such an intermediate piece is the ratio of the work performed by it in driving the following piece, to the energy exerted on it by the preceding piece; and it is evident that *the efficiency of a machine is the product of the efficiencies of the series*

of moving pieces which transmit energy from the driving point to the working point. The same principle applies to a train of *successive machines,* each driving that which follows it.

665. **Mean Efforts and Resistances.**—In Article 515 is given the expression $\int \mathrm{P}\, d\, s$ for the energy exerted by a varying effort whose magnitude at any instant is P; and a corresponding expression $\int \mathrm{R}\, d\, s$ denotes the work performed in overcoming a variable resistance. In a machine moving uniformly, let these expressions have reference to any interval of time, and in a machine moving periodically, to one or any whole number of periods; let s be the space described by the point of application of the effort or resistance in the interval in question; then $\int \mathrm{P}\, d\, s \div s$ or $\int \mathrm{R}\, d\, s \div s$ is the *mean effort* or *mean resistance* as the case may be. The *fluctuations* of the efforts and resistances above and below their mean values concern only the variations of velocity in a machine; and therefore, in the remainder of the present chapter, P and R will be used to denote such mean values only; so that energy exerted and work performed, whether the forces are constant or varying, will be respectively denoted by $\mathrm{P}\, s$ and $\mathrm{R}\, s$. By referring to Articles 517 and 593, it appears, that besides a force and a length, as expressed above, the two factors of a quantity of energy may be a stress and a cubic space, or a couple and an angle, as shown in the following table:—

$$\left.\begin{array}{c}\text{Energy}\\ \text{or}\\ \text{work}\\ \text{in}\\ \text{foot pounds}\end{array}\right\} = \left\{\begin{array}{l}\text{Force in pounds} \times \text{distance in feet;}\\ \text{Couple in foot pounds} \times \text{angular motion to}\\ \quad\text{radius unity; or}\\ \text{Pressure in pounds per square foot} \times \text{space}\\ \quad\text{described by a piston in cubic feet.}\end{array}\right.$$

666. **The General Equations** of the uniform or periodical working of a machine are obtained by introducing the distinction between useful and lost work into the equations of the conservation of energy. Thus, let P denote the mean effort at the driving point, s the space described by it in a given interval of time, being a whole number of periods or revolutions, R_1 the mean useful resistance, s_1 the space through which it is overcome in the same interval, R_2 any one of the prejudicial resistances, s_2 the space through which it is overcome; then

$$\mathrm{P}\, s = \mathrm{R}_1\, s_1 + \Sigma \cdot \mathrm{R}_2\, s_2 \ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

The efficiency of the machine is expressed by

$$\frac{R_1 s_1}{P s} = \frac{R_1 s_1}{R_1 s_1 + \Sigma \cdot R_2 s_2} \ldots\ldots\ldots\ldots\ldots\ldots (2.)$$

667. **Equations in terms of Comparative Motions.**—Let $s_1 : s = n_1$, $s_2 : s = n_2$, &c., be the *ratios* of the spaces described in a whole number of periods by the working point and the several points of resistance, to the space described, in the same interval of time, by the driving point; then equation 1 of Article 666 takes the following form, which expresses the "Principle of Virtual Velocities" (Article 519) as applied to machines:—

$$P = n_1 R_1 + \Sigma \cdot n_2 R_2, \ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

Thus the *mean effort* at the driving point is expressed in terms of the several mean resistances, and of the *comparative motions* alone, which last set of quantities are deduced from the construction of the machine by the principles of the theory of mechanism; so that every proposition in Part IV., respecting the comparative motions of the points of a machine, can at once be converted into a proposition respecting the relation between the mean effort and resistances; and the mean effort required to drive the machine can be determined if the resistances are known.

668. **Reduction of Forces and Couples.** — In calculation it is often convenient to substitute for a force applied to a given point, or a couple applied to a given piece, the *equivalent* force or couple applied to some other point or piece; that is to say, the force or couple, which, if applied to the other point or piece, would exert equal energy, or employ equal work. The principles of this reduction are, that the ratio of the given to the equivalent force is the reciprocal of the ratio of the velocities of their points of application; and the ratio of the given to the equivalent couple is the reciprocal of the ratio of the angular velocities of the pieces to which they are applied.

SECTION 2.—*On the Friction of Machines.*

669. **Co-efficients of Friction.**—The nature and laws of the friction of solid surfaces, and the meanings of co-efficients of friction and angles of repose, have been explained in Articles 189, 190, 191, and 192. The following is a table of the angle of repose φ, the co-efficient of friction $f = \tan \varphi$, and its reciprocal $1 : f$, for the materials of mechanism, condensed from the tables of General Morin, and other sources, and arranged in a few comprehensive classes. The values of those constants which are given in the table have reference to the *friction of motion.* As to the difference between that and the friction of rest, see Article 204.

No.	SURFACES.	φ	f	1 : f
1.	Wood on wood, dry,....................	14° to 26½°	·25 to ·5	4 to 2
2.	„ „ soapy,..................	11½°	·2	5
3.	Metals on oak, dry,....................	26½° to 31°	·5 to ·6	2 to 1·67
4.	„ „ wet,....................	13½° to 14½°	·24 to ·26	4·17 to 3·85
5.	„ „ soapy,..................	11½°	·2	5
6.	Metals on elm, dry,....................	11½° to 14°	·2 to ·25	5 to 4
7.	Hemp on oak, dry,....................	28°	·53	1·89
8.	„ „ wet,....................	18½°	·33	3
9.	Leather on oak,..........................	15° to 19½°	·27 to ·38	3·7 to 2·86
10.	Leather on metals, dry,..................	29½°	·56	1·79
11.	„ „ wet,..................	20°	·36	2·78
12.	„ „ greasy,...............	13°	·23	4·35
13.	„ „ oily,..................	8½°	·15	6·67
14.	Metals on metals, dry,.............. ...	8½° to 11½°	·15 to ·2	6·67 to 5
15.	„ „ wet,.............. ...	16½°	·3	3·33
16.	Smooth surfaces, occasionally greased,	4° to 4½°	·07 to ·08	14·3 to 12·5
17.	„ „ continually greased,	3°	·05	20
18.	„ „ „ best results,	1¾° to 2°	·03 to ·036	33·3 to 27·6

670. **Unguents.**—The last three results in the preceding table, Nos. 16, 17, and 18, have reference to smooth firm surfaces of any kind, greased or lubricated to such an extent that the friction depends chiefly on the continual supply of unguent, and not sensibly on the nature of the solid surfaces; and this ought almost always to be the case in machinery. Unguents should be thick for heavy pressures, that they may resist being forced out, and thin for light pressures, that their viscidity may not add to the resistance.

671. **Limit of Pressure between Rubbing Surfaces.**—The law of the simple proportionality of friction to pressure (Article 190) is only true for dry surfaces, when the pressure is not sufficiently intense to indent or grind the surfaces; and for greased surfaces, when the pressure is not sufficiently intense to force out the unguent from between the surfaces, where it is held by capillary attraction. If the proper limit of intensity of pressure be exceeded, the friction increases more rapidly than in the simple ratio of the pressure. That limit diminishes as the velocity of rubbing increases, according to some law not yet exactly determined. The following are some of its values deduced from experience:—

Railway Carriage Axles.	Limit of Pressure, lb. per square inch.
Velocity of rubbing 1 foot per second, .	392
„ „ 2½ „ . .	224
„ „ 5 „ . .	140
Timber ways for launching ships, about	50

The inclination given to these ways varies from about 1 in 10 for the smallest vessels, to about 1 in 20 for the largest. The co-efficient of friction, when the ways are well lubricated with tallow or soft soap, is probably between ·03 and ·04.

672 **Friction of a Sliding Piece.**—In fig. 262, let A represent a sliding piece, which moves uniformly along the straight guide B B in the direction indicated by the arrow, under two forces which may be direct or oblique, but which are represented as oblique, to make the solution general. The force F_2 opposed to the motion, is the resultant of the *useful resistance* or force which A exerts on the next piece in the train, and of the weight of A itself, and will be called the *given force.* Let the angle which it makes with the guide B B be denoted by i_2. The force F_1 is that which drives the piece; the angle i_1 which its direction makes with the guide B B is supposed to be known; but its magnitude remains to be determined, as well as the friction, which it has to overcome in addition to the useful resistance. Let Q denote the normal pressure of A against B B, so that f Q is the friction. Then we have the two equations of equilibrium :—

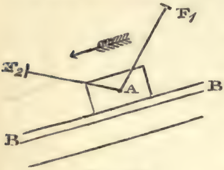

Fig. 262.

$$\left.\begin{aligned} Q &= F_1 \sin i_1 + F_2 \sin i_2; \\ F_1 \cos i_1 &= F_2 \cos i_2 + f Q \\ &= F_1 f \sin i_1 + F_2 (\cos i_2 + f \sin i_2); \end{aligned}\right\} \text{..........(1.)}$$

Substitute value of Q

from which are easily deduced the following equations, solving the problem :— Substitute val. of F_1 in 1st of Eq. (1) [illegible]

$$F_1 = F_2 \cdot \frac{\cos i_2 + f \sin i_2}{\cos i_1 - f \sin i_1}; \ f Q = F_2 \cdot \frac{f \sin (i_1 + i_2)}{\cos i_1 - f \sin i_1} \text{...(2.)}$$

673. The **Moment of Friction** of a rotating piece is the statical moment of the friction relatively to the axis of rotation of the piece, and is the moment of a couple consisting of the friction, and of an equal and opposite component of the pressure exerted by the bearings of the piece against its axle. The moment of friction, being multiplied by the angular motion in a given time, gives the work lost in friction in that time.

674. **Friction of an Axle.**—After a cylindrical axle has run for some time in contact with its bearing, the bearing becomes slightly larger than the axle, so that the point of most intense pressure, which is also the point of resistance, traversed by the resultant of the friction, adapts its position to the direction of the lateral pressure.

In fig. 263, let A A A be a transverse section of the cylindrical axle of a rotating piece, and C its axis of rotation; let R represent the direction and magnitude of what will be called the *given force*, being the resultant of the useful resistance, and of the weight of the piece under consideration. Let P represent the effort required to drive the piece, whose line of action is known, but its magnitude remains to be determined. Let D be the point where the directions of P and R intersect, and D Q the line of action of their resultant, which resultant is equal and opposite to Q, the pressure exerted by the bearing against the axle, and is therefore inclined to the radius C Q by an angle $CQD = \varphi$, being the angle of repose, in such a manner as to resist the rotation, whose direction is indicated by the arrow.

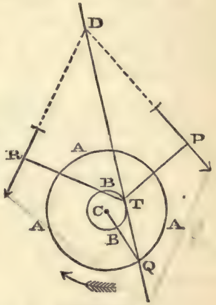

Fig. 263.

Then to find the line of pressure D Q, it is obviously sufficient to describe about the centre C a circle B B whose radius is

$$\overline{\mathrm{C\,T}} = r \cdot \sin \varphi = \frac{f\,r}{\sqrt{1+f^2}} \text{..................(1.)}$$

$r = \overline{\mathrm{C\,Q}}$ being the radius of the axle, and to draw from the known point D a line D T Q touching that circle in T, which point of contact is at that side of the circle which makes a force acting from Q towards T oppose the rotation.

From T draw T R ⊥ R, and T P ⊥ P. Then the magnitude of the effort P is given by the equation

$$\mathrm{P} = \mathrm{R} \cdot \overline{\mathrm{T\,R}} \div \overline{\mathrm{T\,P}} \text{....................(2.)}$$

and that of the pressure Q by the equation

$$\mathrm{Q}^2 = \mathrm{P}^2 + \mathrm{R}^2 + 2\,\mathrm{P\,R} \cdot \cos \angle \mathrm{P\,D\,R} \text{..........(3.)}$$

(the last term of which becomes negative when ∠ P D R is obtuse); while the friction is

$$\mathrm{Q} \sin \varphi = \frac{f\,\mathrm{Q}}{\sqrt{1+f^2}}; \text{.....................(4.)}$$

and its moment

$$\mathrm{Q}\,r \sin \varphi = \mathrm{Q} \cdot \overline{\mathrm{C\,T}} \text{.......................(5.)}$$

When P and R are parallel to each other, Q is their difference or their sum, according as they act at the same or at opposite sides of the axle, and Q T is to be drawn parallel to them both, so that

R T, T P, and C T, lie in one straight line, when equations 2, 4, and 5 will still hold.

In order to diminish the lateral pressure Q, and the friction arising from it, to the least possible amount, the mechanism should be so arranged as to make P and R act parallel to each other at the same side of the axle.

In most actual cases, $\sin \phi = f : \sqrt{1 + f^2}$ differs from $\tan \phi = f$ in a proportion too small to be of any practical importance.

The bearings of axles should be made of materials which, though hard enough to resist the rubbing without abrasion, are not so hard as the axle. Hence for wrought iron axles, bronze bearings are commonly used. Bearings of cast iron, millboard, and hardwood, such as elm, with the grain set radially, have also been used with advantage.

675. **Friction of a Pivot.**—A pivot is the termination of an axle, which presses *endways* against a bearing called a step, or footstep. Pivots require great hardness, and are usually made of steel.

A *flat pivot* is a short cylinder of steel, having a plane circular end for a rubbing surface. If the pressure Q be equally distributed over that surface whose radius is r, the moment of friction is easily found by integration to be

$$\frac{2}{3} f r Q \text{..............................} (1.)$$

In flat pivots, the intensity of the pressure, which is given by the equation

$$p = \frac{Q}{\pi r^2}, \text{..............................} (2.)$$

is usually limited to 2,240 lbs. per square inch.

In the *cup and ball* pivot, the end of the shaft, and the step, present two recesses facing each other, into which are fitted two shallow cups of steel or hard bronze. Between the concave spherical surfaces of those cups is placed a steel ball, being either a complete sphere, or a lens having convex surfaces of a somewhat less radius than the concave surfaces of the cups. The moment of friction of this pivot is at first almost inappreciable, from the extreme smallness of the radius of the circles of contact of the ball and cups; but as they wear, that radius and the moment of friction increase.

676. **Friction of a Collar.**—When it is impracticable or inconvenient to sustain the pressure which acts along a shaft by means of a pivot at its end, that pressure is borne by means of one or more *collars*, or rings projecting from the shaft, and pressing against corresponding ring-shaped bearings, for which, in the case of shafts of screw propellers, hardwood set with the grain endways has been

found a good material amongst others. Let r be the external, and r' the internal radius of a collar; its moment of friction for the pressure Q is given by the formula

See p. 611. [illegible]

$$\frac{2}{3} f \, Q \cdot \frac{r^3 - r'^3}{r^2 - r'^2} \ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

Integrate between [illegible]

677. **Friction of Teeth.**—When a pair of wheels work together, let P be the pressure exerted between each pair of their teeth which comes into action, s the distance through which each pair of teeth slide over each other, as found in Articles 453, 455, 458, and 462 A, and n the number of pairs of teeth which pass the line of centres in a given interval of time. Then in that interval, the work lost by the friction of the teeth is

$$f \, n \, s \, P \ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

678. **Friction of a Band.**—A flexible band, such as a cord, rope, belt, or strap, may be used either to exert an effort or a resistance upon a drum or pulley round which it wraps. In either case, the tangential force, whether effort or resistance, exerted between the band and the pulley, is their mutual friction, caused by and proportional to the normal pressure between them.

In fig. 264, let C be the axis of a pulley A B, round an arc of which there is wrapped a band, T_1 A B T_2; let the outer arrow represent the direction in which the band slides, or tends to slide, relatively to the pulley, and the inner arrow the direction in which the pulley slides, or tends to slide, relatively to the band.

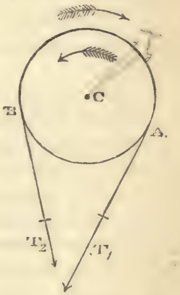

Fig. 264.

Let T_1 be the tension of the free part of the band at that side *towards* which it tends to draw the pulley, or *from* which the pulley tends to draw it; T_2 the tension of the free part at the other side; T the tension of the band at any intermediate point of its arc of contact with the pulley; θ the ratio of the length of that arc to the radius of the pulley; $d\,\theta$ the ratio of an indefinitely small element of that arc to the radius; $R = T_1 - T_2$, the total friction between the band and the pulley; $d\,R$ the elementary portion of that friction due to the elementary arc $d\,\theta$; f the co-efficient of friction between the materials of the band and pulley.

Then according to a principle proved in Articles 179 and 271, it is known that the normal pressure at the elementary arc $d\,\theta$ is

$$T \, d\,\theta;$$

$2T = 2p$ [illegible] $T = p$ [illegible]

$\therefore p = \frac{T}{2}$ [illegible] P = total pressure on arc of contact.

T being the mean tension of the band at that elementary arc; consequently, the friction on that arc is

$$d\,\mathrm{R} = f\,\mathrm{T}\,d\,\theta.$$

Now that friction is also the difference between the tensions of the band at the two ends of the elementary arc; or

$$d\,\mathrm{T} = d\,\mathrm{R} = f\,\mathrm{T}\,d\,\theta;$$

which equation being integrated throughout the entire arc of contact, gives the following formulæ :—

Int. Calc. Art 166.

$$\left.\begin{array}{c} \text{hyp log}\,\dfrac{\mathrm{T}_1}{\mathrm{T}_2} = f\,\theta;\ \mathrm{T}_1 \div \mathrm{T}_2 = e^{f\,\theta}; \\ \mathrm{R} = \mathrm{T}_1 - \mathrm{T}_2 = \mathrm{T}_1\left(1 - e^{-f\,\theta}\right) = \mathrm{T}_2\left(e^{f\,\theta} - 1\right). \end{array}\right\} \quad \ldots\ldots(1.)$$

When a belt connecting a pair of pulleys has the tensions of its two sides originally equal, the pulleys being at rest; and when the pulleys are set in motion, so that one of them drives the other by means of the belt; it is found that the advancing side of the belt is exactly as much tightened as the returning side is slackened, so that the *mean* tension remains unchanged. Its value is given by this formula :—

Substitute value of T, from Eq 1.

Divide both members by R.

$$\frac{\mathrm{T}_1 + \mathrm{T}_2}{2\,\mathrm{R}} = \frac{e^{f\,\theta} + 1}{2\left(e^{f\,\theta} - 1\right)}, \ldots\ldots\ldots\ldots\ldots\ldots(2.)$$

which is useful in determining the original tension required to enable a belt to transmit a given force between two pulleys.

If the arc of contact between the band and pulley, expressed in turns and fractions of a turn, be denoted by n,

$$\theta = 2\,\pi\,n;\ e^{f\,\theta} = 10^{2\cdot7288\,f\,n} \ldots\ldots\ldots\ldots\ldots\ldots(3.)$$

When the band is used to resist the motion of the pulley, it constitutes a kind of brake called a *friction strap.* In this case the rubbing surfaces of the band and pulley may either be both of iron, or may be protected by a covering made of pieces of wood, which is renewed from time to time as it wears out.

679. **In Frictional Gearing,** described in Article 445, it appears that when the angle of the grooves is 40°, and when their surfaces are smooth, clean, and dry, the tangential force transmitted between the wheels is once and a-half the force with which their axes are pressed together. This proportion is much greater than that due to ordinary friction, and must arise partly from adhesion.

680. **Friction Couplings** are used to communicate rotation between pieces having the same axis, where sudden changes of force or of velocity take place; being so adjusted as to limit the force transmitted within the bounds of safety. Contrivances of this kind

are very numerous; one of the most common and most useful is that called a pair of friction cones. . The angle made by the sides of the cones with the axis should not be less than the angle of repose.

681. **Stiffness of Ropes.**—Ropes offer a resistance to being bent, and when bent to being straightened again, which arises from the mutual friction of their fibres. It increases with the sectional area of the rope, and is inversely proportional to the radius of the curve into which it is bent.

The *work lost* in pulling a given length of rope over a pulley, is found by multiplying the length of the rope in feet, by its stiffness in pounds; that stiffness being the excess of the tension at the leading side of the rope above that at the following side, which is necessary to bend it into a curve fitting the pulley, and then to straighten it again.

The following empirical formulæ for the stiffness of hempen ropes have been deduced by General Morin from the experiments of Coulomb :—

Let R be the stiffness in pounds avoirdupois;
d, the diameter of the rope, in inches;
$n = 48\,d^2$ for white ropes, $35\,d^2$ for tarred ropes;
r, the *effective* radius of the pulley, in inches;
T, the tension, in pounds; then,

$$\left.\begin{array}{l}\text{For white ropes, } \mathrm{R} = \dfrac{n}{r}(0{\cdot}0012 + 0{\cdot}001026\,n + 0{\cdot}0012\,\mathrm{T});\\ \text{For tarred ropes, } \mathrm{R} = \dfrac{n}{r}(0{\cdot}006 + 0{\cdot}001392\,n + 0{\cdot}00168\,\mathrm{T}).\end{array}\right\}(1.)$$

682. **Rolling Resistance of Smooth Surfaces.**—By the rolling of two surfaces over each other without sliding, a resistance is caused, which is called rolling friction. It is of the nature of a *couple* resisting rotation; its *moment* is found by multiplying the normal pressure between the rolling surfaces by an *arm* whose length depends on the nature of the rolling surfaces; and the work lost in an unit of time in overcoming it is the product of its moment by the *angular velocity* of the rolling surfaces relatively to each other. The following are approximate values of the arm in *decimals of a foot* :—

Oak upon oak,........................	0·006	(Coulomb).
Lignum-vitæ on oak,..................	0·004	,,
Cast iron on cast iron,................	0·002	(Tredgold).

683. The **Resistance of Carriages on Roads** consists of a constant part, and a part increasing with the velocity. According to General Morin, it is given approximately by the following formula :—

$$R = \frac{Q}{r}\{a + b(v - 3{\cdot}28)\};\ldots\ldots\ldots\ldots(1.)$$

where Q is the gross load, r the radius of the wheels *in inches*, v the velocity in feet per second, and a and b two constants, whose values are

	a	b
For good broken stone roads, ...	·4 to ·55	·024 to ·026
For paved roads, ...	·27	·0684
For the pavement of Paris,................	·39	·03

On gravel roads the resistance is about double, and on sandy and gravelly soft ground, five times the resistance on good broken stone roads.

684. **Resistance of Railway Trains.**—In the following formulæ, which are all empirical—

E denotes the weight of the engine;
T „ the gross load drawn by it;
V „ the velocity, in miles an hour;
r „ the radius of curvature of the line, in miles;
R „ the resistance in pounds;
f „ a co-efficient of friction;
c „ a co-efficient for resistance due to curvature.

Then for single carriages with cylindrical wheels, at velocities up to 12 miles an hour, according to the experiments of Lieutenant David Rankine and the Author,

$$R = f\left(1 + \frac{c}{r}\right)T;\ldots\ldots\ldots\ldots(1.)$$

where $f = 0{\cdot}002$; and $c = 0{\cdot}3$. (*See Experimental Inquiry on the Use of Cylindrical Wheels on Railways*, 1842.)

For an engine and train, the following is an empirical formula deduced from the experiments of various authors:—

$$R = f(T + E)\left(1 + \frac{V^2}{1440}\right)\left(1 + \frac{c}{r}\right);\ldots\ldots\ldots\ldots(2.)$$

where f ranges from ·0027 to ·004, according to the state of the line and carriages, and c from 0·3 to 0·1. (See Rankine's *Manual of Civil Engineering.*)

685. **Heat of Friction.**—The work lost in friction produces heat in the proportion of one British thermal unit, being so much heat as raises the temperature of a pound of water one degree of Fahrenheit, for every 772 foot pounds of lost work.

Excessive heating is prevented by a constant and copious supply of a good unguent.

CHAPTER II.

VARIED MOTIONS OF MACHINES.

686. **The Centrifugal Forces and Couples** exerted by the various rotating pieces of a machine against the bearings of their axles are to be determined by the principles of Articles 540, 592, and 603, and taken into account in determining the lateral pressures which cause friction, and the strength of the axles and framework. As those centrifugal forces and couples cause increased friction and stress, and sometimes also, by reason of their continual change of direction, produce detrimental or dangerous vibration, it is desirable to reduce them to the smallest possible amount; and for that purpose, unless there is some special reason to the contrary, the axis of rotation of every piece which rotates rapidly ought to traverse its centre of gravity, that the resultant centrifugal force may be nothing, and ought to be an axis of inertia, that the centrifugal couple may be nothing. As to axes of inertia, see Article 584.

687. **Actual Energy of a Machine.**—To determine the entire actual energy of a machine at a given instant, it is necessary to know—

(1.) The weight of each of its sliding pieces: let any one of those weights be denoted by W;

(2.) The velocity of translation of each of those pieces at the given instant: let v denote any one of these velocities;

(3.) The moment of inertia of each of its rotating pieces: let any one of these moments be denoted by I;

(4.) The angular velocity of each of those pieces at the given instant; let a be any one of these angular velocities.

These quantities being given, the actual energy of the machine is

[illegible] 2, 533

$$E = \frac{1}{2g}(\Sigma \cdot W v^2 + \Sigma \cdot I a^2); \qquad \text{(1.)}$$

and if the moment of inertia of each rotating piece be expressed in the form $I = W' \rho^2$, W′ being its weight and ρ its radius of gyration, the above expression may be put in the form,

$$E = \frac{1}{2g}(\Sigma \cdot W v^2 + \Sigma \cdot W' \rho^2 a^2) \qquad \text{(2.)}$$

688. **Reduced Inertia.**—The figures, sizes, and connection of the

pieces of a machine being known, the principles of the Theory of Mechanism (Part IV.), enable the comparative motions of all its points to be determined, and in particular, the several ratios of their velocities to that of the driving point at any instant. Let V be the velocity of the driving point, and for any given piece of the machine whose weight is W, let n denote the ratio $v : \mathrm{V}$ if it is a sliding piece, and the ratio $\rho\, a : \mathrm{V}$ if it is a turning piece. Then the sum

$$\Sigma \cdot \mathrm{W}\, n^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

expresses *the weight which, if concentrated at the driving point, would have the same actual energy with the entire machine.* This quantity may be called *the inertia reduced to the driving point.* By Mr. Moseley, who first introduced its consideration into mechanics, it is called the "co-efficient of steadiness."

The actual energy of the machine at any instant may now be expressed by

$$\mathrm{E} = \frac{\mathrm{V}^2\, \Sigma \cdot \mathrm{W}\, n^2}{2\, g} \ldots\ldots\ldots\ldots\ldots\ldots (2.)$$

Another mode of expressing the reduced inertia is with reference to the *driving axis.* Let A represent the angular velocity, at any instant, of the axis of the piece which first receives the motive power; for any shifting piece let $v : \mathrm{A} = l$; and for any rotating piece let $a : \mathrm{A} = n$. Then the *reduced moment of inertia* is

$$\Sigma \cdot \mathrm{W}\, l^2 + \Sigma \cdot \mathrm{I}\, n^2 ; \ldots\ldots\ldots\ldots\ldots\ldots (3.)$$

and the actual energy at any instant,

$$\mathrm{E} = \frac{\mathrm{A}^2}{2\, g} \left\{ \Sigma \cdot \mathrm{W}\, l^2 + \Sigma \cdot \mathrm{I}\, n^2 \right\} \ldots\ldots\ldots\ldots (4.)$$

689. **Fluctuations of Speed** in a machine are caused by the alternate excess of the energy received above the work performed, and of the work performed above the energy received, which produce an alternate increase and diminution of actual energy, according to the law of the conservation of energy explained in Article 552.

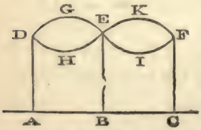

Fig. 265.

To determine the greatest fluctuations of speed in a machine moving periodically, take A B C, in fig. 265, to represent the motion of the driving point during one period; let the effort P of the prime mover at each instant be represented by the ordinate of the curve D G E I F; and let the sum of the resistances, reduced to the driving point, as in Article 668, at each instant, be denoted by R, and represented by the ordinate of the curve D H E K F, which cuts the former curve at the ordinates A D, B E, C F. Then the integral

$$\int (\mathrm{P} - \mathrm{R})\, d\, s,$$

being taken for any part of the motion, gives, as in Article 549, the excess or deficiency of energy, according as it is positive or negative. For the entire period A B C this integral is nothing. For A B, it denotes an *excess of energy received*, represented by the area D G E H; and for B C, an equal *excess of work performed*, represented by the equal area E K F I. Let those equal quantities be each represented by Δ E. Then the actual energy of the machine attains a maximum value at B, and a minimum value at A and C, and Δ E is the difference of these values.

Now let V_0 be the mean velocity, V_1 the greatest velocity, and V_2 the least velocity of the driving point; then

$$\frac{\mathrm{V}_1^2 - \mathrm{V}_2^2}{2\,g} \cdot \Sigma \cdot \mathrm{W}\, n^2 = \Delta\, \mathrm{E}; \ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

which, being divided by twice the *mean actual energy*

$V_0 = \frac{V_1 + V_2}{2}$

$$\frac{\mathrm{V}_0^2}{2\,g} \cdot \Sigma \cdot \mathrm{W}\, n^2 = \mathrm{E}_0,$$

gives

$$\frac{\mathrm{V}_1 - \mathrm{V}_2}{\mathrm{V}_0} = \frac{\Delta\, \mathrm{E}}{2\, \mathrm{E}_0} = \frac{g\, \Delta\, \mathrm{E}}{\mathrm{V}_0^2\, \Sigma \cdot \mathrm{W}\, n^2}; \ldots\ldots\ldots\ldots\ldots (2.)$$

a ratio which may be called the *co-efficient of fluctuation of speed.*

The ratio of the periodical excess and deficiency of energy Δ E to the whole energy exerted in one period or revolution, $\int \mathrm{P}\, d\, s$, has been determined by General Morin for steam engines under various circumstances, and found to be from $\frac{1}{10}$ to $\frac{1}{4}$, for single cylinder engines. For a pair of engines driving the same shaft, with cranks at right angles to each other, the value of this ratio is about one-fourth of its value for single cylinder engines.

690. A **Fly-Wheel** is a wheel with a heavy rim, whose great moment of inertia reduces the co-efficient of fluctuation of speed to a certain fixed amount, being about $\frac{1}{32}$ in ordinary machinery, and $\frac{1}{50}$ or $\frac{1}{60}$ in machinery for fine purposes.

Let $\frac{1}{m}$ be the intended value of the co-efficient of fluctuation of speed, and Δ E, as before, the fluctuation of energy; then if this is to be provided for by the moment of inertia I of the fly-wheel alone, let a_0 be its mean angular velocity; then equation 2 of Article 689 is equivalent to the following:—

$$\frac{1}{m} = \frac{g \Delta \mathrm{E}}{a_0^2 \mathrm{I}}; \ \mathrm{I} = \frac{m g \Delta \mathrm{E}}{a_0^2}; \ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

the second of which equations gives the requisite moment of inertia of the fly-wheel.

691. **Starting and Stopping—Brakes.**—The *starting* of a machine consists in setting it in motion from a state of rest, and bringing it up to its proper mean velocity. This operation requires the expenditure, besides the energy required to overcome the resistance of the machine, of an additional quantity of energy equal to the actual energy of the machine when moving with its mean velocity, as found according to the principles of Article 687.

If, in order to *stop* a machine, the effort of the prime mover is simply suspended, the machine will continue to go until work has been performed in overcoming its resistances equal to the actual energy due to its speed at the time of suspending the effort of the prime mover.

In order to stop the machine in less time than this operation would require, the resistance may be artificially increased by means of a *brake*, which may be a friction-strap, as described in Article 678, or a block pressed against the rim of a wheel, or a grooved sector pressed against a wheel grooved as for frictional gearing (Articles 445, 679).

Let R_1 be the ordinary resistance of the machine, *reduced to the rubbing surface* (Article 668), R_2 the friction produced by the brake, v the velocity of the surface on which it acts at the time when it is first applied, s the distance through which rubbing must take place in order to stop the machine, t the time required for the same effect, E the actual energy of the machine when the brake begins to act. Then

$$s = \mathrm{E} \div (\mathrm{R}_1 + \mathrm{R}_2); \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

and because the mean velocity of rubbing during the operation of stopping is $v \div 2$,

$$t = \frac{2s}{v} = 2\,\mathrm{E} \div v\,(\mathrm{R}_1 + \mathrm{R}_2) \ldots\ldots\ldots\ldots\ldots (2.)$$

CHAPTER III.

ON PRIME MOVERS.

692. A **Prime Mover** is an engine, or combination of moving pieces, which serves to transfer energy from those bodies which naturally develop it, to those by means of which it is to be employed, and to transform energy from the various forms in which it may occur, such as chemical affinity, heat, or electricity, into the form of mechanical energy, or energy of force and motion. The mechanism of a prime mover comprehends all those parts by means of which it regulates its own operations.

The *useful work* of a prime mover is the energy which it transmits to any machine driven by it; and its *efficiency* is the ratio of that useful work to the whole energy received by it from a natural source of energy.

The *effect* or *available power* of a prime mover is its useful work in some given unit of time, such as a second, a minute, an hour, a day.

693. The **Regulator** of a prime mover is some piece of apparatus by which the rate at which it receives energy from the source of energy can be varied; such as the sluice or valve which adjusts the size of the orifice for supplying water to a water-wheel, the apparatus for varying the surface exposed to the wind by windmill-sails, the throttle-valve of a steam engine. In prime movers, whose speed and power have to be varied at will, such as locomotive engines, and winding engines for mines, the regulator is adjusted by hand. In other cases it is adjusted by a self-acting apparatus called a **Governor**—usually consisting of a pair of rotating pendulums, whose angle of deviation from their axis depends upon the speed. (Article 606).

694. **Prime Movers** may be **Classed** according to the forms in which the energy is first obtained. These are—

I. Muscular Strength.
II. The Motion of Fluids.
III. Heat.
IV. Electricity and Magnetism.

695. **Muscular Strength.**—The *daily effect* exerted by the muscular strength of a man or of a beast is the product of three quantities; the useful resistance, the velocity with which that resistance

is overcome, and the number of units of time per day during which work is continued. It is known that for each individual man or animal there is a certain set of values of those three quantities which makes their product a maximum, and is therefore the best for economy of power; and that any departure from that set of values diminishes the daily effect.

The following table of the effects of the strength of men and horses employed in various ways, is compiled from the works of Poncelet and General Morin, and some other sources:—

	MAN.	R lb.	V ft. p. sec.	T″ ÷ 3,600 = hrs. p. day.	R V ft. lb. p. sec.	R V T ft. lb. p. day.
1.	Raising his own weight up stair or ladder, ...	143	0·5	8	72·5	2,088,000
2.	Do. do. do., ...	...	...	10	...	2,616,000
3.	(Tread-wheel, see 1.)					
4.	Hauling up weight with rope,	40	0·75	6	30	648,000
5.	Lifting weights by hand, ...	44	0·55	6	24·2	522,720
6.	Carrying weights up stairs, ...	143	0·13	6	18·5	399,600
7.	Shovelling up earth to a height of 5 feet 3 inches, ...	6	1·3	10	7·8	280,800
8.	Wheeling earth in barrow up slope of 1 in 12, ½ horiz. veloc. 0·9 ft. per sec. (return. empty),	132	0·075	10	9·9	356,400
9.	Pushing or pulling horizontally (capstan or oar), ...	26·5	2·0	8	53	1,526,400
		12·5	5·0	?	62·5	...
10.	Turning a crank or winch, ...	18·0	2·5	8	45	1,296,000
		20·0	14·4	(2 mins.)	288	
11.	Working pump, ...	13·2	2 5	10	33	1,188,000
12.	Hammering, ...	15	?	8 ?	?	480,000
	HORSE.					
13.	Cantering and trotting, drawing a light railway carriage (thoroughbred), ...	min. 22½ mean 30½ max. 50	14⅔	4	447½	6,444,000
14.	Horse drawing cart or boat, walking (draught horse), ...	120	3·6	8	432	12,441,600

696. A **Water Pressure Engine** consists essentially of a working cylinder, in which water moves a piston in the manner stated in Article 499, case 2. Let h be the *virtual fall*, that is, the excess of the dynamic head of the water entering the cylinder above that of the water leaving the cylinder; Q the volume of water supplied per second; ϱ its weight per unit of volume; $1-k$ the efficiency of the engine; then

$$(1-k)\,\varrho\,Q\,h,$$

is its effect per second. In well constructed water pressure engines, $1-k$ varies from ·66 to ·8.

697. **Water-Wheels in General.** — Water may act on a wheel either by its *weight* and pressure, or by its *velocity;* that is, either by its *potential,* or by its *actual energy.* See Article 622.

Let ϱQ denote the weight of water, in *pounds,* supplied to the wheel in a second; h the difference of dynamic head, in *feet,* of the water before and after its action on the wheel; v_1 the velocity of the water, in *feet per second,* just before it begins to press on the wheel, or *supply-velocity;* v_2 the velocity of the water just after it has ceased to act on the wheel, or *discharge-velocity.* Then the total energy of the water, as in Article 622, is

$$\varrho Q \left(h + \frac{v_1^2}{2g}\right) \text{ foot pounds per second;}$$

the energy of the water when discharged,

$$\varrho Q \frac{v_2^2}{2g}, \text{ foot pounds per second;}$$

the total power of the wheel,

$$\varrho Q \left(h + \frac{v_1^2 - v_2^2}{2g}\right) \text{ foot pounds per second;} \quad \text{(1.)}$$

the maximum theoretical efficiency,

$$\left(h + \frac{v_1^2 - v_2^2}{2g}\right) \div \left(h + \frac{v_1^2}{2g}\right); \quad \text{(2.)}$$

the quantity

$$h_1 = h + \frac{v_1^2}{2g} \quad \text{(3.)}$$

may be called the *theoretical fall* or *head.* The available efficiency of a water-wheel falls short of the maximum theoretical efficiency principally from the following causes:—1. The resistance of the channel and orifices by which the water is supplied, which causes the actual height from which the water must descend in order to acquire the supply-velocity v to be greater than $v_1^2 : 2g$. The effect of such resistance is expressed by putting for the *actual fall,*

$$H = h + (1 + \Sigma \cdot f) \frac{v_1^2}{2g}; \quad \text{(4.)}$$

$\Sigma \cdot f$ being the co-efficient of resistance of the channel and orifices of supply, determined according to the principles of Articles 638 to 646. 2. The escape of part of the water before it has completed its action on the wheel. 3. The agitation and mutual friction of the particles of water acting on the wheel; and, 4. The friction of the wheel. The effects of the last three causes are expressed by multiplying the total power and the theoretical efficiency of the wheel by an empirically determined fractional co-efficient k; so that the effect or available power is denoted by

$$(1-k)\,\varrho\,\mathrm{Q}\,h_1;$$

and the *available efficiency* by

$$\left.\begin{matrix} \\ \dfrac{(1-k)\,h_1}{\mathrm{H}}. \end{matrix}\right\}\ \cdots\cdots\cdots\cdots\cdots\cdots(5.)$$

698. **Classes of Water-Wheels.**—Water-wheels may be classed as follows:—*Overshot-wheels* and *breast-wheels, undershot-wheels* and *turbines.*

699. **Overshot and Breast-Wheels.**—The water is supplied to this class of wheels at or below the summit, and acts wholly, or partly by its weight, as it descends in the buckets. (See Article 634). Formerly the buckets used to be closed at their inner sides, but now they are made with openings for the escape and re-entrance of air: an invention of Mr. Fairbairn. A breast-wheel differs from an overshot-wheel chiefly in having the water poured into the buckets at a somewhat lower elevation as compared with the summit of the wheel, and in being provided with a casing or trough, called a *breast,* of the form of an arc of a circle, extending from the regulating sluice to the commencement of the tail-race, and nearly fitting the periphery of the wheel, which revolves within it. The effect of the breast is to prevent the overflow of water from the lips of the buckets until they are over the tail-race. The usual velocity of the periphery of overshot and high breast-wheels is from three to six feet per second; and their available efficiency, when well designed and constructed, is from 0·7 to 0·8.

700. **Undershot-Wheels** are driven by the impulse of water, discharged from an opening at the bottom of the reservoir with the velocity produced by the fall, against *floats* or boards, as to which see Article 649. Every such wheel has a certain *velocity of maximum efficiency,* which does not in any case differ much from half the velocity of the water striking it. In undershot-wheels of the old construction, the floats are flat boards in the direction of radii of the wheel; and the maximum theoretical efficiency is $\frac{1}{2}$. The available efficiency is about 0·3. This class of wheels was much improved by Poncelet, who curved the floats with a concavity backwards, adjusting their position and figure so that the water should be supplied to them without shock, and should drop from them into the tail-race without any horizontal velocity. The available efficiency of such wheels is about 0·6.

701. A **Turbine** is a horizontal water-wheel with a vertical axis, receiving and discharging water in all directions round that axis: that is, driven by a vortex; its efficiency ranges from ·6 to ·8 (see Article 650).

702. **Windmills** are driven by the impulse of the air against oblique surfaces called *sails*, rotating in a plane perpendicular to the direction of the wind.

The best figure and proportions for windmill sails, as determined experimentally by Smeaton, are given by the following formulæ, in which the *whip* means, the length of an arm, or the distance of the tip of a sail from the axis:—length of sail, $\frac{5}{6}$ whip:—breadth at end nearest axis, $\frac{1}{5}$ whip:—at tip, $\frac{1}{3}$ whip:—angles made by the surface of the sail with the plane of rotation—at the end nearest the axis, 18°:—at the tip, 7°. The efficiency of a good windmill is about 0·29. (See Smeaton on Windmills, in Tredgold's *Hydraulic Tracts.*)

703. The **Efficiency of Heat Engines** is the subject of a peculiar branch of science, *Thermodynamics;* and an outline only of the principles on which it depends can here be given.

If the number of British Fahrenheit units of heat produced by the combustion of one pound of a given kind of fuel, be multiplied by Joule's equivalent, 772 foot pounds, the result is the *total heat of combustion* of the fuel in question, expressed in foot pounds. For different kinds of coal, it varies from 6,000,000 to 12,000,000 foot pounds. This total heat is expended, in any given engine, in producing the following effects, whose sum is equal to the heat so expended:—

1. The *waste heat of the furnace,* being from 0·15 to 0·6 of the total heat, according to the construction of the furnace, and the skill with which the combustion is regulated.

2. The *necessarily rejected heat of the engine,* being $= \frac{t_2}{t_1} \times$ the heat received by the elastic fluid: t_1 being the upper, and t_2 the lower limits of *absolute* temperature, which is measured from the absolute zero, 493°·2 Fahrenheit below the melting point of ice.

3. The *heat wasted by the engine,* whether by conduction, or by non-fulfilment of the conditions of maximum efficiency.

4. The *useless work of the engine,* employed in overcoming friction and other prejudicial resistances.

5 The *useful work.* The efficiency of a thermodynamic engine is improved by diminishing as far as possible the first four of these effects, so as to increase the fifth.

The efficiency of a heat engine is the product of three factors; viz.:—the efficiency of the furnace, being the ratio of the heat transferred to the elastic fluid to the total heat of combustion;—the efficiency of the fluid, being the fraction of the heat received by it which is transformed into mechanical energy;—and the efficiency of the mechanism, being the fraction of that energy which is available for driving machines. The maximum efficiency of the fluid between given limits of absolute temperature is expressed by

$$\frac{t_1 - t_2}{t_1} \dotfill (1.)$$

As to the mechanical action of an elastic fluid on a piston, see Article 656.

704. **Steam Engines.**—Formulæ for the mechanical action of steam on a piston, both exact and approximate, have been given in Article 656, equations 6 to 13.

The efficiency of the steam lies between the limits ·02 and ·2 in extreme cases, and ·04 and ·1 in ordinary cases.

The details of the construction and working of steam engines can be explained in a special treatise only.

The *duty* of an engine is the work performed by a given quantity of fuel, such as one pound. The duty of a pound of coal varies in different classes of engines from about 100,000 to 1,900,000 foot pounds. These are extreme results, as respects wastefulness on the one hand, and economy on the other. In good ordinary engines, the duty varies from 200,000 to 700,000.

705. **Electrodynamic Engines,** though capable of higher efficiency than heat engines, are not so economical commercially, on account of the greater cost of the materials consumed in them. Their theoretical efficiency, according to a law demonstrated by Mr. Joule, is given by the formula

$$\frac{\gamma_1 - \gamma_2}{\gamma_1}; \dotfill (1.)$$

where γ_1 is the strength which the electric current would have if the machine performed no mechanical work, and γ_2 is the actual strength of the current.

This law, and the law of the maximum efficiency of heat engines, are particular cases of a general law which regulates all transformations of energy, and is the basis of the Science of Energetics.*

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* *Edinburgh Philosophical Journal*, July, 1855; *Proceedings of the Philosophical Society of Glasgow*, 1853-5.

APPENDIX.

ARTICLE 634, p. 579.

Motion of Water in Waves.—I. *Rolling Waves.*—In waves which are not accompanied by permanent translation of the particles of water, it is known by observation that those particles revolve in orbits situated in vertical planes which are perpendicular to the ridges and furrows of the waves, and parallel to their direction of advance; also, that each revolving particle moves forward while on the crest of a wave, downward when on the back slope, backward when in the trough, and upward when on the front slope. The *length* of a wave is the distance, in the direction of advance, from crest to crest; the *height* is equal to the vertical diameter of the orbit of a surface particle. Each particle makes one revolution while the wave advances through a wave-length; the interval of time thus occupied is called the *period.* Let L denote the wave-length, T the period, a the velocity of advance; then $a = \frac{L}{T}$; and also, mean velocity of revolution of a particle = circumference of orbit ÷ T.

The orbits of the particles are approximately elliptic, with the longer axis horizontal. In going from the surface towards the bottom, the dimensions of the orbits are found to diminish, the vertical axis diminishing faster than the horizontal axis, as shown at A, B, C, in fig. A. At the bottom the particles move back and forward in a straight line, as at D.

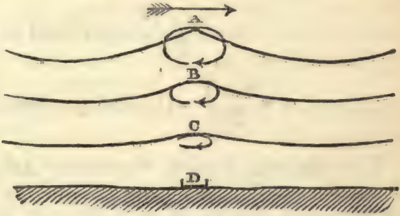


Fig. A.

The deeper the water is, as compared with the length of a wave, the more nearly equal are the two axes of the orbit of a surface particle; and in water whose depth is half a wave-length and upwards, those axes are sensibly equal, and the orbit of a surface particle sensibly circular.

II. *Relation between Figure of Surface and Velocity of Advance.*—In fig. 252, page 578, let C be the centre, and C B the radius of the circular orbit of a particle. Lay off C A vertically upwards, of a length equal to that of the *equivalent pendulum* (that is, the pendulum whose period is T)—viz.,

$$\mathrm{C\,A} = \frac{g\mathrm{T}^2}{4\,\pi^2} = \frac{\mathrm{T}^2\ \text{(seconds)}}{0{\cdot}815\ \text{foot nearly}} \quad \ldots\ldots\ldots\ldots\ldots (1.)$$

Then we have gravity : centrifugal force : : A C : C B ; and A B represents (as in Article 634, page 578) the resultant of gravity and centrifugal force ; so that *a surface of uniform pressure* traversing B is normal to A B. The upper surface of the wave is such a surface ; and in order to fulfil that condition its profile must be *a trochoid traced by the point* B *while a circle of the radius* C A *rolls on the under side of a horizontal straight line traversing* A. The *length* of such a wave, and its *velocity of advance*, are given by the following equations :—

$$L = 2\,\pi\,C\,A = \frac{gT^2}{2\pi} = \text{(in feet) } 5{\cdot}12\ T^2\,; \ldots\ldots\ldots\ldots(2.)$$

$$a = \frac{L}{T} = \frac{gT}{2\,\pi} = \text{(in feet per second) } 5{\cdot}12\ T \ldots\ldots\ldots\ldots(3.)$$

When the orbits of the surface particles are elliptic, let m be the ratio in which the vertical axis is less than the horizontal axis. Then it is evident that in order that the surface of the wave may still be everywhere normal to the resultant of gravity and re-action, we must have

$$L = \frac{m\,g\,T^2}{2\,\pi} = \text{(in feet) } 5{\cdot}12\ m\ T^2\,; \ldots\ldots\ldots\ldots(4.)$$

$$a = \frac{m\,g\,T}{2\,\pi} = \text{(in feet per second) } 5{\cdot}12\ m\ T \ldots\ldots\ldots\ldots(5.)$$

III. *Relation between Velocity of Advance and Depth of Uniform Disturbance.*—Let h be the height of a wave ; that is, the vertical diameter of the orbit of a surface particle. Then, in an indefinitely short interval of time, the front slope of the wave advances through the distance $a\ d\ t$, and the volume of water contained between the original and new positions of the front slope, per unit of breadth, is $h\ a\ d\ t$. In the same interval of time there passes into the space vertically below the front slope, per unit of breadth, the volume of water $2\ u\ c\ d\ t$, where u is the forward velocity of a surface particle at the crest, $-u$ the equal backward velocity of a surface particle in the trough, and c a depth which may be called the *depth of uniform disturbance*, because it is equal to the mean depth of a canal in which the volume of water displaced per second would be equal to that displaced per second in the actual wave, if the horizontal velocity of disturbance were the same from surface to bottom. Equating the two volumes just given, we have $h\ a = 2\ u\ c$; but u can b shown to be $= g\,h \div 2\,a$; therefore $c = a^2 \div g$. Hence the velocity of advance of a wave of any figure in which the volume displaced horizontally per second is equivalent to that due to a horizontal velocity of disturbance equal to the surface velocity down to the depth c, is given by the equation

$$a = \sqrt{g\,c} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.)$$

For waves rolling in deep water, without interference by external forces, it can be shown that the diameters of the orbits of particles at different depths vary proportionally to $e^{-\frac{z}{c}}$; where z is the depth of the centre of the orbit of the particle in question below the centre of the orbit of a surface particle.

In water of the depth k, let $\mathrm{L} \div 2\,\pi = b$; then it can be shown that at the surface, $m = \left(e^{\frac{k}{b}} - e^{-\frac{k}{b}}\right) \div \left(e^{\frac{k}{b}} + e^{-\frac{k}{b}}\right)$; that $c = m\,b$; and that the horizontal and vertical diameters of an orbit vary respectively as $e^{\frac{k-z}{b}} + e^{\frac{z-k}{b}}$, and as $e^{\frac{k-z}{b}} - e^{\frac{z-k}{b}}$. In very deep water, m sensibly $= 1$, and $c = b$.

In very shallow water the horizontal disturbance is sensibly uniform from the surface to the bottom, so that c represents the actual depth; and the vertical disturbance is sensibly proportional to the height above the bottom.

IV. *Waves of Translation* are those which are accompanied by a permanent travelling of the particles of water, and are said to be positive or negative according as that travelling is forward or backward. Their motions may be expressed by taking two different quantities, u' and $-u''$, to denote respectively the forward velocity of a particle at the crest of a wave, and the backward velocity of a particle in the trough; when the velocity of advance will be given by the formula

$$a + \tfrac{1}{2}\,(u' - u'') \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7.)$$

V. *Authorities on Waves.*—Weber's *Wellenlehre;* Scott Russell, in *Reports of the British Association*, 1844; Airy, *On Tides and Waves;* Stokes, *Cambridge Transactions*, 1842, 1850; Earnshaw, *Ib.*, 1845; Froude, *Trans. of the Institution of Naval Architects*, 1862; Rankine, *Philos. Trans.*, 1863; Do., *Philos. Mag.*, November, 1864; Do., *Proceedings of the Royal Society*, 1868; Watts, Rankine, Napier, and Barnes, *On Shipbuilding;* Thomas Stevenson, *On Harbours;* Caligny, *Liouville's Journal*, June and July, 1866; Cialdi, *Sul Moto Ondoso del Mare.*

308 A.* **Continuous Girders.**—The fundamental principle of the theory of continuous girders, with the load distributed in any manner, is the "Theorem of Three Moments," due originally to Clapeyron and Bresse and improved by Heppel (*See* Bresse, *Mécanique Appliquée*, Part III., and the *Proceedings of the Royal Society* for 1869).

The subject is treated of in Art. 178 of the tenth edition of *Civil Engineering*. The following demonstration, with deduced formulæ, is abstracted from a paper communicated by Mr. Mansfield Merriman, C.E., to the *Philosophical Magazine*, September, 1875.

The elastic curve (Art. 319) has the following equation:—

$$\frac{d^2 y}{d x^2} = \frac{\mathrm{M}}{\mathrm{EI}}.$$

The equation for any particular case is obtained by substituting the values of M and I (constant) in terms of x, and integrating twice.

Let l be length of first span, l' of second. Let M_0, M_1, M_2, be the moments at three points of support. Let W be a single concentrated load at a distance a from support 0, and W′ at a distance a' from 1.

Since equilibrium prevails, we have for a section between W and 1, the equation

$$\mathrm{M}_0 - \mathrm{F}\,x + \mathrm{W}\,(x - a) - \mathrm{M} = 0, \ldots\ldots\ldots\ldots\ldots \text{(I.)}$$

making $x = l$, M becomes M_1, and

$$\mathrm{F} = \frac{\mathrm{M}_0 - \mathrm{M}_1}{l} + \frac{\mathrm{W}(l-a)}{l} = \frac{\mathrm{M}_0 - \mathrm{M}_1}{l} + \mathrm{W}\,(1 - k), \ldots\ldots \text{(I′.)}$$

k being any fraction. Insert now the value of M in the differential equation of the curve, and integrate it twice: t_0 the tangent of the angle which the curve at the origin makes with the axis of abscissa is the constant in the first integration, and zero in the second.

The required equation is—

$$y = t_0\,x + \frac{1}{6\,\mathrm{EI}}\left\{3\,\mathrm{M}_0 x^2 - \mathrm{F} x^3 + \mathrm{W}\,(x-a)^3\right\} \ldots\ldots\ldots \text{(II.)}$$

Substituting the value of F in terms of M_0, M_1, and W, making $x = l$, and $a = k\,l$; $y = 0$, and we obtain

$$6\,\mathrm{EI}\,t = -2\,\mathrm{M}_0\,l - \mathrm{M}_1\,l + \mathrm{W}\,l^2\,(2\,k - 3\,k^2 + k^3) \ldots\ldots (\alpha.)$$

---

* See Article 308, p. 338.

Now in the value of $\frac{d\,y}{d\,x}$ make $x = l$; $\frac{d\,y}{d\,x}$ becomes t_1, the tangent at 1, and we obtain

$$6\ \mathrm{EI}\ t_1 = \mathrm{M}_0\, l + 2\ \mathrm{M}_1\, l - \mathrm{W}\ l^2\,(k - k^3).\ \ldots\ldots\ldots(\mathrm{III.})$$

If the origin be taken at 1, we obtain an equation analagous to (α).

$$6\ \mathrm{EI}\ t_1 = \ -2\ \mathrm{M}_1\, l' - \mathrm{M}_2\, l' + \mathrm{W}'\ l'^2\ (2\ k - 3\ k^2 + k^3),\ (\mathrm{III}'.)$$

where k denotes $\frac{a'}{l'}$, and is not necessarily the same in the two expressions.

We thus obtain the *Theorem of Three Moments* for concentrated loads

$$\mathrm{M}_0 l + 2\ \mathrm{M}_1(l + l') + \mathrm{M}_2\ l' = \mathrm{W}\ l^2\,(k - k^3) + \mathrm{W}'\,l'^2(2k - 3k^2 + k^3), (\mathrm{IV.})$$

For many loads Σ has to be prefixed to the terms involving W and W′. For uniformly distributed loads w and w' per unit of length, we place $\Sigma\,\mathrm{W} = \int w\,d\,(kl)$ and $\Sigma\ \mathrm{W}' = \int w'\,d(k\,l')$, integrating between the required limits. If the loads extend over the whole span, the first integral is taken between $k\,l = 0$ and $k\,l = l$, the second between $k\,l' = 0$, and $k\,l' = l'$. Then

$$\mathrm{M}_0\ l + 2\ \mathrm{M}_1\ (l + l') + \mathrm{M}_2\ l' = \tfrac{1}{4}\ w\ l^3 + \tfrac{1}{4}\ w'\ l'^3,\ \ldots\ldots(\mathrm{V.})$$

which is the theorem as first deduced by Clapeyron.

The following are the formulæ deduced by Mr. Merriman—

CASE 1.—*Ends resting freely upon abutments.*

Let the girder consist of any number of unequal spans, the rth only being loaded. Let s = the number of spans, and l_1, l_2, &c., their lengths; 1 being the first and $s + 1$ the last support, the index n will refer to any support. A single load in the rth span is called W, and its distance from the rth support $k\ l_r$ or a.

Referring to (IV.) it is seen there are two functions of W and $k\ l_r$ of frequent occurrence. Denoting these by A and B for the supports r and $r + 1$.

$$\left.\begin{array}{r} \mathrm{A} = \mathrm{W}\ l_r^2\ (2\ k - 3\ k^2 + k^3)\ d\ k \\ \mathrm{B} = \mathrm{W}\ \ l_r^2\ (k - k^3),\ldots\ldots\ldots \end{array}\right\}\ \text{for a single load in the } r\text{th span.}$$

$$\left.\begin{array}{l} \mathrm{A} = \int_{k_1}^{k_2} w\ l_r^3\,(2\ k - 3\ k^2 + k^3)\ d\ k \\ \mathrm{B} = \int_{k_1}^{k_2} w\ l_r^3\ (k - k^3)\ d\ k, \end{array}\right\}\ \begin{array}{l}\text{for an uniform load whose ends} \\ \text{are distant } k_1\,l_r \text{ and } k_2\,l_r \text{ from} \\ \text{the support } r. \end{array}$$

From the equations of moments, and the solution of these equations by the method of indeterminate co-efficients, the two following equations are derived. When $n < r + 1$:—

$$\mathrm{M}_n = c_n \mathrm{M}_2 = c_n \frac{\mathrm{A} d_{s-r+2} + \mathrm{B} d_{s-r+1}}{d_{s-1}\, l_2 + 2\, d_s\, (l_1 + l_2)}, \;\ldots\ldots\ldots\ldots(1.)$$

also

$$\mathrm{M}_{s-1} = d_3\, \mathrm{M}_s, \quad \mathrm{M}_{s-2} = d_4\, \mathrm{M}_s.$$

When $n > r$

$$\mathrm{M}_n = d_{s-n+2}\, \mathrm{M}_s = d_{s-n+2} \frac{\mathrm{A}\, c_r + \mathrm{B} c_{r+1}}{c_{s-1}\, l_{s-1} + 2\, c_s (l_{s-1} + l_s)} \;\ldots\ldots(2.)$$

The values of the quantities c, d are

$$\left.\begin{array}{ll}
c_1 = 0 & d_1 = 0, \\
c_2 = 1 & d_2 = 1, \\
c_3 = -2\,\dfrac{l_1 + l_2}{l_2} & d_3 = -2\,\dfrac{l_s + l_{s-1}}{l_{s-1}} \\
c_4 = -2\,c_3\,\dfrac{l_2 + l_3}{l_3} - c_2\,\dfrac{l_2}{l_3} & d_4 = -2\,d_3\,\dfrac{l_{s-1} + l_{s-2}}{l_{s-2}} - d_2\,\dfrac{l_{s-1}}{l_{s-2}} \\
c_5 = -2\,c_4\,\dfrac{l_3 + l_4}{l_4} - c_3\,\dfrac{l_3}{l_4} & d_5 = -2\,d_4\,\dfrac{l_{s-2} + l_{s-3}}{l_{s-3}} - d_3\,\dfrac{l_{s-2}}{l_{s-3}}
\end{array}\right\} (3.)$$

Let the shearing stress in the span l_r, at a point infinitely near to the rth support, be denoted by F_r, and to the $r + 1$th support by F'_r; then

$$\left.\begin{array}{ll}
\mathrm{F}_r = \dfrac{\mathrm{M}_r - \mathrm{M}_{r+1}}{l_r} + a & \left\{\begin{array}{l}\text{for the right hand shear at the } r\text{th} \\ \quad \text{support, } \;.\;\;.\;\;.\;\;.\;\;.\end{array}\right. \\
\mathrm{F}'_r = \dfrac{\mathrm{M}_{r+1} - \mathrm{M}_r}{l_r} + b & \left\{\begin{array}{l}\text{for the left hand shear at the } r+1\text{th} \\ \quad \text{support, } \;.\;\;.\;\;.\;\;.\;\;.\end{array}\right. \\
\mathrm{F}_n = \dfrac{\mathrm{M}_n - \mathrm{M}_{n+1}}{l_n} & \left\{\begin{array}{l}\text{for the right hand shear at all supports} \\ \quad \text{except } r, \;.\;\;.\;\;.\;\;.\;\;.\;\;.\end{array}\right. \\
\mathrm{F}_{n-1} = \dfrac{\mathrm{M}_n - \mathrm{M}_{n-1}}{l_{n-1}} & \left\{\begin{array}{l}\text{for the left hand shear at all supports} \\ \quad \text{except } r+1, \;.\;\;.\;\;.\;\;.\end{array}\right.
\end{array}\right\} (4.)$$

then the reaction at any support is

$$\mathrm{R}_n = \mathrm{F}'_{n-1} + \mathrm{F}_n; \; \mathrm{R}_r = \mathrm{F}'_{r-1} + \mathrm{F}_r, \;\&\text{c.}, \;\ldots\ldots\ldots\ldots(5.)$$

Let M and F denote the moment and shearing force at any section; then

$$\left.\begin{array}{l} \mathrm{M} = \mathrm{M}_r - \mathrm{F}_r\, x + \mathrm{W}\,(x - a) \text{ for a section between W and} \\ \qquad\qquad\qquad\qquad \text{the following support:—} \\ \mathrm{M} = \mathrm{M}_n - \mathrm{F}_n\, x \text{ for any other section,} \end{array}\right\} (6.)$$

$$\left.\begin{array}{l} \mathrm{F} = \mathrm{W} - \mathrm{F}_r \text{ for a section between W and the } r+1\text{th} \\ \qquad\qquad \text{support,} \\ \mathrm{F} = -\,\mathrm{F}_n \text{ for any other section,} \end{array}\right\} (7.)$$

Equations (6) and (7) refer to a concentrated load; for an uniform load for W substitute $\int w\, d\, a$.

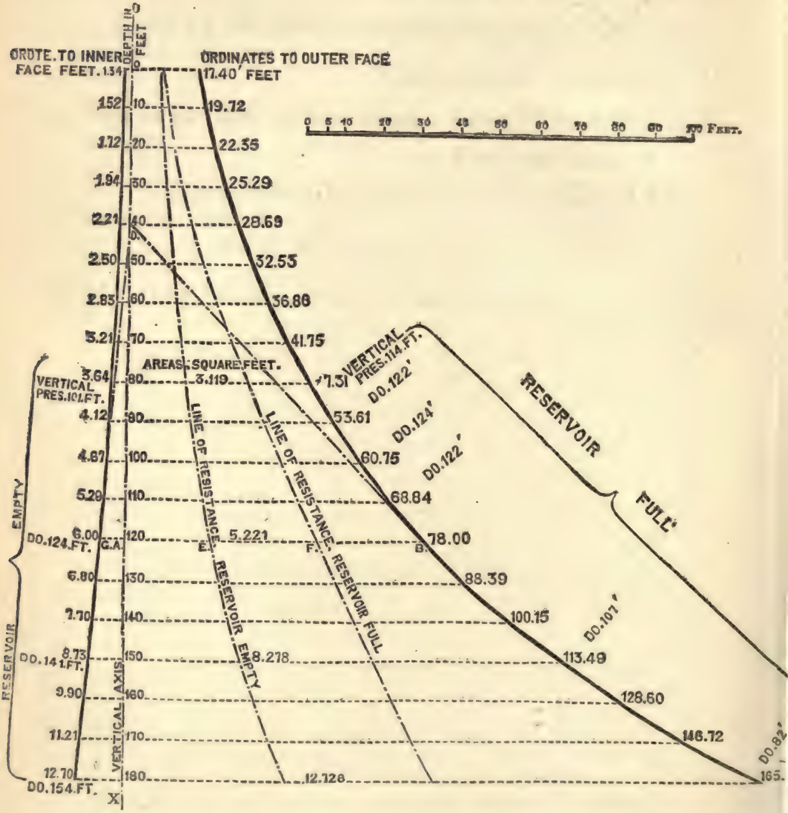
CASE 2.—*One end free and the other fixed horizontally.*

In the application of the above formulæ (1) to (7) to this case, make $l_s = 0$, and let $s - 1$ be the number of spans.

CASE 3.—*Both ends fixed horizontally.*

In the above formulæ make $l_1 = 0$ and $l_s = 0$, and let $s - 2$ be the number of spans.

The diagram annexed shows the form and proportions adopted by Professor Rankine for reservoir-walls of great height.



For detailed description see *The Engineer* for January 5, 1872 (a reprint of this paper is embodied in the Rankine Memorial Volume of Selected Papers).

The lines of resistance lie within, or near to, the middle third of the thickness of the wall. The outer and inner faces are logarithmic curves. It is desirable to give such walls a curvature in plan convex towards the reservoir, to counteract the tendency of the wall to being bent by the pressure of the water into a curved shape, concave towards the water.

The following notes on American Bridge Practice are taken from *The Transactions of the American Society of Civil Engineers*, Vol. VIII. :—

"American bridges are generally built up from the following individual members, most, if not all the mechanical work upon them being done in the shop. 1st. Chord and web eye-bars; round, square, or flat bars, with a head at each end, formed by some process of forging. These are tension members. 2nd. Lateral, diagonal, and counter rods. 3rd. Floor-beam hangers. 4th. Pins. 5th. Lateral struts. 6th. Posts. 7th. Top chord sections. The last three being columns formed by rivetting together various rolled forms; plates, angles, channels, I beams, &c. Some are square-ended, others pin-connected. These are compression members. 8th. Floor-beams and stringers. These consist either of rolled beams, rivetted plate girders, or occasionally of latticed or trussed girders. The proportion of depth to span in American bridges is from one-fifth to one-seventh.

"In top chords, posts, and struts the strains are calculated by a modification of Rankine's formula, as follows :—

$$p = \frac{8000}{1 + \frac{l^2}{40000\,r^2}}$$ for square-end compression members.

$$p = \frac{8000}{1 + \frac{l^2}{30000\,r^2}}$$ for compression members with one pin and one square end.

$$p = \frac{8000}{1 + \frac{l^2}{20000\,r^2}}$$ for compression members with pin bearings.

where p = the allowed compression per square inch of cross section.

l = the length of compression member, in inches.

r = the least radius of gyration of the section, in inches."

ARTICLE 240, p. 215.

The correctness of the value for wind pressure, as adopted by Professor Rankine, has been lately proved in the severe storms which have visited this country, a recent committee of enquiry having fixed this pressure on bridge surfaces at 56 lbs. per square foot.

CONTINUOUS BRAKES FOR RAILWAY TRAINS.—The use of brake power is now considerably extended in railway traffic, and instead of the brakes being only applied on tender and guard's van, the application has been extended to the carriages composing the train. Very considerable resistance is thus obtained, and consequent cessation of motion at a much earlier period. Various forms of continuous brake have been tried recently, and the results of the experiments are familiar to engineers. Some of the various forms are the screw-brake, chain-brake, vacuum-brake, hydraulic-brake, and compressed-air-brake, in all of which, by means of mechanism extending below the carriages and actuated by the engine-driver or guard, the whole or part of the wheels of the train can be braked. In the first two methods, rigid or flexible bodies are employed to transmit the power required, whilst, in the others, the same object is accomplished through the medium of fluids. In the hydraulic-brake, water at a high pressure from a pump on the engine is conveyed by a pipe; in the vacuum-brake the air is removed, and in the air-brake the air is forced under pressure to the points required. In the automatic arrangements, whether of air or vacuum, there are reservoirs. It has been found desirable to adopt reservoirs or vessels having pistons immediately in connection with the brake blocks, the object in the automatic arrangements being to keep up a certain condition in the chambers, whether of pressure or vacuum, by which, if destroyed either intentionally or accidentally (as through the breakage of a pipe), the braking action may at once take place.

In some cases 1½ seconds is sufficient to apply the brakes, and fast trains can be stopped in about 300 yards.

ARTICLE 271, p. 289.

Boilers are now largely made of steel, and when iron is used in the shell, the flues and ends are sometimes made of steel. Iron rivets, however, appear to be still in favour, as the shearing strength of mild steel appears less in proportion to tensile strength than in the case of iron. The tensile strength of steel boiler plates is about 29 tons per square inch, and the elastic limit appears to be about half of that. Punching out the rivet holes weakens the metal about 30 per cent. This loss, however, may be restored by annealing. Drilling the rivet holes does not affect the strength, and is the usual method adopted. Some advantage should arise, so far as steaming properties are concerned through the use of steel, as lighter sections can be used for the plates.

The Pulsometer Pump, recently introduced for engineering work, is a steam pump with few or no moving parts, the water is forced to the height of delivery by the direct action of the steam, the supply being raised by the condensation of the same steam. The action is regulated by a ball valve at the upper part, and clack valves at the lower part, of the working chamber.

ARTICLE 346, p. 377.

Steel is now largely used for rails, boiler and ship plates, &c., and combines great strength with ductility, the ultimate tensile strength varying from 27 to 33 tons per square inch, with an elongation of about 20 per cent. The limit of elasticity is about one-half of the ultimate or breaking strength.

The Board of Trade allows 6½ tons per square inch as the safe working strength for bridge structures.

It appears that by the use of steel in the construction of ships a saving of weight of about 16 per cent. is obtained.

Some peculiarities exist as to the behaviour of steel, and care must be taken both in the working of it from the ingot into plates, and in the workshop or yard, one special point being, that it should not be worked at a "black heat," or about 550° F. The steel referred to is known as "Mild Steel" or "Ingot Metal," and safety in working lies above or below this temperature. The question as to the comparative wear of iron and steel by corrosion seems still undecided, but so far no practical difference has been observed.

Steel rails and tires are now used for railway traffic, and articles of cast steel can now be manufactured possessing considerable strength and ductility.

ARTICLE 653, p. 598.

Of late years much experimental information has been obtained as to the resistance of both vessels and their models, Mr. Froude, of Torquay, having determined the relations which exist between the ship and her model. Again, trials of the same vessel, at different speeds, have shown clearly that the law of resistance varies for different speeds. This subject is one of much importance, especially in connection with new and untried forms of vessels, and has already received much advancement from experimental research.*

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* For authorities on this subject, see *Transactions of the Institution of Engineers and Shipbuilders in Scotland*, and *Transactions of the Institute of Naval Architects.*

SECTION 6, p. 464.

Hydraulic pressure is now largely used for engineering work, such as rivetting, flanging, punching, shearing, &c., and for the motive power for moving heavy loads, such as swing bridges, cranes, &c. By the introduction of flexible pipes and portable rivetting machines, hydraulic rivetting can be performed on bridges and ships. To accomplish this a high pressure is used, varying from 700 to 1,500 lbs. per square inch, and this is obtained by means of an "accumulator," or cylinder containing a loaded piston, against which water is pumped in by a steam engine. This water under pressure being connected with the die of, say, a rivetting machine, is free to exert its pressure on the opening of the connecting valve, and a fall of the loaded plunger taking place at same time the rivet is closed up. This powerful closing pressure is very serviceable when a number of plates are to be bound together, as is the case for some of the keels of our large steam vessels.

ARTICLE 637, p. 581.

In a Report on Safety Valves,* drawn up by a Committee, of which Professor Rankine was a member, but whose decease happened prior to the completion of the experiments which were being carried out, it is stated that the weight of steam in pounds discharged per minute per square inch of opening, with square-edged entrance, corresponds very nearly with three-fourths of the absolute pressure in the boiler, the range being from 25·37 lbs. to 100 lbs. per square inch. And the requisite area for a safety valve is shown to be—

$$a = \frac{4 \times \text{square feet of fire-grate}}{p_1},$$

where a = area of orifice in square inches, and p_1 = the absolute pressure.

ARTICLE 626, p. 572.

The Editor appends the following investigation by the late Professor Rankine of the value of the theoretical co-efficient of contraction in a jet of water issuing from a large cistern with a pipe going into it. The investigation was laid before the Professor's Class of Civil Engineering and Mechanics in Glasgow University, Session 1866-67.

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* See *Transactions of the Institution of Engineers and Shipbuilders in Scotland*, vol. xviii.

Let a = area of orifice in feet.
,, v = velocity of outflow in feet per second.
,, $c\,a\,v$ = number of cubic feet per second.
,, D = weight of a cubic foot of water.
then, $D\,c\,a\,v$ = weight of flow per second.

Now, the reaction or backward pressure exerted against the reservoir $= \frac{D\,c\,a\,v^2}{g}$; the pressure in the reservoir $= \frac{D\,v^2}{2g}$; multiplying the latter expression by a, and equating, we have—

$$\frac{D\,c\,a\,v^2}{g} = \frac{D\,v^2\,a}{2g}, \text{ or } c = \frac{1}{2}.$$

Gas Engines have now been introduced with success, especially for the lighter classes of work. These engines have the great advantage of being easily managed, as the starting and stopping of them only entails the opening or shutting of the tap regulating the gas supply. The action in the cylinder is an explosive one, the explosion taking place at certain intervals of the revolutions, a heavy fly-wheel being added to store up the excess of energy over work done when the explosion takes place.

ARTICLE 684, p. 620.

From recent experiments made with trains brought to rest simply by their own resistance, it appears that the latter amounts to about 9½ lbs. per ton of weight, or ½ per cent.

The effect of handbrakes as applied to tender and guards' vans was found to be about 2⅓ per cent. of the weight. A train going about 45 miles per hour could be stopped in 1,000 yds., or at 60 miles per hour in about 1 mile. With the continuous brake system, as much as 10 per cent. of the train weight can be obtained as a retarding force, and a train travelling 60 miles per hour can be stopped in 400 yards. Brake-blocks are of cast-iron, steel, or wood.

Some difference of opinion appears as to the effect of skidding the wheels; no doubt, greater wear and tear is caused when the wheels are skidded.

ARTICLE 704, p. 630.

In Compound Engines as now used for marine and also land purposes, the consumpt of coal is less than 2 lbs. per indicated horse power, or probably about 1¾ lb., being a duty of over 1,000,000 foot-pounds.

The economy of the marine engine is largely due to high pressure steam, about 90 lbs. per square inch being now often carried, to surface condensation, and the large ratio of expansion obtained by the compound system where the steam passes from one cylinder into one or two others, before reaching the condenser; our best engines, however, are only yielding an efficiency of from about $\frac{1}{8}$ to about $\frac{1}{10}$. This appears to be made up more or less as follows :—

Efficiency of furnace and boiler, $\frac{6}{10}$; efficiency of the steam, $\frac{2}{10}$; or total efficiency $\frac{6}{10} \times \frac{2}{10}$ = about $\frac{1}{8}$; again, if we take the efficiency of the propeller as $\frac{5}{10}$, we shall have about $\frac{1}{16}$ as the final efficiency.

DIMENSIONS AND STABILITY OF THE OUTER SHELL OF THE GREAT CHIMNEY OF ST. ROLLOX.

Divisions of Chimney.	Heights above Ground.	External Diameters.		Thicknesses.		Greatest pressure of Wind consistent with Security.
	Feet.	Feet.	Inches.	Feet.	Inches.	lb. per square foot.
	435½	13	6			
V.				1	2	
	350½	16	9	—		77
IV.				1	6	
	210½	24	0	—		55*
III.				1	10½	
	114½	30	6	—		57
II.				2	3	
	54½	35	0	—		63
I.				2	7½	
	0	40	0	—		71

Foundation.	Depth below Ground.	External Diameter.	Thicknesses. Concrete.		Brick.
	Feet.	Feet.	Feet.	Inches.	Feet.
I.	0	50	5	0	3
II.	8	50	4	8	3
III.	14	50	25	0	12
	20	50			0

Total height from base of foundation to top of chimney, 455½ feet.

TOWNSEND'S CHIMNEY, GLASGOW. BUILT, 1857–59.

Total height 468 feet. Height from surface of ground to cope,

---

* Joint of least stability.

454 feet. Extra height of 20 feet of ornamental iron work since added, and connected with the lightning conductor.

Outside diameter at foundations, 50 feet; outside diameter at surface of ground, 32 feet; outside diameter at top of cope, 12 feet 8 inches. The sides have a straight batter. The thickness varies from 7 bricks at base to 1½ brick at top.

I.

TABLE OF THE RESISTANCE OF MATERIALS TO STRETCHING AND TEARING BY A DIRECT PULL, *in pounds avoirdupois per square inch.*

MATERIALS.	Tenacity, or Resistance to Tearing.	Modulus of Elasticity, or Resistance to Stretching.
STONES, NATURAL AND ARTIFICIAL:		
Brick, Cement,	280 to 300	
Glass,	9,400	8,000,000
Slate,	9,600 to 12,800	13,000,000 to 16,000,000
Mortar, ordinary,	50	
METALS:		
Brass, cast,	18,000	9,170,000
„ wire,	49,000	14,230,000
Bronze or Gun Metal (Copper 8, Tin 1),	36,000	9,900,000
Copper, cast,	19,000	
„ sheet,	30,000	
„ bolts,	36,000	
„ wire,	60,000	17,000,000
Iron, cast, various qualities,	13,400 to 29,000	14,000,000 to 22,900,000
„ average,	16,500	17,000,000
Iron, wrought, plates,	51,000	
„ joints, double rivetted,	35,700	
„ „ single rivetted,	28,600	
„ bars and bolts,	60,000 to 70,000	29,000,000
„ hoop, best-best,	64,000	
„ wire,	70,000 to 100,000	25,300,000
„ wire-ropes,	90,000	15,000,000
Lead, sheet,	3,300	720,000
Steel bars,	100,000 to 130,000	29,000,000 to 42,000,000
Steel plates, average,	80,000	
Tin, cast,	4,600	
Zinc,	7,000 to 8,000	

MATERIALS.	Tenacity, or Resistance to Tearing.	Modulus of Elasticity, or Resistance to Stretching.
TIMBER AND OTHER ORGANIC FIBRE:		
Acacia, false. See "Locust."		
Ash (*Fraxinus excelsior*),	17,000	1,600,000
Bamboo (*Bambusa arundinacea*),	6,300	
Beech (*Fagus sylvatica*),	11,500	1,350,000
Birch (*Betula alba*),	15,000	1,645,000
Box (*Buxus sempervirens*),	20,000	
Cedar of Lebanon (*Cedrus Libani*),	11,400	486,000
Chestnut (*Castanea Vesca*),	10,000 to 13,000	1,140,000
Elm (*Ulmus campestris*),	14,000	700,000 to 1,340,000
Fir: Red Pine (*Pinus sylvestris*),	12,000 to 14,000	1,460,000 to 1,900,000
„ Spruce (*Abies excelsa*),	12,400	1,400,000 to 1,800,000
„ Larch (*Larix Europœa*),	9,000 to 10,000	900,000 to 1,360,000
Hoxen Yarn, about	25,000	
Hazel (*Corylus Avellana*),	18,000	
Hempen Ropes, from	12,000 to 16,000	
Hide, Ox, undressed,	6,300	
Hornbeam (*Carpinus Betulus*),	20,000	
Lancewood (*Guatteria virgata*),	23,400	
Leather, Ox,	4,200	24,300
Lignum-Vitæ (*Guaiacum officinale*),	11,800	
Locust (*Robinia Pseudo-Acacia*),	16,000	
Mahogany (*Swietenia Mahagoni*),	8,000 to 21,800	1,255,000
Maple (*Acer campestris*),	10,600	
Oak, European (*Quercus sessiliflora* and *Quercus pedunculata*),	10,000 to 19,800	1,200,000 to 1,750,000
„ American Red (*Quercus rubra*),	10,250	2,150,000
Silk Fibre,	52,000	1,300,000
Sycamore (*Acer Pseudo-Platanus*),	13,000	1,040,000
Teak, Indian (*Tectona grandis*),	15,000	2,400,000
„ African, (?)	21,000	2,300,000
Whalebone,	7,700	
Yew (*Taxus baccata*),	8,000	

II.

TABLE OF THE RESISTANCE OF MATERIALS TO SHEARING AND DISTORTION, *in pounds avoirdupois per square inch.*

MATERIALS.	Resistance to Shearing.	Transverse Elasticity, or Resistance to Distortion.
METALS:		
Brass, wire-drawn,		5,330,000
Copper,		6,200,000
Iron, cast,	27,700	2,850,000
„ wrought,	50,000	8,500,000 to 10,000,000
TIMBER:		
Fir: Red Pine,	500 to 800	62,000 to 116,000
„ Spruce,	600	......
„ Larch,	970 to 1,700	......
Oak,	2,300	82,000
Ash and Elm,	1,400	76,000

III.

TABLE OF THE RESISTANCE OF MATERIALS TO CRUSHING BY A DIRECT THRUST, *in pounds avoirdupois per square inch.*

MATERIALS.	Resistance to Crushing.
STONES, NATURAL AND ARTIFICIAL:	
Brick, weak red,	550 to 800
„ strong red,	1,100
„ fire,	1,700
Chalk,	330
Granite,	5,500 to 11,000
Limestone, marble,	5,500
„ granular,	4,000 to 4,500
Sandstone, strong,	5,500
„ ordinary,	3,300 to 4,400
„ weak,	2,200
Rubble masonry, about four-tenths of cut stone.	
METALS:	
Brass, cast,	10,300
Iron, cast, various qualities,	82,000 to 145,000
„ „ average,	112,000
„ wrought,	about 36,000 to 40,000

MATERIALS.	Resistance to Crushing.
TIMBER,* Dry, crushed along the grain:	
Ash,	9,000
Beech,	9,360
Birch,	6,400
Blue-Gum (*Eucalyptus Globulus*),	8,800
Box,	10,300
Bullet-tree (*Achras Sideroxylon*),	14,000
Cabacalli,	9,900
Cedar of Lebanon,	5,860
Ebony, West Indian (*Brya Ebenus*),	19,000
Elm,	10,300
Fir: Red Pine,	5,375 to 6,200
" American Yellow Pine (*Pinus variabilis*),	5,400
" Larch,	5,570
Hornbeam,	7,300
Lignum-Vitæ,	9,900
Mahogany,	8,200
Mora (*Mora excelsa*),	9,900
Oak, British,	10,000
" Dantzic,	7,700
" American Red,	6,000
Teak, Indian,	12,000
Water-Gum (*Tristania nerifolia*),	11,000

IV.

TABLE OF THE RESISTANCE OF MATERIALS TO BREAKING ACROSS, *in pounds avoirdupois per square inch.*

MATERIALS.	Resistance to Breaking, or Modulus of Rupture.†
STONES:	
Sandstone,	1,100 to 2,360
Slate,	5,000

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* The resistances stated are for *dry* timber. Green timber is much weaker, having sometimes only half the strength of dry timber against crushing.

† The modulus of rupture is eighteen times the load which is required to break a bar of one inch square, supported at two points one foot apart, and loaded in the middle between the points of support.

MATERIALS.	Resistance to Breaking, or Modulus of Rupture.
METALS:	
Iron, cast, open-work beams, average, ...	17,000
„ „ solid rectangular bars, var. qualities,	33,000 to 43,500
„ „ „ „ average,......	40,000
„ wrought, plate beams, ...	42,000
TIMBER:	
Ash,..	12,000 to 14,000
Beech,..	9,000 to 12,000
Birch, ...	11,700
Blue-Gum,.......................................	16,000 to 20,000
Bullet-tree, ...	15,900 to 22,000
Cabacalli,..	15,000 to 16,000
Cedar of Lebanon,.............................	7,400
Chestnut,...	10,660
Cowrie (*Dammara australis*), ...	11,000
Ebony, West Indian, ...	27,000
Elm,..	6,000 to 9,700
Fir: Red Pine, ...	7,100 to 9,540
„ Spruce,..	9,900 to 12,300
„ Larch,..	5,000 to 10,000
Greenheart (*Nectandra Rodiœi*),..................	16,500 to 27,500
Lancewood, ...	17,350
Lignum-Vitæ,...................................	12,000
Locust, ...	11,200
Mahogany, Honduras,..........................	11,500
„ Spanish, ...	7,600
Mora,...	22,000
Oak, British and Russian,......................	10,000 to 13,600
„ Dantzic, ...	8,700
„ American Red,................................	10,600
Poon,..	13,300
Saul,..	16,300 to 20,700
Sycamore, ...	9,600
Teak, Indian,....................................	12,000 to 19,000
„ African,.......................................	14,980
Tonka (*Dipteryx odorata*), ...	22,000
Water-Gum, ...	17,460
Willow (*Salix*, various species),..................	6,600

V.—COMPARATIVE TABLE OF FRENCH AND BRITISH MEASURES.

	No.	Log.	Log.	No.	
Grains in a gramme,...	15·43235	1·188432	$\bar{2}$·811568	0·064799	Gramme in a grain.
Pounds avoird. in a kilogramme,	2·20462	0·343334	$\bar{1}$·656666	0·453593	Kilog. in a lb. avoirdupois.
Ton in a tonne,....................	0·984206	$\bar{1}$·993086	0·006914	1·01605	Tonnes in a ton.
Feet in a mètre, ...	3·2808693	0·515989	$\bar{1}$·484011	0·3047972I	Mètres in a foot.
Inch in a millimètre,............	0·03937043	$\bar{2}$·595170	1·404830	25·39977	Millimètres in an inch.
Mile in a kilomètre, ...	0·621377	$\bar{1}$·793355	0·206645	1·60933	Kilomètres in a mile.
Square feet in a square mètre,..	10·7641	1·031978	$\bar{2}$·968022	0·0929013	Square mètre in a square foot.
Square inch in a square millimètre, ...	0·00155003	$\bar{3}$·190340	2·809660	645·148	Square millim. in a square inch.
Cubic feet in a cubic mètre,.....	35·3156	1·547967	$\bar{2}$·452033	0·0283161	Cubic mètre in a cubic foot.
Foot-pounds in a kilogrammètre,	7·23308	0·859323	$\bar{1}$·140677	0·138254	Kilogrammètre in a foot-pound.
Pounds-to-the-foot in a kilogramme-to-the-mètre,......	0·671963	$\bar{1}$·827345	0·172655	1·48818	Kilogrammes-to-the-mètre in a pound-to-the-foot.
Pounds-to-the-square-foot in a kilogramme-to-the-square-mètre, ...	0·204813	$\bar{1}$·311356	0·688644	4·88252	Kilogrammes-to-the-square-mètre in a pound-to-the-square-foot.
Pounds-to-the-square-inch in a kilog.-to-the-square-millimètre,......................	1422·31	3·152994	$\bar{4}$·847006	0·000703083	Kilog.-to-the-square-millimètre in a pound-to-the-square-inch.
Pounds-to-the-cubic-foot in a kilogramme-to-the-cubic-mètre,......................	0·062426	$\bar{2}$·795367	1·204633	16·019	Kilogrammes-to-the-cubic-mètre in a pound-to-the-cubic-foot.
Fahrenheit-degrees in a centigrade-degree,................	1·8	0·255273	$\bar{1}$·744727	0·5555$\dot{5}$	Centigrade-degree in a Fahrenheit degree.
British units of heat in a French unit,.................	3·96832	0·598607	$\bar{1}$·401393	0·251996	French units of heat in a British unit.

VI.

TABLE OF SPECIFIC GRAVITIES OF MATERIALS.

	Weight of a cubic foot in lb. avoirdupois.
GASES, at 32° Fahr., and under the pressure of one atmosphere, of 2116·4 lb. on the square foot:	
Air,	0·080728
Carbonic Acid,	0·12344
Hydrogen,	0·005592
Oxygen,	0·089256
Nitrogen,	0·078596
Steam (ideal),	0·05022
Æther vapour (ideal),	0·2093
Bisulphuret-of-carbon vapour (ideal),	0·2137
Olefiant gas,	0·0795

	Weight of a cubic foot in lb. avoirdupois.	Specific gravity, pure water = 1.
LIQUIDS at 32° Fahr. (except Water, which is taken at 39°·4 Fahr.):		
Water, pure, at 39°·4,	62·425	1·000
„ sea, ordinary,	64·05	1·026
Alcohol, pure,	49·38	0·791
„ proof spirit,	57·18	0·916
Æther,	44·70	0·716
Mercury,	848·75	13·596
Naphtha,	52·94	0·848
Oil, linseed,	58·68	0·940
„ olive,	57·12	0·915
„ whale,	57·62	0·923
„ of turpentine,	54·31	0·870
Petroleum,	54·81	0·878
SOLID MINERAL SUBSTANCES, non-metallic:		
Basalt,	187·3	3·00
Brick,	125 to 135	2 to 2·167
Brickwork,	112	1·8
Chalk,	117 to 174	1·87 to 2·78
Clay,	120	1·92
Coal, anthracite,	100	1·602
„ bituminous,	77·4 to 89·9	1·24 to 1·44
Coke,	62·43 to 103·6	1·00 to 1·66
Felspar,	162·3	2·6
Flint,	164·2	2·63

	Weight of a cubic foot in lb. avoirdupois.	Specific gravity, pure water = 1.
Solid Mineral Substances—*continued.*		
Glass, crown, average,	156	2·5
" flint, "	187	3·0
" green, "	169	2·7
" plate, "	169	2·7
Granite,	164 to 172	2·63 to 2·76
Gypsum,	143·6	2·3
Limestone (including marble),	169 to 175	2·7 to 2·8
" magnesian,	178	2·86
Marl,	100 to 119	1·6 to 1·9
Masonry,	116 to 144	1·85 to 2·3
Mortar,	109	1·75
Mud,	102	1·63
Quartz,	165	2·65
Sand (damp),	118	1·9
" (dry),	88·6	1·42
Sandstone, average,	144	2·3
" various kinds,	130 to 157	2·08 to 2·52
Shale,	162	2·6
Slate,	175 to 181	2·8 to 2·9
Trap,	170	2·72
Metals, solid:		
Brass, cast,	487 to 524·4	7·8 to 8·4
" wire,	533	8·54
Bronze,	524	8·4
Copper, cast,	537	8·6
" sheet,	549	8·8
" hammered,	556	8·9
Gold,	1186 to 1224	19 to 19·6
Iron, cast, various,	434 to 456	6·95 to 7·3
" average,	444	7·11
Iron, wrought, various,	474 to 487	7·6 to 7·8
" average,	480	7·69
Lead,	712	11·4
Platinum,	1311 to 1373	21 to 22
Silver,	655	10·5
Steel,	487 to 493	7·8 to 7·9
Tin,	456 to 468	7·3 to 7·5
Zinc,	424 to 449	6·8 to 7·2

TIMBER:*	Weight of a cubic foot in lb. avoirdupois.	Specific gravity, pure water = 1.
Ash,	47	0·753
Bamboo,	25	0·4
Beech,	43	0·69
Birch,	44·4	0·711
Blue-Gum,	52·5	0·843
Box,	60	0·96
Bullet-tree,	65·3	1·046
Cabacalli,	56·2	0·9
Cedar of Lebanon,	30·4	0·486
Chestnut,	33·4	0·535
Cowrie,	36·2	0·579
Ebony, West Indian,	74·5	1·193
Elm,	34	0·544
Fir: Red Pine,	30 to 44	0·48 to 0·7
„ Spruce,	30 to 44	0·48 to 0·7
„ American Yellow Pine,...	29	0·46
„ Larch,	31 to 35	0·5 to 0·56
Greenheart,	62·5	1·001
Hawthorn,	57	0·91
Hazel,	54	0·86
Holly,	47	0·76
Hornbeam,	47	0·76
Laburnum,	57	0·92
Lancewood,	42 to 63	0·675 to 1·01
Larch. See "Fir."		
Lignum-Vitæ,	41 to 83	0·65 to 1·33
Locust,	44	0·71
Mahogany, Honduras,	35	0·56
„ Spanish,	53	0·85
Maple,	49	0·79
Mora,	57	0·92
Oak, European,	43 to 62	0·69 to 0·99
„ American Red,	54	0·87
Poon,	36	0·58
Saul,	60	0·96
Sycamore,	37	0·59
Teak, Indian,	41 to 55	0·66 to 0·88
„ African,	61	0·98
Tonka,	62 to 66	0·99 to 1·06
Water-Gum,	62·5	1·001
Willow,	25	0·4
Yew,	50	0·8

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* The Timber in every case is supposed to be dry.

INDEX.

ABSOLUTE Unit of Force, 486.
Abutments of Arches, 261.
,, —Open and Hollow, 263.
,, —Stability of, 226, 235.
,, —Strength of, 268.
Accelerating Effect of Gravity, 485.
,, Force, 490.
,, Impulse, 483.
Acceleration, 386.
Accumulator, 641.
Actual Energy, 499, 507.
,, of a Machine, 621.
,, of Rotation, 532.
Adhesion, 209.
Aggregate Combinations, 466.
Air—Apparent Weight of Bodies in, 123.
,, —Expansion of, 123.
,, —Velocity of Sound in, 563.
,, —Weight of, 123, 652.
Angle of Repose, 210.
,, of Rotation, 391.
,, of Rupture, 204, 259.
,, of Torsion, 356.
Angular Impulse, 506.
,, Momentum, 505, 529.
,, Velocity, 391, 492.
Arch—Abutments of, 261.
,, —Angle, Joint, and Point of Rupture of, 259.
,, —Circular Linear, 183, 200.
,, —Clustered, 263.
,, —Distorted, 202.
,, —Distorted Elliptic Linear, 186.
,, —Elliptic Linear, 184.
,, —Geostatic approximate, 196, 207.
,, —Hydrostatic, 190, 207, 353.
,, —Iron-ribbed, 376.
,, —Line of Pressures in, 257.
,, —Linear—or Equilibrated Rib, 162, 175, 182.
,, —Linear—for Normal Pressure, 189.
,, —Piers of, 263.
,, —Pointed, 203.
,, —Skew, 261.
,, —Stability of, 226, 257.
,, —Stereostatic, 198.
,, —Strength of, 268.
Arch—Total Thrust of, 203, 260.
Areas—Conservation of, 507.
,, —Measurement of, 58.
Atmospheric Pressure, 69.
Authorities on Waves, 633.
Axes—Conjugate, 77, 79.
,, of Elasticity, 278.
,, of Inertia, 524, 526.
,, of Stress, 93, 98.
Axis—Fixed, 545.
,, —Instantaneous, 397.
,, of Angular Momentum, 505, 529.
,, of Rotation, 390.
Axle—Friction of, 614.
,, —Resilience of, 357.
,, —Strength of, 353.
,, —Torsion of, 356.
,, with Crank—Strength of, 358.

BALANCE, 15.
,, of any System of Forces, 41.
,, of Couples, 21.
,, of Floating Bodies, 120.
,, of Fluids, 116.
,, of Forces in One Line, 19.
,, of Inclined Forces, 35.
,, of Parallel Forces, 21, 25.
,, of Stress and Weight, 112.
,, of Structures, 129.
Balanced Forces—Motion under, 476.
Ballistic Pendulum, 548.
Bands—Friction of, 617.
,, in Mechanism, 454.
Bars—Strength of Iron and Steel, 377.
Beams, 133.
,, —Allowance for Weight of, 346.
,, —Cast-iron, 318.
,, Deflection under any Load, 328.
,, —Direct Vertical Stress in, 342.
,, —Expansion and Contraction of, 348.
,, fixed at both ends, 332.
,, —Limiting Length of, 347.
,, —Lines of Principal Stress in, 341.
,, of Uniform Strength, 320.
,, originally Curved, 348.
,, —Partially loaded, 344.
,, —Proof Deflection of, 322.

Beams—Proportion of Depth to Span of, 327.
,, —Resilience of, 330.
,, —Shearing Stress in, 338, 342.
,, —Sloping, 348.
,, —Strength of, 307, 315, 318, 634.
Bearings, 422.
Belts, 288, 454.
Bending—Moment of, 308.
,, —Resistance to, 307.
Bevel-wheels, 428, 448.
Blocks—Stability of a series of, 230.
,, and Tackle, 462.
Bodies, 13.
Boiler Stays, 296.
Boilers—Strength of, 289, 296, 299, 306, 640.
Bond in Brickwork, 222.
,, Masonry, 222.
Bowstring Girder, 369.
Bracing of Frames, 142.
Brakes, 242, 624, 640, 643.
Breaking across—Resistance to, 307.
Breast-wheel, 628.
Brickwork, 222, 226, 242.
Bridges, 149, 153, 263, 344, 639.
,, —Suspension, 149, 171, 286, 370.
Buckling, 302.
Bulging, 302.
Buoyancy, 120.
,, —Centre of, 121, 601.
Buttresses, 228, 235.

Cables—Strength of, 288.
Cam, 449.
Cast-iron Beams, 318.
,, —Strength of, 362, 646.
Catenary, 177.
Cells—Strength of, 364.
Centre of Buoyancy, 121, 601.
,, of Gravity, 49, 180, 505.
,, of Mass, 482.
,, of Oscillation, 546.
,, of Parallel Forces, 31.
,, of Percussion, 520, 544.
,, of Pressure, 71, 76, 125, 227.
,, of Resistance, 131.
Centrifugal Couple, 537, 621.
,, Force, 387, 491, 546.
,, Pump, 597.
Chains—Equilibrium of, 162.
,, for Pulleys, 454.
Channel—Flow in, 411.
Chimneys—Stability of, 228, 235, 240, 644.
Cinematics, 15, 421.
,, Principles of, 379.
Click, 462.
Coefficient of Contraction, 572, 642.
Coefficient of Elasticity, 277, 279.
,, of Friction, 210, 612.
,, of Pliability, 277, 279.
Collapsing—Resistance to, 306.
Collar—Friction of, 616.
Collision, 508.
Columns—Strength of, 360, 638.
Comparative Motion, 384, 389.
Components, 19, 37, 381.
Composition of Couples, Forces, Motions, 23, 381.
Compound Engines, 642, 643.
,, Screws, 467.
Compressibility of Liquids, 271.
Compression—Resistance to, 302.
Cones—Speed, 457.
Conjugate Axes, 79.
,, Stress, 85.
Connected Bodies—Motions of, 420, 421.
Connecting Rods—Strength of, 363.
Conservation of Angular Momentum, or of Areas, 506.
,, of Energy, 478, 501, 508.
,, of Momentum, 505.
Continuity—Equations of—in Gases, 417.
,, —Equations of—in Liquids, 411, 413.
Continuous Brakes, 640, 643.
,, Girders, 338, 634.
Contracted Vein, 572.
Contraction—Coefficient of, 572, 642.
Cord—Equilibrium of, 162.
,, —Motion of, 408.
Cords, 454.
,, —Reduplication of, 462.
Counterforts, 255.
Couples—Centrifugal, 537.
,, —Deviating, 535.
,, —Energy and Work of, 537.
,, —Polygon of, 25.
,, —Reduction of, 612.
,, —Statical—Theory of, 21.
,, with Inclined Axes, 24.
,, with Parallel Axes, 21.
Coupling—Friction, 618.
,, —Hooke's, 461.
,, —Oldham's, 453.
,, of Parallel Axes, 459.
Crank and Axle—Motion of, 458.
,, ,, —Strength of, 358.
Cross-breaking—Resistance to, 314.
Crushing by Bending—Resistance to, 360.
,, —Direct Resistance to, 302,
,, ,, —Table of, 648.
Cup and Ball Pivot, 616.
Current, 412.
,, —Pressure of—on a Solid Body, 598.
,, —Radiating, 412, 574.
Cycloid, 398.

Cycloidal Pendulum, 497.
Cylinders—Strength of, 289, 294.

DAMS—Stability of, 235, 243.
Day—Mean Solar, 382.
,, —Sidereal, 380, 381.
Dead Points, 458.
Deflection of Beam, 312, 322, 328.
Deviating Couple, 535.
,, Force, 491, 492, 545.
Deviation (of Motion)—Moment of, 528.
,, —Uniform, 387.
,, —Varying, 388.
Differential Windlass, 466.
Differentiation, 386.
Direction—Fixed and nearly fixed, 379.
Distributed Forces, 48.
Domes—Stability of, 265.
Driving Point, 610.
Drums in Mechanism, 455.
Ductility, 273, 376.
Duty of Engines, 630.
Dynamic Head, 568, 579.
Dynamics, 15.
,, —General Equations of, 484.
,, —Principles of, 475.
Dynamometer, 478.

EARTH—Friction of, 211.
,, Foundations, 219.
,, —Pressure of, 218, 249, 277.
,, —Stability of, 212.
,, ,, —Table of Examples of, 221.
Eccentric—Motion of, 460.
Eddy, 412.
Effect of a Machine, 610.
Efficiency, 609, 610.
,, of Heat Engine, 629, 644.
,, of Waterwheels, 628.
,, of Windmills, 629.
Effort, 476, 611.
Elastic Curve, 349.
,, Force, 270.
Elasticity—Coefficients of, 277.
,, —Fluid, 270.
,, —Liquid, 271.
,, —Modulus of, 279, 646.
,, —Potential Energy of, 277.
,, —Theory of, 270, 275.
Electro-dynamic Engine—Efficiency of, 630.
Elementary Combinations, 465.
Ellipsoid of Inertia, 526.
Energy, 477.
,, —Actual, 499, 507.
,, —Actual—of a Rotating Body, 532.
,, —Components of, 480, 499.
,, —Conservation of—Motion being Uniform, 478.
Energy — Conservation of — in Varied Motion, 501, 508.
,, —Initial, 503.
,, of Couples, 537.
,, —Potential, 477.
,, —Total, 503, 569.
,, —Transformation of, 499.
Engines, 625, 643.
Epicycloid, 401.
Epicycloidal Teeth, 444.
Epitrochoid, 401.
Equilibrated Arch, 162, 175, 182.
Equilibrium of forces, 25, 43.
,, of Structures, 129.
,, —Stable and Unstable, 128.
Expansion of Air, 123, 606.
,, of Metals, Stones, Brick, Glass, Timber, 349.
,, of Steam, 606.
,, of Water, 125.
Extrados of Arch, 173.

FACTORS of Safety, 274, 362, 365.
Fall, or Head, 627.
Falling Body, 485, 486.
Fan—Blowing, 598.
First Law of Motion, 476.
Fixed Direction, 379.
,, Point, 14, 381.
Flexure—Moment of, 311.
,, —Resistance of, 312.
Floating Bodies, 120, 600.
,, —Oscillation of, 603.
Floats of Waterwheels, 628.
Flow of Fluids, 417.
,, of Liquid, 410.
Flues—Strength of, 306.
Fluid, 100.
,, —Elasticity of, 285.
,, —Equilibrium of, 116.
,, —Flow of, 417.
,, —Impulse of—on a Solid Surface, 591.
,, —Motion of, 410, 566.
,, —Pressure of, 99.
,, Resistance, 598.
Fly-wheel, 623.
Foot-pound, 477.
Force, 15, 17.
,, —Absolute Unit of, 486.
,, —Centrifugal, 387, 491.
,, —Deviating 491, 492, 595.
,, —Distributed, 48.
,, —Reciprocating, 503.
,, —Representation of, 19.
,, —Unbalanced—Measures of, 501.
Forces—Action of—on a System of Bodies, 510.
,, —Parallelogram of, 35.
,, —Parallelopiped of, 37.

Forces—Polygon of, 86.
,, —Reduction of, 612.
,, —Residual, 498, 511.
,, —Resolution of, 37.
Foundation, 129.
Foundations—Earth, 219, 255.
Fracture, 272.
Frames—Bracing of, 142.
,, —Equilibrium and Stability of, 132.
,, of two Bars, 136.
,, —Polygonal, 139.
,, —Resistance of—at a Section, 150.
,, —Triangular, 137.
Free Rotation, 533.
,, Surface, 570.
Friction, 209.
,, —Coefficient of, 210, 612.
,, Couplings, 618.
,, —Heat of, 620.
,, —Internal, 377.
,, —Moment of, 614.
,, of Gas, 590.
,, of Liquids, 584.
,, of Machines, 612, 614.
,, of Solid Bodies—Law of, 209.
,, Strap, 618.
,, —Tables of, 211, 613.
Frictional Gearing, 618.
,, Stability, 209.
,, Tenacity, 222.
Furnace—Waste Heat of, 629.

GAS, 13.
,, —Action of—on a Piston, 604.
,, —Dynamic Head in, 579.
,, Engine, 642.
,, —Equation of Continuity in, 581.
,, —Flow of—from an Orifice, 581, 642.
,, —Flow of—with Friction, 590.
,, —Motion of, 566.
,, —Motion of—without Friction, 579.
Geostatic Arch, 196.
Girder—Bowstring, 369.
,, —Cellular, 367.
,, —Compound, 366.
,, —Half-lattice, 153, 369.
,, —Lattice, 160, 369.
,, —Plate, 366.
,, —Stiffening—for Suspension Bridges, 370.
,, —Tubular, 366, 367.
,, —Warren, 153.
Governor, 548, 625.
Gravity—Accelerating Effect of, 485.
,, —Centre of, 15, 49, 61, 180.
,, —Motion under, 485, 486.
,, —Specific, 49, 124.
,, —Specific—Table of, 652.
Grease, 613.
Groined Vaults, 262.
Gyration, 515, 542.
,, —Radius of, 515.
,, —Table of Radii of, 518.

HEAD—Dynamic—of Gas, 579.
,, ,, —of Liquid, 568, 627.
,, —Equal—Surfaces of, 573.
Headers in Masonry, 223.
Heat Engine—Efficiency of, 629.
Heat of Friction, 620.
,, of Steam, 607.
,, —Specific—of Gases at Constant Pressure, 580.
Height due to Velocity, 487.
Helical Motion, 394.
Hooke's Double Joint, 462.
,, Gearing, 451.
,, Law, 275.
,, Universal Joint, 461.
Hoop-tension, 290.
Horse-power, 610.
Horse—Work of, 626.
Hunting Cog, 434.
Hydraulic Hoist, 465.
,, Mean Depth, 587.
,, Press, 462.
,, Rivetting, 642.
Hydraulics, 585.
Hydrodynamics, 475, 566.
Hydrostatics—Principles of, 100, 112, 117.
Hydrostatic Arch, 190, 206, 353.

IMMERSED Body—Pressure on, 122.
,, Plane—Pressure on, 125.
Impact, 564.
Impulse, 483.
,, —Angular, 506.
,, between Solids and Fluids, 591.
,, and Momentum—Law of, 484.
Inclined Plane, 489.
Indicator, 478.
Inertia, or Mass, 482.
,, —Axes of, 524.
,, —Ellipsoid of, 526, 532.
,, —Moment of, 77, 514, 518.
,, —Reduced, 621.
Inside Gearing, 441.
Instantaneous Axis, 397, 404, 467.
Integrals—Approximate Computation of, 58, 386.
Intensity of Distributed Force, 48.
,, of Pressure, 69.
,, of Stress, 68.
Internal Equilibrium of Stress and Weight, 112.
Internal Stress, 280.

Intrados of Arch, 173.
Isochronous Vibration, 553.
Isotropic Solid, 278.

JET—Impulse of, 591.
Joints of a Structure, 129, 131.
,, of Masonry, 211.
,, of Rupture, 259.

KEYS—Friction of, 226.

LATERAL Force, 476.
Lattice Girder, 153, 160.
Least Resistance—Principle of, 215.
Leather—Strength of, 288.
Length—Measure of, 13, 14.
Lever, 26.
Line, 13.
Linear Arch, 162, 182, 189, 203, 258.
Link Motion, 468.
Linkwork in Mechanism, 424, 458.
Liquid, 13.
,, —Dynamic Head of, 568.
,, —Equilibrium of, 118.
,, —Flow of—from an Orifice, 570.
,, —Flow of—in a Pipe, 411, 588.
,, —Flow of—in a Stream, 586.
,, —Free Surface of, 570.
,, —Motion of, 410.
,, —Motion of—in Plane Layers, 570.
,, —Motion of—with Friction, 584.
,, —Surface of Equal Pressure in, 570.
,, without Friction—Motion of, 567.

MACHINES, 5, 15, 421.
,, —Actual Energy of, 621.
,, —Pieces of, 422.
,, —Reduced Inertia of, 621.
,, —Theory of, 609.
,, —Varied Motion of, 621.
,, —Work of—with Uniform or Periodic Motion, 610.
Man—Work of, 626.
Masonry and Brickwork—Bond of, 222.
,, ,, —Friction of, 211, 222.
,, —Stability of, 230.
Mass, 482, 484, 485.
,, —Centre of, 482.
Matter, 13.
Measures—Comparative Table of British and French, 651.
,, of Length, 13, 14.
,, of Stress, 69.
,, of Time, 381.
,, of Velocity, 382.
,, of Weight, 18.
Mechanical Equivalent of Heat, 620.
Mechanics—Applied, 13.
,, —Dissertation on, 1.
Mechanism—Aggregate Combinations in, 425, 466,
,, —Elementary Combinations in, 423, 426.
,, —Principle of Connection in, 424.
,, —Theory of, 421.
Mercury—Weight of, 69.
Metacentre, 601.
Modulus of Elasticity, 279, 646.
,, of Resilience, 287.
,, of Rupture, 316, 649.
Moment—Bending, 307.
,, of a Couple, 22.
,, of Deviation, 528.
,, of Flexure, 311.
,, of Friction, 614.
,, of Inertia, 514.
,, of Inertia of a Surface, 77.
,, of Inertia—Tables of, 82, 229, 518.
,, of Stability, 233.
,, of Stress, 73.
,, of Torsion, 353.
,, Statical, 27, 29.
Momentum, 482.
,, —Angular, 505, 529.
,, —Conservation of, 505.
,, and Impulse—Law of, 484.
,, of a Rotating Body, 529.
Motion, 14.
,, —Comparative, 384, 389.
,, —Component and Resultant, 381, 383.
,, —First Law of, 476.
,, —Friction of, 226, 612.
,, of a System of Bodies, 505.
,, of Fluids—Dynamics of 475, 566.
,, of Gases, 417.
,, of Liquids, 410, 566.
,, of Pliable Bodies and Fluids, 408.
,, of Pliable Bodies—Dynamics of, 552.
,, of Points, 379.
,, of Points—Varied, 385.
,, of Rigid Bodies, 390.
,, —Second Law of, 484.
,, —Uniform — Dynamical Principles of, 476.
,, —Varied—Dynamical Principles of, 482.
Muscular Strength—Work of, 625.

NEUTRAL axis, 73.
Notch—Flow through, 573.

OIL, 613.
Oldham's Coupling, 453.
Orifice—Flow through, 571.

Oscillation, 416.
,, —Angular, 542.
,, —Centre of, 546.
,, —Elliptical, 495.
,, —Straight, 494.
Oscillating Pendulum, 496, 546.
Overshot Wheel, 628.

PARABOLA—Formulæ relating to, 165.
Parallel Forces, 25.
,, Motion, 469.
,, Projection, 45, 61, 127.
Pendulum—Ballistic, 548.
,, —Compound Oscillating, 546.
,, —Compound Revolving, 547.
,, —Cycloidal, 497.
,, —Rotating, 547.
,, —Simple Oscillating, 496.
,, —Simple Revolving, 492.
Percussion—Centre of, 520, 544.
Periodical Motion of Machines, 501.
Pieces of a Structure, 129.
Piers—Stability of, 228.
,, of Arches, 263.
,, —Open and Hollow, 263.
Pile-driving, 564.
Pillars—Strength of Long, 360, 639.
,, —Strength of Short, 302.
Pinion, 434, 443.
Pinnacle on a Buttress, 239.
Pipes—Flow in, 411, 588.
,, —Friction in, 585, 588.
,, —Resistance caused by Sudden Enlargement in, 589.
,, —Resistance of Curves and Knees in, 589.
,, —Resistance of Mouthpieces of, 589.
,, —Strength of, 289.
Piston, 413.
,, —Action of a Fluid upon, 604.
Piston Rods—Strength of, 363.
Pitch, 394, 433, 449.
,, Surface, 426, 454.
Pivot—Friction of, 616.
Plasticity, 272.
Plate-iron Girders, 365, 366.
,, Joints, 299.
Plates—Strength of Iron and Steel, 377, 641.
Pliability, 273.
,, —Coefficients of, 277.
Point—Fixed, 14, 381.
,, —Motions of, 379.
,, —Physical, 13, 475.
Pointed Arch, 203.
Posts—Timber—Strength of, 365.
Potential Energy, 477.
,, of Elasticity, 277.
Pound—Standard, 18.

Power, 610.
Preliminary Dissertation, 1.
Press—Hydraulic, 464.
,, —Strength of, 290.
Pressure, 20, 69, 564.
,, between Rubbing Surfaces, 613.
,, in a Sloping Solid Mass, 126.
,, —Internal, 289.
,, of Earth, 218, 249, 277.
,, of Fluids, 99.
Prime Movers, 609, 625.
Principle of D'Alembert, 511.
Projectiles, 487, 491, 599.
Projection—Parallel, 45, 61, 127.
Proof Strength, 273, 274.
Pulsometer Pump, 641.
Pull, 69.
Pulleys and Belts, 454.
,, and Cords, 462.
,, —Speed, 457.
Pump—Centrifugal, 597.
,, —Pulsometer, 641.
,, Rods—Strength of, 297.

RACK—Motion of, 427.
,, —Teeth of, 438.
Radiating Current, 574.
Radius of Gyration, 515.
,, Vector, 392.
Railways—Resistance on, 620.
Reciprocating Force, 503.
Reduced Inertia, 621.
Reduction of Forces and Couples in Machines to the Driving Point, 612.
Regulator of a Prime Mover, 625.
Repose—Angle of, 210.
Reservoir Walls—Stability of, 243, 638.
Resilience, 273.
,, of Axle, 357.
,, of Beam, 330.
,, of Tie-bar, 287.
Resistance, 476.
,, —Centre of, 131.
,, —Line of, 131.
,, of Carriages on Roads, 619.
,, of Fluids, 598.
,, of Machines, 610.
,, of Materials, 273, 646.
,, of Railway Trains and Engines, 620, 643.
,, of Rolling, 619.
,, of Ships, 598, 641.
,, —Point of, 610.
Resolution of Forces, 37.
,, of Internal Stress, 82.
Rest, 14.
,, —Friction of, 226, 612.
Resultant, 18.
,, Momentum, 482.
,, of any System of Forces, 41.

Resultant of Couples, 23, 24.
" of Inclined Forces, 35.
" of Motions, 381.
" of Parallel Forces, 26, 28, 30.
" of Stress, 70.
" of Weight, 49.
Retaining Walls, 227, 249, 638.
Retardation, 386.
Revêtements, 227, 249.
Revolving Simple Pendulum, 492.
Rib—Arch, 182.
Ribbed Arches, 376.
Rigid Body—Action of a Single Force on, 543.
" —Motion of, 390, 394, 513.
Rigidity or Stiffness, 271.
" of a Truss, 144.
" —Supposition of Perfect, 18.
Rivets—Strength of, 299.
Rivetted Joints—Strength of, 289, 299.
Rivetting—Hydraulic, 642.
Roads—Resistance of, 619.
Rolling Cones, 405, 535.
" Contact in Mechanism, 426.
" Load, 332.
" of Cylinder on Cylinder, 400.
" of Cylinder on Plane, 398.
" of Plane on Cylinder, 398.
" of Ships, 604.
" Resistance, 619.
Roof, 142, 145.
Ropes—Stiffness of, 619.
" —Strength of, 288.
Rotating Body—Comparative Motion of Points in, 393.
" —Relative Motion of a Pair of Points in, 392.
Rotation, 390.
" —Actual Energy of, 532.
" and Force—Analogy of, 405.
" —Angular Velocity of, 391.
" —Axis of, 390.
" Combined with Translation, 394.
" —Comparative Motions in Compound, 406.
" —Compound, 399.
" —Dynamical Principles of, 513.
" —Free, 533.
" —Instantaneous Axis of, 397, 543.
" —Uniform, 535.
" —Varied, 406, 538.
" —Varied — Combined with Translation, 543.
Rupture—Modulus of, 316, 649.
" —Angle of, 204, 259.
" —Point of, 204, 259.

SAFETY—Factors of, 274, 641.
" Valves, 642.
Screw-like Motion, 394.
" Gearing, 451.
Screws—Compound, 467.
" —Friction of, 226.
" in Mechanism, 449.
Second Law of Motion, 484.
Sections—Method of—Applied to Framework, 150.
Set, 271.
Shafts and Axles—Strength of, 353.
Shear, 69, 87.
Shearing Force in Beams, 307.
" —Resistance to, 298;
" Stress in Beams, 338.
" —Table of, 648.
Shifting or Translation, 390.
Ship Resistance, 599, 641.
Shrunk Rings, 294.
Skew Arches, 261.
" Bevel Wheels, 430, 449.
Sliding Contact in Mechanism, 436.
Solid, 13.
Sound—Velocity of, 563.
Spandril Wall, 257.
Specific Gravity, 42, 124, 652.
Speed-cones, 457.
Speed—Fluctuations of, 622.
Spheres—Strength of, 290.
Spiral, 398.
Stability, 128.
" —Frictional, 230.
" of Floating Bodies, 600.
" of Structures, 130, 131.
Standard Measure of Length, 14.
" Measure of Weight, 18.
Starting of Machines, 624.
Statics, 15.
" —Principles of, 17.
Stays, 133, 136.
Steady Motion of a Gas, 419.
" of a Liquid, 412, 414.
Steam—Action of, 606.
" Boilers, 289.
" Engine—Efficiency of, 629, 630, 643.
Steel—Strength of, 377, 631, 641, 646.
Stereostatic Arch, 198.
Stiffness, 130, 270, 273.
" of Beams, 322.
Stopping of Machines, 624.
Strain, 272.
" and Stress—Relations between, 280.
" —Ellipse of, 280.
" —Resolution and Composition of, 275.
Stream—Friction of, 586.
" —Hydraulic Mean Depth of, 587.
" Lines, 600.
" of Gas, 417.
" of Liquid, 411, 586.

Stream—Varying, 587.
Strength, 130, 270.
  „ of Abutments and Vaults, 268.
  „ of Axles, 353, 358.
  „ of Beams, 307, 315.
  „ of Boilers, Pipes, and Cylinders, 289, 299, 306.
  „ of Bolts, Pins, Keys, and Rivets, 299.
  „ of Iron and Steel, 377,640, 646.
  „ of Iron — Effects of Repeated Melting on, 376.
  „ of Leathern Belts, 288.
  „ of Long Pillars and Struts, 360, 639.
  „ of Masonry and Brickwork, 268, 302.
  „ of Pump-rods, 298.
  „ of Ropes and Cables, 288.
  „ of Short Pillars, 304.
  „ of Spheres, 290, 295.
  „ of Teeth, 359.
  „ of Tie-bar, 286.
  „ of Tubes and Flues, 306.
  „ —Proof, 273.
  „ —Tables of, 377, 646.
  „ Transverse, 315.
  „ —Ultimate, 273.
Stress, 68.
  „ and Strain — Relations between, 280.
  „ —Internal, 82.
Stretchers in Masonry, 223.
Stretching—Resistance to, 286.
Stroke—Length of—in Mechanism, 460.
Structures, 15.
  „ —Theory of, 129.
  „ —Transformation of, 129.
Struts, 133.
  „ —Strength of, 302, 360, 365.
  „ „ Wrought-iron, 364.
Superposition of Small Motions, 555.
Surface, 13.
Suspension Bridge, 149, 165, 168, 171.
  „ —Stiffened, 370.
  „ —Strength of, 288, 301.
  „ with Sloping Rods, 171.
  „ with Vertical Rods, 168.
System of Bodies—Motion of, 505.

TABLES of Bending Moments, 310, 311.
  „ of Coefficient of Friction, 211, 613.
  „ of Compressibilty of Liquids, 471.
  „ of Expansion by Heat, 349.
  „ of Factors of Safety, 274, 356.
  „ of Figures of Beams, 321.
  „ of French and British Measures, 651.
Tables of Measures of Velocity, 382.
  „ of Moments of Inertia, 82, 229, 518.
  „ of Shearing Forces, 310, 311.
  „ of Specific Gravities, 652.
  „ of Stability of Earth, 221.
  „ of Strength of Iron Pillars, 363.
  „ „ of Materials, 274, 288, 646-650.
  „ of Work of Men and Horses, 626.
Tangential Stress, 69, 87.
Tearing—Resistance to, 286.
  „ —Tables of Resistance to, 288, 289, 377, 646.
Teeth of Wheels, 432.
  „ —Dimensions of, 447.
  „ —Epicycloidal, 444.
  „ —Friction of, 617.
  „ —Form of, 438.
  „ —Involute, 441.
  „ of Bevel Wheels, 448.
  „ of Wheel and Trundle, 447.
  „ —Pitch and Number of, 432.
  „ —Strength of, 359.
Temperature—Effects of, 376.
Tenacity, 286.
Tension, 69.
Testing Strength, 273.
Theory and Practice in Mechanics—Harmony of, 1, 10.
Thrust, 69.
Tie, 132.
  „ —Flexible, 169.
  „ —Strength of, 286.
Timber Struts, 365.
  „ Ties, 301.
Time—Measure of, 381.
Torsion—Moment of, 353.
Toughness, 273.
Towers—Stability of, 240.
Trains of Mechanism, 465.
  „ „ —Efficiency of, 610.
  „ „ —Epicyclic, 473.
  „ of Wheels, 434.
Transformation, 66, 127.
  „ of Cords and Chains, 180.
  „ of Energy, 501.
  „ of Frames, 162.
  „ of Stress, 92.
  „ of Structures in Masonry, 332, 268.
Translation or Shifting, 390.
  „ —Varied, 482.
Transverse Strength, 315.
  „ —Table of, 649.
Trochoid, 398.
Trundle, 447.
Truss, 144.
  „ —Compound, 148.
Trussing—Secondary, 145.

Tunnels, 264.
Turbine, 595, 628, 629.
Turning, 390.
Twisting and Bending, 358.
,, Moment, 353.

UNBALANCED Force—Measures of, 501.
Undershot Wheel, 628.
Unguents, 613.
Uniform Deviation, 387.
,, Effort or Resistance—Effect of, 490.
,, Motion, 382.
,, Motion under Balanced Forces, 476.
,, Velocity, 382.
Unit of Force, 18, 486.
,, of Length, 14.
,, of Specific Gravity, 49.
,, of Time, 381.
Universal Joint, 461.
,, ,, Double, 462.
Unsteady Motion of Fluid, 413, 415.

VANES—Impulse of Liquid on, 593.
Varied Rotation, 538.
Vaults—Groined, 262.
,, —Stability of, 226.
Velocity, 382.
,, —Angular, 391.
,, of Sound, 563.
,, Ratio, 463.
,, —Uniform, 382.
,, —Uniformly-varied, 386.
,, —Varied, 385.
,, —Varied Rate of Variation of, 387.
Velocities—Virtual, 479.
Vibration, 552.
,, —Isochronous, 553.
,, —Not Isochronous, 557.
,, of Elastic Body, 557.
Virtual Velocities, 479.
Viscous Liquid, 273.
Vis-viva, 499.
Volume, 13.
Vortex, 412, 574.
,, —Action of—on Wheel, 595, 629.
,, —Combined, 576.
,, —Forced, 576.
,, —Free Circular, 574.
,, —Free Spiral, 576.
Vortex Wheel, 596.

WALLS—Retaining, 227, 249.
,, —Stability of, 226, 638.
Warren Girder, 153.
Water—Apparent Weight of Bodies immersed in, 125.
,, —Expansion of, 125.
,, —Flow of, 585.
,, Pipes, 289, 585.
,, Pressure Engine, 626.
,, —Velocity of Sound in, 563.
,, —Weight of, 125.
Water-wheel, 578, 627.
,, —Action of Vortex on, 595, 629.
,, —Efficiency of, 627.
,, —Impulse of Water on Floats of, 593.
Waves—Motion in, 416, 579, 631.
,, of Vibration, 562.
Wedges—Friction of, 226.
Weight, 49, 485.
,, —Apparent—of Body immersed in Fluid, 123.
,, —Measures of, 18, 651.
,, —Table of, 652.
Weir, 243.
Wheel and Screw, 452.
Wheels—Bevel, 428, 448.
,, —Grooved, 431.
,, —Motion of, 426.
,, —Non-circular, 428, 449.
,, —Skew-bevel, 430, 449.
,, —Teeth of, 432.
,, —Train of, 434.
White's Tackle, 463.
Wind—Action of—on Towers and Chimneys, 240.
,, —Pressure of, 240, 639.
Windlass—Differential, 466.
Windmills, 629.
Wiper or Cam, 449.
Work, 477.
,, of Machines, 610.
,, —Useful and Lost, 610.
Working Point, 611.
,, Stress, 274.
Wrenching—Resistance to, 353.
Wrought-iron—Strength of, 362, 377, 646.

YARD—Standard, 14.

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